ON GENERALIZED VECTOR QUASIVARIATIONAL-LIKE INEQUALITY PROBLEMS

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We introduce a class of generalized vector quasivariational-like inequality problems in Banach spaces. We derive some new existence results by using KKM-Fan theorem and an equivalent fixed point theorem. As an application of our results, we have obtained as special cases the existence results for vector quasi-equilibrium problems, generalized vector quasivariational inequality and vector quasi-optimization problems. The results of this paper generalize and unify the corresponding results of several authors and can be considered as a significant extension of the previously known results.

1. Introduction

Let K be a nonempty subset of a space X and $f : K \times K \to \mathbb{R}$ be a bifunction. The equilibrium problem introduced and studied by Blum and Oettli [4] in 1994 is defined to be the problem of finding a point $x \in K$ such that $f(x, y) \geq 0$ for each $y \in K$. If we take $f(x, y) = \langle T(x), y - x\rangle$, where $T : K \to X^*$ (dual of X) and $\langle \cdot, \cdot \rangle$ is the pairing between X and X^* then the equilibrium problem reduces to standard variational inequality, introduced and studied by Stampacchia [20] in 1964. In recent years this theory has become very powerful and effective tool for studying a wide class of linear and nonlinear problems arising in mathematical programming, optimization theory, elasticity theory, game theory, economics, mechanics, and engineering sciences. This field is dynamic and has emerged as an interesting and fascinating branch of applicable mathematics with wide range of applications in industry, physical, regional, social, pure, and applied sciences. The papers by Harker and Pang [9] and M. A. Noor, K. I. Noor, and T. M. Rassias [18, 19] provide some excellent survey on the developments and applications of variational inequalities whereas for comprehensive bibliography for equilibrium problems we refer to Giannessi [8], Daniele, Giannessi, and Maugeri [5], Ansari and Yao [3] and references therein.

In the present paper, we consider a general type of variational inequality problem which contains equilibrium problems as a special case. So it is interesting to compare these two ways of the problem setting. We establish some existence results for solution to this type of variational inequality problem by using KKM-Fan theorem and an equivalent

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fixed point theorem. From special cases, we obtain various known and new results for solving various classes of equilibrium problems, variational inequalities and related problems. Our results generalizes and improves the corresponding results in the literature.

2. Preliminaries

Let X and Y be real Banach Spaces. A nonempty subset P of X is called *convex cone* if $\lambda P \subseteq P$ for all $\lambda \geq 0$ and $P + P = P$. A cone P is called *pointed cone* if P is a cone and $P \bigcap (-P) = \{0\}$, where 0 denotes the zero vector. Also, a cone P is called *proper* if it is properly contained in X. Let K be a non-empty subset of X. We will denote by 2^K the set of all nonempty subsets of K, $\mathrm{cl}_X(K)$ the closure of K in X, $L(X,Y)$ the space of all continuous linear operators from X to Y and $\langle u, x\rangle$ the evaluation of $u \in L(X,Y)$ at $x \in X$. Let $T : X \to 2^Y$ be a multifunction, the *graph* of T denoted by $\mathcal{G}(T)$, is the set $\{(x,y) \in X \times Y : x \in X,\ y \in T(x)\}$. The *inverse* of T denoted by T^{-1} is a multifunction from $R(T)$, range of T, to X defined by $x \in T^{-1}(y)$ if and only if $y \in T(x)$. Also T is said to be *upper semicontinuous* on X if for each $x \in X$ and each open set U in Y containing $T(x)$, there exists an open neighbourhood V of x in X such that $T(y) \subseteq U$, for each $y \in V$. T is said to be *upper hemicontinuous* at x if for each $y \in X$, $\lambda \in [0,1]$, the multifunction $\lambda \to T(\lambda y + (1-\lambda)x)$ is upper semicontinuous at 0^+. A multifunction $T : K \to 2^{L(X,Y)}$ is called *generalized upper hemicontinuous* at $x \in K$ if for each $y \in K$, $\lambda \to \langle T(\lambda y + (1-\lambda)x), \eta(y,x)\rangle$ is upper semicontinuous at 0^+, where $\eta : K \times K \to X$ is a bifunction. Let $C : K \to 2^Y$ be a multifunction such that for each $x \in K, C(x)$ is a closed, convex moving cone with $\operatorname{int} C(x) \neq \emptyset$, where $\operatorname{int} C(x)$ denotes the interior of $C(x)$. The partial order $\preceq_{C_x}$ on Y induced by $C(x)$ is defined by declaring $y \preceq_{C_x} z$ if and only if $z - y \in C(x)$ for all $x,y,z \in K$. We will write $y \prec_{C_x} z$ if $z - y \in \operatorname{int} C(x)$ in the case $\operatorname{int} C(x) \neq \emptyset$. Let $f : K \times K \to Y$, $\eta : K \times K \to X$ be bifunctions and $T : K \to 2^{L(X,Y)}$, $S : K \to 2^X$ be multifunctions. The purpose of this paper is to consider the *generalized vector quasi-variatonal-like inequality problem* of finding $x^* \in K \cap \mathrm{cl}_X S(x^*)$ such that, for each $x \in S(x^*)$ there exists $t^* \in T(x^*)$ such that

