CONVERGENCE THEOREMS FOR A COMMON FIXED POINT OF A FINITE FAMILY OF NONSELF NONEXPANSIVE MAPPINGS

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Let K be a nonempty closed convex subset of a reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Let $T_i : K \to E$, $i = 1,\dots,r$, be a family of nonexpansive mappings which are weakly inward. Assume that every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive mappings. A strong convergence theorem is proved for a common fixed point of a family of nonexpansive mappings provided that T_i, $i = 1,2,\dots,r$, satisfy some mild conditions.

1. Introduction

Let K be a nonempty closed convex subset of a real Banach space E. A mapping $T : K \to E$ is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$. Let $T : K \to K$ be a nonexpansive self-mapping. For a sequence $\{\alpha_n\}$ of real numbers in $(0,1)$ and an arbitrary $u \in K$, let the sequence $\{x_n\}$ in K be iteratively defined by $x_0 \in K$,

$$x_{n+1} := \alpha_{n+1} u + (1 - \alpha_{n+1}) T x_n, \quad n \ge 0. \tag{1.1}$$

Halpern [5] was the first to study the convergence of the algorithm (1.1) in the framework of Hilbert spaces. Lions [6] improved the result of Halpern, still in Hilbert spaces, by proving strong convergence of $\{x_n\}$ to a fixed point of T if the real sequence $\{\alpha_n\}$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\lim_{n\to\infty}((\alpha_n - \alpha_{n-1})/\alpha_n^2) = 0$.

It was observed that both Halpern's and Lions' conditions on the real sequence $\{\alpha_n\}$ excluded the natural choice, $\alpha_n := (n+1)^{-1}$. This was overcome by Wittmann [12] who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ if $\{\alpha_n\}$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii)* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

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Reich [9] extended this result of Wittmann to the class of Banach spaces which are uniformly smooth and have weakly sequentially continuous duality maps. Moreover, the sequence $\{\alpha_n\}$ is required to satisfy conditions (i) and (ii) and to be decreasing (and hence also satisfying (iii)*). Subsequently, Shioji and Takahashi [10] extended Wittmann's result to Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex subset of K has the fixed point property for nonexpansive mappings and $\{\alpha_n\}$ satisfies conditions (i), (ii), and (iii)*.

Xu [13] showed that the results of Halpern holds in *uniformly smooth Banach spaces* if $\{\alpha_n\}$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii)** $\lim_{n\to\infty}((\alpha_n - \alpha_{n-1})/\alpha_n) = 0$.

As has been remarked in [13], conditions (iii) and (iii)* are not comparable. Also conditions (iii)* and (iii)** are not comparable. However, condition (iii) does not permit the natural choice $\alpha_n := (n+1)^{-1}$ for all integers $n \geq 0$. Hence, conditions (iii)* and (iii)** are preferred.

In [2], Chidume et al. extended the results of Xu to Banach spaces which are more general than uniformly smooth spaces.

Next consider r nonexpansive mappings $T_1, T_2, \ldots, T_r$. For a sequence $\{\alpha_n\} \subseteq (0,1)$ and an arbitrary $u_0 \in K$, let the sequence $\{x_n\}$ in K be iteratively defined by $x_0 \in K$,

$$x_{n+1} := \alpha_{n+1} u + (1 - \alpha_{n+1}) T_{n+1} x_n, \quad n \geq 0, \tag{1.2}$$

where $T_n = T_{n(\bmod r)}$.

In 1996, Bauschke [1] defined and studied the iterative process (1.2) in Hilbert spaces with conditions in (i), (ii), and (iii)* on the parameter $\{\alpha_n\}$.

Recently, Takahashi et al. [11] extended Bauschke's result to uniformly convex Banach spaces. More precisely, they proved the following result.