$$\langle t^*, \eta(x,x^*)\rangle + f(x^*,x) \notin -\operatorname{int}_Y C(x^*). \tag{2.1}$$

If we take T as single valued mapping then as corollary, we consider the problem of finding $x^* \in K \cap \mathrm{cl}_X S(x^*)$ such that, for each $x \in S(x^*)$,

$$\langle T(x^*), \eta(x,x^*)\rangle + f(x^*,x) \notin -\operatorname{int}_Y C(x^*). \tag{2.2}$$

If $\eta(x,y) = x - g(y)$, for all $x,y \in K$, where $g : K \to K$ is a mapping, then as corollary, we consider the problem of finding $x^* \in K \cap \mathrm{cl}_X S(x^*)$ such that, for each $x \in S(x^*)$ there exists $t^* \in T(x^*)$ such that

$$\langle t^*, x - g(x^*)\rangle + f(x^*,x) \notin -\operatorname{int}_Y C(x^*). \tag{2.3}$$

Problems (2.2) and (2.3) also appears to be new.

If $f \equiv 0$ and $S : K \to 2^K$ be a multifunction with closed values, then (2.1) reduces to the problem of finding $x^* \in S(x^*)$ such that, for each $x \in S(x^*)$ there exists $t^* \in T(x^*)$ such that

$$\langle t^*, \eta(x,x^*)\rangle \notin -\operatorname{int}_Y C(x^*). \tag{2.4}$$

It is called *generalized vector quasi-variational-like inequality problem* considered and studied by Ding [6].

If $f \equiv 0$ and $\operatorname{cl}_X S(x) = K$ for each $x \in K$, (2.1) becomes the *generalized vector variational-like inequality problem* of finding $x^* \in K$ such that for each $x \in K$ there exists $t^* \in T(x^*)$ such that

$$\langle t^*, \eta(x,x^*)\rangle \notin -\operatorname{int}_Y C(x^*). \tag{2.5}$$

This problem was introduced and studied by Ansari [1, 2] and B.-S. Lee and G.-M. Lee [16], and if $\eta(x,y) = x - y$ for each $x, y \in K$, then (2.5) was considered by Lin, Yang, and Yao [17] and Konnov and Yao [15].

When $T \equiv 0$ and $S : K \to 2^K$, problem (2.1) reduces to the *vector quasi-equilibrium problem* of finding $x^* \in K$ such that

$$x^* \in \operatorname{cl}_X S(x^*), \quad f(x^*,x) \notin -\operatorname{int}_Y C(x^*) \quad \forall y \in S(x^*). \tag{2.6}$$

This problem was considered and studied by Khaliq and Krishan [11]. If $\eta(x,y) = x - y$ for each $x, y \in K$ and $S : K \to 2^K$, problem (2.1) reduces to the problem of finding $x^* \in K$ such that for each $x \in S(x^*)$ there exists $t^* \in T(x^*)$ such that

$$x^* \in \operatorname{cl}_X S(x^*) \quad \text{and} \quad \langle t^*, x - x^*\rangle + f(x^*,x) \notin -\operatorname{int}_Y C(x^*), \tag{2.7}$$

which is known as *vector quasi-variational inequality problem* studied by Khaliq, Siddiqi, and Krishan [13].

From the above special cases, it is clear that our generalized vector quasi-variational-like inequality problem (2.1) is a more general format of several classes of variational inequalities and equilibrium problems. It includes as special cases the generalized vector quasi-variational and variational-like inequality problems in [1, 2, 6, 8, 12, 13, 14, 15, 16, 17] as well as the vector quasi-equilibrium problems in [3, 4, 5, 8, 11].

Now, we mention some more definitions which will be used in the sequel.

Definition 2.1. A multifunction $F : X \to 2^X$ is called KKM-*map*, if for every finite subset $\{x_1,\dots,x_n\}$ of X, $\operatorname{con}\{x_1,\dots,x_n\} \subset \bigcup_{i=1}^n F(x_i)$, where $\operatorname{con}\{x_1,\dots,x_n\}$ is the convex hull of $\{x_1,\dots,x_n\}$.

Definition 2.2. Let $C : K \to 2^Y$ be a multifunction such that $C(x)$ is a proper closed and convex moving cone with $\operatorname{int}_Y C(x) \neq \emptyset$, then a mapping $g : K \to Y$ is called *C_x-convex* if for each $x, y \in K$ and $\lambda \in [0,1]$, $(1-\lambda)g(x) + \lambda g(y) - g((1-\lambda)x + \lambda y) \in C(x)$ and is called *affine* if for each $x, y \in K$ and $\lambda \in \mathbb{R}$,

$$g(\lambda x + (1-\lambda)y) = \lambda g(x) + (1-\lambda)g(y). \tag{2.8}$$

Remark 2.3. If $g : K \to Y$ is a C_x-convex vector-valued function then, $\sum_{i=1}^n \lambda_i g(y_i) - g(\sum_{i=1}^n \lambda_i y_i) \in C(x)$, for all $y_i \in K$, $t_i \in [0,1]$, $i = 1, \ldots, n$ with $\sum_{i=1}^n \lambda_i = 1$.

Definition 2.4. Let $f : K \times K \to Y$, $\eta : K \times K \to X$ be bifunctions and $T : K \to 2^{L(X,Y)}$ be a multifunction, then the pair (T, f) is called *$\eta - C_x$-pseudomonotone* in K if for all $x, y \in K$,

$$\begin{aligned} &\exists u \in T(x), \quad \langle u, \eta(y,x)\rangle + f(x,y) \notin -\operatorname{int}_Y C(x) \\ &\quad \Longrightarrow \forall v \in T(y), \quad \langle v, \eta(y,x)\rangle + f(x,y) \notin -\operatorname{int}_Y C(x), \end{aligned} \tag{2.9}$$

and the pair (T, f) is called *weakly $\eta - C_x$-pseudomonotone* in K if for all $x, y \in K$,

$$\begin{aligned} &\exists u \in T(x), \quad \langle u, \eta(y,x)\rangle + f(x,y) \notin -\operatorname{int}_Y C(x) \\ &\quad \Longrightarrow \exists v \in T(y), \quad \langle v, \eta(y,x)\rangle + f(x,y) \notin -\operatorname{int}_Y C(x). \end{aligned} \tag{2.10}$$

We also need the following KKM-Fan theorem [7] and a fixed point theorem which is a weaker version of Tarafdar's theorem in [21].

Theorem 2.5. *Let K be a nonempty subset of a topological vector space X and $F : K \to 2^K$ be a* KKM*-mapping with closed values. If there is a subset D contained in a compact convex subset of K such that $\cap_{x\in D} F(x)$ is compact then $\cap_{x\in D} F(x) \neq \emptyset$.*

Theorem 2.6. *Let K be a nonempty subset of a Hausdorff topological vector space X and $F : K \to 2^K$ be a multifunction with nonempty convex values such that $F^{-1}(y)$ is open in K for each $y \in K$. If there exists a nonempty subset D contained in a compact convex subset of K such that $K \setminus \cup_{y\in D} F(y)$ is compact or empty. Then there exists $x^* \in K$ such that $x^* \in F(x^*)$.*

Remark 2.7. Theorem 2.5 has many equivalent formulations in terms of fixed points and is also equivalent to Theorem 2.6.