Theorem 1.1 [11]. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E which has a uniformly Gâteaux differentiable norm. Let $T_i : K \to K$, $i = 1, \ldots, r$, be a family of nonexpansive mappings with $F := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and $\bigcap_{i=1}^{r} F(T_i) = F(T_r T_{r-1} \cdots T_1) = F(T_1 T_r \cdots T_2) = \cdots = F(T_{r-1} T_{r-2} \cdots T_1 T_r)$. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1} u + (1 - \alpha_{n+1}) T_{n+1} x_n, \quad n \geq 0, \tag{1.3}$$

where $T_n := T_{n(\bmod r)}$ and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* (iii)* $\sum_{n=1}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$. *Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_r\}$. Further, if $Px_0 = \lim_{n\to\infty} x_n$ for each $x_0 \in K$, then P is a sunny nonexpansive retraction of K onto F.*

More recently, O'Hara et al. [8] proved the following complementary result to Bauschke's theorem [1] with condition (iii)* replaced with (iii)** $\lim_{n\to\infty}((\alpha_{n+r} - \alpha_n)/\alpha_{n+r}) = 0$ (or equivalently, $\lim_{n\to\infty}(\alpha_n/\alpha_{n+r}) = 1$).

Theorem 1.2 [8]. *Let K be a nonempty closed convex subset of a Hilbert space H and let $T_i : K \to K$, $i = 1,\dots,r$, be a family of nonexpansive mappings with $F := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and $\bigcap_{i=1}^{r} F(T_i) = F(T_r T_{r-1} \cdots T_1) = F(T_1 T_r \cdots T_2) = \cdots = F(T_{r-1} T_{r-2} \cdots T_1 T_r)$. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1} u + (1 - \alpha_{n+1}) T_{n+1} x_n, \quad n \geq 0, \tag{1.4}$$

where $T_n := T_{n(\mathrm{mod}\, r)}$ and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* (iii)** $\lim_{n\to\infty}(\alpha_n/\alpha_{n+r}) = 1$. *Then $\{x_n\}$ converges strongly to Pu, where P is the projection of K onto F.*

In the above work, the mappings $T_1, T_2, \dots, T_r$ remain self-mappings of a nonempty closed convex subset K either of a Hilbert space or a uniformly convex space. If, however, the domain of $T_1, T_2, \dots, T_r$, $D(T_i) = K$, $i = 1, 2, \dots, r$, is a proper subset of E and T_i maps K into E, then the iteration process (1.4) may fail to be well defined (see also (1.3)).

It is our purpose in this paper to define an algorithm for nonself-mappings and to obtain a strong convergence theorem to a fixed point of a family of nonself nonexpansive mappings in Banach spaces more general than the spaces considered by Takahashi et al. [11] with $\{\alpha_n\}$ satisfying conditions (i), (ii), and (iii)*. We also show that our result holds if $\{\alpha_n\}$ satisfies conditions (i), (ii), and (iii)**. Our results extend and improve the corresponding results of O'Hara et al. [8], Takahashi et al. [11], and hence Bauschke [1] to more general Banach spaces and to the class of *nonself*-maps.

2. Preliminaries

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx := \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single valued. In the sequel, we will denote the single-valued normalized duality map by j.

The norm is said to be *uniformly Gâteaux differentiable* if for each $y \in S_1(0) := \{x \in E : \|x\| = 1\}$, $\lim_{t\to 0}((\|x + ty\| - \|x\|)/t)$ exists uniformly for $x \in S_1(0)$. It is well known that L_p spaces, $1 < p < \infty$, have uniformly Gâteaux differentiable norm (see, e.g., [4]). Furthermore, if E has a uniformly Gâteaux differentiable norm, then the duality map is norm-to-w^* uniformly continuous on bounded subsets of E.

A Banach space E is said to be *strictly convex* if $\|(x + y)/2\| < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space E, we have that if $\|x\| = \|y\| = \|\lambda x + (1 - \lambda) y\|$, for $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$.

Let K be a nonempty subset of a Banach space E. For $x \in K$, the *inward set* of x, $I_K(x)$, is defined by $I_K(x) := \{x + \lambda(u - x) : u \in K,\ \lambda \geq 1\}$. A mapping $T : K \to E$ is called *weakly inward* if $Tx \in \mathrm{cl}[I_K(x)]$ for all $x \in K$, where $\mathrm{cl}[I_K(x)]$ denotes the closure of the inward set. Every self-map is trivially weakly inward.