3. Existence results

Throughout this section and next section, unless otherwise specified, we assume that K is a nonempty closed convex subset of real Banach space X and Y is a real Banach space. We assume that $C : K \to 2^Y$ is a multifunction such that for each $x \in K, C(x)$ is a proper closed and convex moving cone with $\operatorname{int}_Y C(x) \neq \emptyset$. Consider a multifunction $S : K \to 2^X$ such that for each $x \in K$, $K \cap S(x) \neq \emptyset$, $S^{-1}(x)$ is weakly open in K, $\operatorname{cl} S(x)$ is weakly closed and for all $\alpha \in (0,1]$, $(1-\alpha)x+\alpha y \in S(x)$ and set $E = \{x \in K : x \in \operatorname{cl} S(x)\}$. Assume that the mapping $x \to Y \setminus (-\operatorname{int}_Y C(x))$ for each $x \in K$, is a weakly closed mapping, that is, its graph is closed in $X \times Y$ with weak topologies of X and Y.

Theorem 3.1. *Let $f : K \times K \to Y$ and $\eta : K \times K \to X$ be bifunctions and $T : K \to 2^{L(X,Y)}$ be a multifunction. Suppose the following assumptions holds:*

(i) *for each $x \in K$, $\eta(x,x) = 0$ and $f(x,x) \in C(x) \cap -C(x)$,*

(ii) *T is generalized upper hemicontinuous in K with nonempty compact values,*

(iii) *$\eta(\cdot,\cdot)$ is affine in the first argument and is continuous in the second argument, f is C_x-convex in second argument and the pair (T, f) is weakly $\eta - C_x$-pseudomonotone for each $x \in K$,*

(iv) *for each $x, y \in K$ and $x_\lambda \in K$ such that $x_\lambda \xrightarrow{w} x$ (weak), there exists a subnet x_μ of x_λ and $s \in f(x,y) - C(x)$ such that $f(x_\mu, y) \xrightarrow{w} s$,*

(v) *there is a nonempty weakly compact subset D of K and a subset D_o of a weakly compact convex subset of K such that for all $x \in K \setminus D$, there exists $z \in D_o \cap S(x)$, $\langle T(x), \eta(z,x)\rangle + f(x,z) \subset -\operatorname{int}_Y C(x)$.*

Then there exists $x^* \in K \cap \operatorname{cl}_X S(x^*)$ such that for each $x \in S(x^*)$ there exists $t^* \in T(x^*)$ such that

$$\langle t^*, \eta(x,x^*)\rangle + f(x^*,x) \notin -\operatorname{int}_Y C(x^*). \tag{3.1}$$

Proof. To prove the theorem, we first define the multifunctions P_1 and P_2 for each $x, y \in K$ by

$$\begin{aligned} P_1(x) &= \{z \in K : \langle T(x), \eta(z,x)\rangle + f(x,z) \subset -\operatorname{int}_Y C(x)\}, \\ P_2(x) &= \{z \in K : \langle T(z), \eta(z,x)\rangle + f(x,z) \subset -\operatorname{int}_Y C(x)\}. \end{aligned} \tag{3.2}$$

Now for $i = 1,2$ set

$$\Phi_i(x) = \begin{cases} S(x) \cap P_i(x) & \text{if } x \in E \\ K \cap S(x) & \text{if } x \in K \setminus E \end{cases} \tag{3.3}$$

and $Q_i(y) = K \setminus \Phi_i^{-1}(y)$. Then

$$\begin{aligned} Q_i(y) &= K \setminus \{x \in K : y \in \Phi_i(x)\} \\ &= K \setminus [\{x \in E : y \in S(x) \cap P_i(x)\} \cup \{x \in K \setminus E : y \in S(x)\}] \\ &= K \setminus [\{E \cap S^{-1}(y) P_i^{-1}(y)\} \cup \{(K \setminus E) \cap S^{-1}(y)\}] \\ &= K \setminus [\{E \cap P_i^{-1}(y) \cup (K \setminus E)\} \cap S^{-1}(y)] \\ &= K \setminus [\{(K \setminus E) \cup P_i^{-1}(y)\} \cap S^{-1}(y)] \\ &= [K \setminus \{(K \setminus E) \cup P_i^{-1}(y)\}] \cup [K \setminus S^{-1}(y)] \\ &= [E \cap \{K \setminus P_i^{-1}(y)\}] \cup [K \setminus S^{-1}(y)]. \end{aligned} \tag{3.4}$$

We divide the proof into six steps.

Step 1. E is nonempty and weakly closed: Since $K \cap S(x) \neq \emptyset$ for all $x \in K$, $\cup_{y\in K} S^{-1}(y) = K$. By the given assumption and condition (v), $S^{-1}(y)$ is open in K for each $y \in K$ and $K \setminus D \subset \cup_{y\in D_o} S^{-1}(y) \subset K$. Hence $K \setminus \cup_{y\in D_o} S^{-1}(y)$ is contained in D and is weakly compact. Thus Theorem 2.6 implies that S has a fixed point in K and hence $E \neq \emptyset$. Also weakly closedness of $\operatorname{cl} S(\cdot)$ implies that E is weakly closed.

Step 2. Q_1 is KKM mapping in K: Suppose that there exists a finite subset $\{y_1,\dots,y_n\}$ of K and $\lambda_i \geq 0$, $i = 1,\dots,n$, with $\sum_{i=1}^n \lambda_i = 1$, such that

$$x_o = \sum_{i=1}^n \lambda_i y_i \notin \bigcup_{i=1}^n Q_1(y_i), \tag{3.5}$$

then we have $x_o \in \Phi_1^{-1}(y_i)$, which implies that $y_i \in \Phi_1(x_o)$ for all $i = 1,\dots,n$. If $x_o \in E$, then $\Phi_1(x_o) = S(x_o) \cap P_1(x_o)$. Hence $y_i \in P_1(x_o)$, which implies that

$$\langle T(x_o), \eta(y_i,x_o)\rangle + f(x_o,y_i) \subset -\operatorname{int}_Y C(x_o). \tag{3.6}$$

This implies that for all $u \in T(x_o)$,

$$\langle u, \eta(y_i,x_o)\rangle + f(x_o,y_i) \in -\operatorname{int}_Y C(x_o) \quad i = 1,\dots,n. \tag{3.7}$$

Which implies

$$\sum_{i=1}^n \lambda_i \langle u, \eta(y_i,x_o)\rangle + \sum_{i=1}^n \lambda_i f(x_o,y_i) \in -\operatorname{int}_Y C(x_o). \tag{3.8}$$

Using (3.8), C_x-convexity of f and assumption (i), we have for all $u \in T(x_0)$

$$\begin{aligned}
0 &= \langle u, \eta(x_o,x_o)\rangle \\
&= \left\langle u, \eta\left(\sum_{i=1}^n \lambda_i y_i, x_o\right)\right\rangle \\
&= \sum_{i=1}^n \lambda_i \langle u, \eta(y_i,x_o)\rangle + \sum_{i=1}^n \lambda_i f(x_o,y_i) + f\left(x_o, \sum_{i=1}^n \lambda_i y_i\right) \\
&\quad - \sum_{i=1}^n \lambda_i f(x_o,y_i) - f(x_o,x_o) \in -\operatorname{int} C(x_o) - C(x_o) - C(x_o) \\
&= -\operatorname{int} C(x_o).
\end{aligned} \tag{3.9}$$

Which implies $C(x_o) = Y$, a contradiction. If $x_o \in K \setminus E$, then $\Phi_1(x_o) = K \cap S(x_o)$. Hence $x_o = \sum_{i=1}^n \lambda_i y_i \in S(x_o)$, a contradiction again. Thus Q_1 is KKM mapping.

Step 3. Q_2 is KKM mapping in K: Using the definition of P_i $(i=1,2)$ and weakly $\eta - C_x$-pseudomonotonicity of the pair (T,f) we have $K\setminus P_1^{-1}(y)\subset K\setminus P_2^{-1}(y)$. Thus $Q_1(y)\subset Q_2(y)$ for all $y\in K$ and hence Q_2 is also KKM-mapping.
Step 4. $Q_2(y)$ for each $y\in K$ is weakly closed: Weakly closedness of $Q_2(y)$ follows from (3.4), if we prove that for each $y\in K$

$$\begin{aligned} K\setminus P_2^{-1}(y) &= \{x\in K : y\notin P_2(x)\} \\ &= \{x\in K : \langle T(y),\eta(y,x)\rangle + f(x,y)\not\subseteq -\operatorname{int}_Y C(x)\} \end{aligned} \tag{3.10}$$

is weakly closed. Assume that $x_\lambda \xrightarrow{w} x$ and $x_\lambda\in K\setminus P_2^{-1}(y)$. Which implies that there exists $t_\lambda\in T(y)$ such that

$$\langle t_\lambda,\eta(y,x_\lambda)\rangle + f(x_\lambda,y)\notin -\operatorname{int}_Y C(x_\lambda). \tag{3.11}$$