Let $K \subseteq E$ be closed convex and Q a mapping of E onto K. Then Q is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in E$ and $t \geq 0$. A mapping Q of E into E is said to be a *retraction* if $Q^2 = Q$. If a mapping Q is a retraction, then $Qz = z$ for every $z \in R(Q)$, *range of* Q. A subset K of E is said to be a *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E onto K and it is said to be a *nonexpansive retract* of E if there exists a nonexpansive retraction of E onto K. *If $E = H$, the metric projection P_K is a sunny nonexpansive retraction from H to any closed convex subset of H.*

In the sequel, we will make use of the following lemma.

Lemma 2.1. *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the relation*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n, \quad n \geq 0, \tag{2.2}$$

where (i) $0 < \alpha_n < 1$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Suppose, either* $(a)\sigma_n = o(\alpha_n)$, *or* $(b)\sum_{n=1}^{\infty} \sigma_n < \infty$, *or* $(c)\limsup_{n\to\infty} \sigma_n \leq 0$. *Then $a_n \to 0$ as $n \to \infty$ (see, e.g.,* [13]*).*

We will also need the following results.

Lemma 2.2 (see, e.g., [7]). *Let E be a real Banach space. Then the following inequality holds. For each $x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle \quad \forall j(x + y) \in J(x + y). \tag{2.3}$$

Theorem 2.3 [7, Theorem 1, Proposition 2(v)]. *Let K be a nonempty closed convex subset of a reflexive Banach space E which has a uniformly Gâteaux differentiable norm. Let $T : K \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive mappings. Then there exists a continuous path $t \to z_t$, $0 < t < 1$, satisfying $z_t = tu + (1 - t)Tz_t$, for arbitrary but fixed $u \in K$, which converges strongly to a fixed point of T. Further, if $Pu = \lim_{t\to 0} z_t$ for each $u \in K$, then P is a sunny nonexpansive retraction of K onto $F(T)$.*

3. Main results

We now prove the following theorem.

Theorem 3.1. *Let K be a nonempty closed convex subset of a reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Assume that every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive mappings. Let $T_i : K \to E$, $i = 1, \ldots, r$, be a family of nonexpansive mappings which are weakly inward with $F := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and $\bigcap_{i=1}^{r} F(QT_i) = F(QT_rQT_{r-1}\cdots QT_1) = F(QT_1QT_r\cdots QT_2) = \cdots = F(QT_{r-1}QT_{r-2}\cdots QT_1QT_r)$. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})QT_{n+1}x_n, \quad n \geq 0, \tag{3.1}$$

where $T_n := T_{n(\bmod r)}$ and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$; *and either* (iii)* $\sum_{n=1}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$, *or* (iii)** $\lim_{n\to\infty}((\alpha_{n+r} - \alpha_n)/\alpha_{n+r}) = 0$. *Then $\{x_n\}$ converges strongly to a common fixed point* *of* $\{T_1, T_2, \ldots, T_r\}$. *Further, if* $Pu = \lim_{n\to\infty} x_n$ *for each* $u \in K$, *then* P *is a sunny nonexpansive retraction of* K *onto* F.

Proof. For $x^* \in F$, one easily shows by induction that $\|x_n - x^*\| \le \max\{\|x_0 - x^*\|, \|u - x^*\|\}$, for all integers $n \ge 0$, and hence $\{x_n\}$ and $\{QT_{n+1}x_n\}$ are bounded. But this implies that $\|x_{n+1} - QT_{n+1}x_n\| = \alpha_{n+1}\|u - QT_{n+1}x_n\| \to 0$ as $n \to \infty$. Now we show that

$$\|x_{n+r} - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.2}$$

From (3.1), we get that

$$\begin{aligned}
\|x_{n+r} - x_n\| &= \|(\alpha_{n+r} - \alpha_n)(u - QT_n x_{n-1}) + (1-\alpha_{n+r})(QT_{n+r}x_{n+r-1} - QT_n x_{n-1})\| \\
&= \|(\alpha_{n+r} - \alpha_n)(u - QT_n x_{n-1}) + (1-\alpha_{n+r})(QT_n x_{n+r-1} - QT_n x_{n-1})\| \\
&\le (1-\alpha_{n+r})\|x_{n+r-1} - x_{n-1}\| + |\alpha_{n+r} - \alpha_n| M,
\end{aligned} \tag{3.3}$$

for some $M > 0$. We consider two cases.