Since $T(y)$ is compact, without loss of generality, we can assume that there exists $t\in T(y)$ such that $t_\lambda\to t$. Also

$$\begin{aligned} \langle t_\lambda,\eta(y,x_\lambda)\rangle &= \langle t_\lambda - t,\eta(y,x_\lambda)\rangle + \langle t,\eta(y,x_\lambda)\rangle, \\ \|\langle t_\lambda - t,\eta(y,x_\lambda)\rangle\| &\le \|t_\lambda - t\|\|\eta(y,x_\lambda)\| \longrightarrow 0. \end{aligned} \tag{3.12}$$

Since t is also continuous when X and Y are equipped by the weak topologies and η is continuous in the second argument,

$$\langle t,\eta(y,x_\lambda)\rangle \xrightarrow{w} \langle t,\eta(y,x)\rangle. \tag{3.13}$$

Thus (3.11)–(3.13), yields

$$\langle t_\lambda,\eta(y,x_\lambda)\rangle \longrightarrow \langle t,\eta(y,x)\rangle. \tag{3.14}$$

By assumption (iv) there exists a subnet x_μ of x_λ and $s\in f(x,y)-C(x)$ such that $f(x_\mu, y)\xrightarrow{w} s$. Therefore, using (3.11), (3.14), assumption (iv), and weak closedness of $x\to Y\setminus(-\operatorname{int} C(x))$ in K, we have

$$\langle t,\eta(x,y)\rangle + s\in Y\setminus(-\operatorname{int}_Y C(x)). \tag{3.15}$$

Thus

$$\begin{aligned} \langle t,\eta(y,x)\rangle + f(x,y) = \langle t,\eta(y,x)\rangle + s + f(x,y) - s &\in Y\setminus(-\operatorname{int}_Y C(x)) + C(x) \\ &= Y\setminus(-\operatorname{int}_Y C(x)). \end{aligned} \tag{3.16}$$

Which implies that $x\in K\setminus P_2^{-1}(y)$ and hence $K\setminus P_2^{-1}(y)$ is weakly closed.

Step 5. There exists $x^* \in K \setminus \cup_{y \in K} \Phi_2^{-1}(y)$. By (v) for each $x \in K \setminus D$, there exists $z \in D_o \cap S(x)$ such that $z \in \Phi_2(x)$. Which implies that $K \setminus D \subset \cup_{z \in D_o} \Phi_2^{-1}(z)$. Hence

$$D \supset \bigcap_{z \in D_o} K \setminus \Phi_2^{-1}(z) = \bigcap_{z \in D_o} Q_2(z). \tag{3.17}$$

Thus all the assumptions of Theorem 2.5 are satisfied and hence there exists

$$x^* \in \bigcap_{y \in K} K \setminus \Phi_2^{-1}(y) = K \setminus \bigcup_{y \in K} \Phi_2^{-1}(y). \tag{3.18}$$

Step 6. x^* is a solution of (2.1). If $x^* \in K \setminus E$, (3.18) implies that $\Phi_2(x^*) = \varnothing$. But given assumption implies $\Phi_2(x^*) = K \cap S(x^*) \neq \varnothing$, which is a contradiction. If $x^* \in E$, then $\Phi_2(x^*) = P_2(x^*) \cap S(x^*) = \varnothing$. Which implies that for each $y \in S(x^*)$, $y \notin P_2(x^*)$. That is for each $y \in S(x^*)$,

$$\langle T(y), \eta(y, x^*) \rangle + f(x^*, y) \not\subset -\operatorname{int}_Y C(x^*). \tag{3.19}$$

Suppose that x^* is not solution of (2.1). Which implies that there exists $y^* \in S(x^*)$,

$$\langle T(x^*), \eta(y^*, x^*) \rangle + f(x^*, y^*) \subset -\operatorname{int}_Y C(x^*). \tag{3.20}$$

Since T is generalized upper hemicontinuous for $\alpha > 0$, small enough

$$\langle T(\alpha y^* + (1-\alpha)x^*), \eta(y^*, x^*) \rangle + f(x^*, y^*) \subset -\operatorname{int}_Y C(x^*). \tag{3.21}$$

On the other hand using assumption (ii), (3.19), $\eta(x,x) = 0$ and C_x-convexity of f, we have

$$\begin{aligned}
&\langle T(\alpha y^* + (1-\alpha)x^*), \eta(y^*, x^*) \rangle + f(x^*, y^*) \\
&\quad = \frac{1}{\alpha}\{\langle T(\alpha y^* + (1-\alpha)x^*), \eta(\alpha y^* + (1-\alpha)x^*, x^*) \rangle + f(x^*, \alpha y^* + (1-\alpha)x^*)\} \\
&\qquad + \frac{1}{\alpha}\{\alpha f(x^*, y^*) + (1-\alpha) f(x^*, x^*) - f(x^*, \alpha y^* + (1-\alpha)x^*)\} \\
&\qquad - \frac{1-\alpha}{\alpha} f(x^*, x^*) \\
&\qquad \subset Y \setminus \{-C(x^*)\} + C(x^*) + \{C(x^*) \cap (-C(x^*))\} \\
&\quad = Y \setminus \{-C(x^*)\}.
\end{aligned} \tag{3.22}$$

Which contradicts (3.21). Hence x^* must be a solution of (2.1). □

Corollary 3.2. *If in Theorem 3.1 we take T as single valued mapping then there exists $x^* \in K \cap \mathrm{cl}_X S(x^*)$ such that, for each $x \in S(x^*)$,*

$$\langle T(x^*), \eta(x,x^*)\rangle + f(x^*,x) \notin -\mathrm{int}_Y C(x^*). \tag{3.23}$$

Corollary 3.3. *If in Theorem 3.1 we take $\eta(x,y) = x - g(y)$, for all $x, y \in K$, where $g : K \to K$ is a mapping, then there exists $x^* \in K \cap \mathrm{cl}_X S(x^*)$ such that, for each $x \in S(x^*)$ there exists $t^* \in T(x^*)$ such that*

$$\langle t^*, x - g(x^*)\rangle + f(x^*,x) \notin -\mathrm{int}_Y C(x^*). \tag{3.24}$$

Theorem 3.4. *If we avoid compactness of $T(x)$ for each $x \in K$ and replace the weakly $\eta - C_x$-pseudomonotonicity of the pair (T, f) by $\eta - C_x$-pseudomonotonicity and the assumption (v) by*

(v)^o *there is a nonempty weakly compact subset D of K and a subset D_o of a weakly compact convex subset of K such that for all $x \in K \setminus D$, there exists $z \in D_o \cap S(x)$, $\langle T(x), \eta(z,x)\rangle + f(x,z) \cap -\mathrm{int}_Y C(x) \neq \emptyset$*

in Theorem 3.1, then there exists $x^* \in K \cap \mathrm{cl}_X S(x^*)$ such that for each $x \in S(x^*)$ there exists $t^* \in T(x^*)$ such that

$$\langle t^*, \eta(x,x^*)\rangle + f(x^*,x) \notin -\mathrm{int}_Y C(x^*). \tag{3.25}$$

Proof. We first define a multifunction P_3 for each $x \in K$ by

$$P_3(x) = \{z \in K : \exists t \in T(z) : \langle t, \eta(z,x)\rangle + f(x,z) \in -\mathrm{int}_Y C(x)\}. \tag{3.26}$$

Using P_1, P_3 with the corresponding Φ_i and Q_i, $i = 1,3$ analogously to the proof of Theorem 3.1, we can show that Q_1 is a KKM-mapping. By the $\eta - C_x$-pseudomonotonicity of the pair (T, f), $K \setminus P_1^{-1}(y) \subset K \setminus P_3^{-1}(y)$ and hence $Q_1(y) \subset Q_3(y)$ for all $x \in K$. Thus Q_3 is also a KKM mapping in K. Now weakly closedness of $Q_3(y)$ follows from (3.4), if we prove that for each $y \in K$

$$\begin{aligned} K \setminus P_3^{-1}(y) &= \{x \in K : y \notin P_3(x)\} \\ &= \{x \in K : \exists t \in T(y) : \langle t, \eta(y,x)\rangle + f(x,y) \notin -\mathrm{int}_Y C(x)\} \end{aligned} \tag{3.27}$$

is weakly closed. Assume that $x_\lambda \xrightarrow{w} x$ and $x_\lambda \in K \setminus P_3^{-1}(y)$. Which implies that for all $t \in T(y)$ we have

$$\langle t, \eta(y,x_\lambda)\rangle + f(x_\lambda, y) \notin -\mathrm{int}_Y C(x_\lambda). \tag{3.28}$$