Case 1. Condition (iii)* is satisfied. Then,

$$\|x_{n+r} - x_n\| \le (1-\alpha_{n+r})\|x_{n+r-1} - x_{n-1}\| + \sigma_n, \tag{3.4}$$

where $\sigma_n := M|\alpha_{n+r} - \alpha_n|$ so that $\sum_{n=1}^{\infty}\sigma_n < \infty$.

Case 2. Condition (iii)** is satisfied. Then,

$$\|x_{n+r} - x_n\| \le (1-\alpha_{n+r})\|x_{n+r-1} - x_{n-1}\| + \sigma_n, \tag{3.5}$$

where $\sigma_n := \alpha_{n+r}\beta_n$ and $\beta_n := (|\alpha_{n+r} - \alpha_n|M/\alpha_{n+r})$ so that $\sigma_n = o(\alpha_{n+r})$.

In either case, by Lemma 2.1, we conclude that $\lim_{n\to\infty}\|x_{n+r} - x_n\| = 0$. Next we prove that

$$\lim_{n\to\infty}\|x_n - QT_{n+r}\cdots QT_{n+1}x_n\| = 0. \tag{3.6}$$

In view of (3.2), it suffices to show that $\lim_{n\to\infty}\|x_{n+r} - QT_{n+r}\cdots QT_{n+1}x_n\| = 0$. Since $\|x_{n+r} - QT_{n+r}x_{n+r-1}\| = \alpha_{n+r}\|u - QT_{n+r}x_{n+r-1}\|$ and $\lim_{n\to\infty}\alpha_n = 0$, we have that $x_{n+r} - QT_{n+r}x_{n+r-1} \to 0$. From

$$\begin{aligned}
\|x_{n+r} - QT_{n+r}QT_{n+r-1}x_{n+r-2}\| &\le \|x_{n+r} - QT_{n+r}x_{n+r-1}\| \\
&\quad + \|QT_{n+r}x_{n+r-1} - QT_{n+r}QT_{n+r-1}x_{n+r-2}\| \\
&\le \|x_{n+r} - QT_{n+r}x_{n+r-1}\| + \|x_{n+r-1} - QT_{n+r-1}x_{n+r-2}\| \\
&= \|x_{n+r} - QT_{n+r}x_{n+r-1}\| + \alpha_{n+r-1}\|u - QT_{n+r-1}x_{n+r-2}\|,
\end{aligned} \tag{3.7}$$

we also have $x_{n+r} - QT_{n+r}QT_{n+r-1}x_{n+r-2} \to 0$. Similarly, we obtain the conclusion. Let $z_t^n \in K$ be a continuous path satisfying

$$z_t^n = tu + (1-t)QT_{n+r}QT_{n+r-1}\cdots QT_{n+1}z_t^n \tag{3.8}$$

guaranteed by Theorem 2.3. Also by Theorem 2.3, $z_t^n \to Pu$ as $t \to 0^+$, where P is the sunny nonexpansive retraction of K onto $\bigcap_{i=1}^r F(QT_i)$ (notice $\bigcap_{i=1}^r F(QT_i) = F(QT_{n+r}QT_{n+r-1}\ldots QT_{n+1})$) and hence as T_i, $i = 1,\ldots,r$, is weakly inward by [2, Remark 2.1], $Pu \in F = \bigcap_{i=1}^r F(T_i)$. Let $a = \limsup_{n\to\infty}\langle u - Pu, j(x_n - Pu)\rangle$. Now we show that $a \le 0$. We can find a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $a = \lim_{i\to\infty}\langle u - Pu, j(x_{n_i} - Pu)\rangle$. We assume that $n_i \equiv k(\operatorname{mod} r)$ for some $k \in \{1,2,\ldots,r\}$. Using Lemma 2.2, we have that

$$
\begin{aligned}
\|z_t^k - x_{n_i}\|^2 &= \|t(u - x_{n_i}) + (1-t)(QT_{n_i+r}QT_{n+r-1}\cdots QT_{n_i+1}z_t^k - x_{n_i})\|^2 \\
&\le (1-t)^2\|QT_{n_i+r}QT_{n_i+r-1}\cdots QT_{n_i+1}z_t^k - x_{n_i}\|^2 + 2t\langle u - x_{n_i}, j(z_t^k - x_{n_i})\rangle \\
&\le (1-t)^2\big(\|QT_{n_i+r}QT_{n_i+r-1}\cdots QT_{n_i+1}z_t^k - QT_{n_i+r}QT_{n_i+r-1}\cdots QT_{n_i+1}x_{n_i}\| \\
&\qquad\qquad + \|QT_{n_i+r}QT_{n_i+r-1}\cdots QT_{n_i+1}x_{n_i} - x_{n_i}\|\big)^2 \\
&\quad + 2t\Big(\|z_t^k - x_{n_i}\|^2 + \langle u - z_t^k, j(z_t^k - x_{n_i})\rangle\Big) \\
&\le (1+t^2)\|z_t - x_{n_i}\|^2 + \|QT_{n_i+r}QT_{n_i+r-1}\cdots QT_{n_i+1}x_{n_i} - x_{n_i}\| \\
&\quad \times (2\|z_t^k - x_{n_i}\| + \|QT_{n_i+r}QT_{n_i+r-1}\cdots QT_{n_i+1}x_{n_i} - x_{n_i}\|) \\
&\quad + 2t\langle u - z_t^k, j(z_t^k - x_{n_i})\rangle,
\end{aligned}
\tag{3.9}
$$

and hence,

$$
\begin{aligned}
\langle u - z_t^k, j(x_{n_i} - z_t^k)\rangle \le \frac{t}{2}\|z_t^k - x_{n_i}\|^2 &+ \frac{\|QT_{n_i+r}QT_{n_i+r-1}\cdots QT_{n_i+1}x_{n_i} - x_{n_i}\|}{2t} \\
&\times (2\|z_t^k - x_{n_i}\| + \|QT_{n_i+r}QT_{n_i+r-1}\cdots QT_{n_i+1}x_{n_i} - x_{n_i}\|).
\end{aligned}
\tag{3.10}
$$

Since $\{x_{n_i}\}$ is bounded, we have that $\{QT_{n+r}QT_{n_i+r-1}\cdots QT_{n_i+1}x_{n_i}\}$ is bounded and by (3.6), $\|x_{n_i} - QT_{n_i+r}QT_{n_i+r-1}\cdots QT_{n_i+1}x_{n_i}\| \to 0$ as $i \to \infty$, then it follows from the last inequality that

$$
\limsup_{t\to 0^+}\limsup_{i\to\infty}\langle u - z_t^k, j(x_{n_i} - z_t^k)\rangle \le 0. \tag{3.11}
$$

Moreover, j is norm-to-w^* uniformly continuous on bounded subsets of E. Thus, we obtain from (3.11) that

$$
\limsup_{i\to\infty}\langle u - Pu, j(x_{n_i} - Pu)\rangle \le 0, \tag{3.12}
$$

and hence $\limsup_{n\to\infty}\langle u - Pu, j(x_n - Pu)\rangle \le 0$. Furthermore, from (3.1), we have $x_{n+1} - Pu = \alpha_{n+1}(u - Pu) + (1-\alpha_{n+1})(QT_{n+1}x_n - Pu)$. Thus using Lemma 2.2, we obtain that

$$
\begin{aligned}
\|x_{n+1} - Pu\|^2 &\le (1-\alpha_{n+1})^2\|QT_{n+1}x_n - Pu\|^2 + 2\alpha_{n+1}\langle u - Pu, j(x_{n+1} - Pu)\rangle \\
&\le (1-\alpha_{n+1})\|x_n - Pu\|^2 + \sigma_{n+1},
\end{aligned}
\tag{3.13}
$$

where $\sigma_{n+1} := \alpha_{n+1}\beta_{n+1}$ and $\limsup_{n\to\infty}\sigma_{n+1} \le 0$, for $\beta_{n+1} := \langle u - Pu, j(x_{n+1} - Pu)\rangle$. Thus, by Lemma 2.1, $\{x_n\}$ converges strongly to a common fixed point Pu of $\{T_1, T_2, \ldots, T_r\}$. The proof is complete. □

If in Theorem 3.1, T_i, $i = 1,\dots,r$, are self-mappings then the projection operator Q is replaced with I, the identity map on E. Moreover, each T_i for $i \in \{1,2,\dots,r\}$ is weakly inward. Thus, we have the following corollary.

Corollary 3.2. *Let K be a nonempty closed convex subset of a reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Assume that every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive mappings. Let $T_i : K \to K$, $i = 1,\dots,r$, be a family of nonexpansive mappings with $\bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and $\bigcap_{i=1}^{r} F(T_i) = F(T_r T_{r-1} \cdots T_1) = F(T_1 T_r \cdots T_2) = \cdots = F(T_{r-1} T_{r-2} \cdots T_1 T_r)$. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1} u + (1-\alpha_{n+1}) T_{n+1} x_n, \quad n \geq 0, \tag{3.14}$$

where $T_n := T_{n(\operatorname{mod} r)}$ and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$; and either (iii) $\sum_{n=1}^{\infty} |\alpha_{n+r} - \alpha_n| < \infty$, or (iii)** $\lim_{n\to\infty}((\alpha_{n+r} - \alpha_n)/\alpha_{n+r}) = 0$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2,\dots,T_r\}$. Further, if $Pu = \lim_{n\to\infty} x_n$ for each $u \in K$, then P is a sunny nonexpansive retraction of K onto F.*

In the sequel, we will use the following lemma.

Lemma 3.3. *Let K be a nonempty closed convex subset of a strictly convex real Banach space E. Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Let $T_i : K \to E$, $i = 1,\dots,r$, be a family of nonexpansive mappings which are weakly inward with $\bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $S_i : K \to E$, $i = 1,\dots,r$, be a family of mappings defined by $S_i := (1-\lambda_i)I + \lambda_i T_i$, $0 < \lambda_i < 1$ for each $i = 1,2,\dots,r$. Then $\bigcap_{i=1}^{r} F(T_i) = \bigcap_{i=1}^{r} F(S_i) = \bigcap_{i=1}^{r} F(QS_i)$ and $\bigcap_{i=1}^{r} F(S_i) = F(QS_r QS_{r-1} \cdots QS_1) = F(QS_1 QS_r \cdots QS_2) = \cdots = F(QS_{r-1} QS_{r-2} \cdots QS_1 QS_r)$.*

Proof. We note that, since T_i for each $i \in \{1,2,\dots,r\}$ is weakly inward, then by [3, Remark 3.3], S_i, is weakly inward. Moreover, by [2, Remark 2.1], $F(QS_i) = F(S_i)$. Furthermore, one easily shows that $F(S_i) = F(T_i)$ for each $i = 1,2,\dots,r$. Now we show that $\bigcap_{i=1}^{r} F(S_i) = F(QS_r QS_{r-1} \cdots QS_1) = F(QS_1 QS_r \cdots QS_2) = \cdots = F(QS_{r-1} QS_{r-2} \cdots QS_1 QS_r)$. For simplicity, we prove for $r = 2$. It is clear that $F(S_1) \bigcap F(S_2) \subseteq F(QS_2 QS_1)$. Now, we show that $F(QS_2 QS_1) \subseteq F(S_1) \bigcap F(S_2)$. Let $z \in F(QS_2 QS_1)$ and $w \in F(S_1) \bigcap F(S_2) = F(T_1) \bigcap F(T_2)$. Then,

$$\begin{aligned}
\|z - w\| &= \|QS_2 QS_1 z - w\| \\
&\leq \|(1-\lambda_2) Q[(1-\lambda_1)z + \lambda_1 T_1 z] + \lambda_2 T_2 (Q[(1-\lambda_1)z + \lambda_1 T_1 z]) - w\| \\
&\leq (1-\lambda_2)\|(1-\lambda_1)z + \lambda_1 T_1 z - w\| + \lambda_2 \|(1-\lambda_1)z + \lambda_1 T_1 z - w\| \\
&= \|(1-\lambda_1)(z - w) + \lambda_1 (T_1 z - w)\| \\
&\leq (1-\lambda_1)\|z - w\| + \lambda_1 \|T_1 z - w\| \leq \|z - w\|.
\end{aligned} \tag{3.15}$$

Thus from the preceding inequalities and strict convexity of E, we obtain that $z - w = T_1 z - w$ and $T_2(Q[(1-\lambda_1)z + \lambda_1 T_1 z]) - w = z - w$. Therefore, we obtain that $z = T_1 z = T_2 z$. This completes the proof. $\square$

Theorem 3.4. *Let K be a nonempty closed convex subset of a strictly convex reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Assume that every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive mappings. Let $T_i : K \to E$, $i = 1,\ldots,r$, be a family of nonexpansive mappings which are weakly inward with $\bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $S_i : K \to E$, $i = 1,\ldots,r$, be a family of mappings defined by $S_i := (1-\lambda_i)I + \lambda_i T_i$, $0 < \lambda_i < 1$ for each $i = 1,2,\ldots,r$. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1}u + (1-\alpha_{n+1})QS_{n+1}x_n, \quad n \geq 0, \tag{3.16}$$

where $S_n := S_{n(\bmod r)}$ and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n\to\infty}\alpha_n = 0$; (ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$; and either (iii) $\sum_{n=1}^{\infty}|\alpha_{n+r} - \alpha_n| < \infty$, or (iii)** $\lim_{n\to\infty}((\alpha_{n+r} - \alpha_n)/\alpha_{n+1}) = 0$. Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_r\}$. Further, if $Pu = \lim_{n\to\infty} x_n$ for each $u \in K$, then P is a sunny nonexpansive retraction of K onto F.*

Proof. By Lemma 3.3, $\bigcap_{i=1}^{r} F(T_i) = \bigcap_{i=1}^{r} F(S_i) = \bigcap_{i=1}^{r} F(QS_i)$ and $\bigcap_{i=1}^{r} F(QS_i) = F(QS_rQS_{r-1}\cdots QS_1) = F(QS_1QS_r\cdots QS_2) = \cdots = F(QS_{r-1}QS_{r-2}\cdots QS_1QS_r)$. Thus, as in the proof of Theorem 3.1, $x_n \to x^* \in \bigcap_{i=1}^{r} F(T_i)$. The proof is complete. □

If in Theorem 3.4, T_i, $i = 1,\ldots,r$, are self-mappings, the following corollary follows.

Corollary 3.5. *Let K be a nonempty closed convex subset of a strictly convex reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Assume that every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive mappings. Let $T_i : K \to K$, $i = 1,\ldots,r$, be a family of nonexpansive mappings with $\bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. Let $S_i : K \to K$, $i = 1,\ldots,r$, be a family of mappings defined by $S_i := (1-\lambda_i)I + \lambda_i T_i$, $0 < \lambda_i < 1$ for each $i = 1,2,\ldots,r$. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := \alpha_{n+1}u + (1-\alpha_{n+1})S_{n+1}x_n, \quad n \geq 0, \tag{3.17}$$

where $S_n := S_{n(\bmod r)}$ and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions: (i) $\lim_{n\to\infty}\alpha_n = 0$; (ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$; and either (iii) $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, or (iii)** $\lim_{n\to\infty}((\alpha_{n+1} - \alpha_n)/\alpha_{n+r}) = 0$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_r\}$. Further, if $Pu = \lim_{n\to\infty} x_n$ for each $u \in K$, then P is a sunny nonexpansive retraction of K onto F.*

Remark 3.6. Corollaries 3.2 and 3.5 are improvements of Theorems 1.1 and 1.2 to more general Banach spaces (having a uniformly Gâteaux differentiable norm) than uniformly convex spaces. Moreover, If E is a Hilbert space, Corollary 3.2 reduces to the result of Bauschke [1].

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