Thus assumption (iv) implies that there is a subnet x_μ and $s \in f(x,y) - C(x)$ such that $f(x_\mu, y)$
$\xrightarrow{w} s$. Using (3.28), continuity of η in the second argument and of t in the weak topolo-

gies and weak closedness of $x \to Y \setminus -\operatorname{int} C(x)$ in K, we have $\langle t, \eta(y,x)\rangle + s \notin -\operatorname{int}_Y C(x)$. Thus

$$\begin{aligned}\langle t,\eta(y,x)\rangle + f(x,y) &= \langle t,\eta(y,x)\rangle + s + f(x,y) - s \in Y \setminus -\operatorname{int} C(x) + C(x)\\ &= Y \setminus -\operatorname{int} C(x),\end{aligned} \tag{3.29}$$

which shows that $K \setminus P_3^{-1}(y)$ is weakly closed and so is $Q_3(y)$. Similarly as for Q_2, using $(\mathrm{v})^o$, $\cap_{z\in D_o} Q_3(z)$ is weakly compact. Thus all the assumptions of Theorem 2.5 are satisfied and hence there exists

$$x^* \in \bigcap_{y\in K} K \setminus \Phi_3^{-1}(y) = K \setminus \bigcup_{y\in K} \Phi_3^{-1}(y). \tag{3.30}$$

Now it remains to show that x^* is a solution of (2.1), which follows directly from Step 4 of Theorem 3.1 with minor modifications. □

Theorem 3.5. *Suppose that all the assumptions of Theorem 3.4 are satisfied except weak $\eta - C_x$-pseudomonotonicity of the pair (T, f) in (iii) and the condition that generalized upper hemicontinuity of T is strengthened to the upper semicontinuity of T in the weak topology of X and norm topology of $L(X,Y)$. Then there exists $x^* \in K \cap \operatorname{cl}_X S(x^*)$ such that, for each $x \in S(x^*)$ there exists $t^* \in T(x^*)$ such that*

$$\langle t^*, \eta(x,x^*)\rangle + f(x^*,x) \notin -\operatorname{int}_Y C(x^*). \tag{3.31}$$

Proof. To prove this theorem it is sufficient to prove that there exists $x^* \in \cap_{y\in K} Q_1(y)$. To apply Theorem 2.5 for Q_1, it remains to check only the weak closedness of $Q_1(y)$ for each $y \in K$, which follows from (3.4), if we prove that for each $y \in K$

$$\begin{aligned}K \setminus P_1^{-1}(y) &= \{x \in K : y \notin P_1(x)\}\\ &= \{x \in K : \langle T(x), \eta(y,x)\rangle + f(x,y) \nsubseteq -\operatorname{int}_Y C(x)\}\end{aligned} \tag{3.32}$$

is weakly closed. Assume that $x_\lambda \xrightarrow{w} x$ and $x_\lambda \in K \setminus P_1^{-1}(y)$. Which implies that there exists $t_\lambda \in T(x_\lambda)$ such that

$$\langle t_\lambda, \eta(y,x_\lambda)\rangle + f(x_\lambda,y) \notin -\operatorname{int}_Y C(x_\lambda). \tag{3.33}$$

Upper semi-continuity of T implies that for each $\epsilon > 0$, there exists a weak neighborhood $N(x)$ such that $T(N(x)) \subset B(T(x),\epsilon)$. We can take $x_\lambda \in N(x)$ and hence there is $t'_\lambda \in T(x)$ such that $\|t_\lambda - t'_\lambda\| < \epsilon$. Since $T(x)$ is compact, without loss of generality, we can assume that there exists $t \in T(x)$ such that $t'_\lambda \to t$. Consequently, $\|t_\lambda - t\| \to 0$. Thus using arguments similar to those used in Theorem 3.1, $Q_1(y)$ is closed and hence the proof is complete. □

Remark 3.6. Theorem 3.1 improves and generalizes [11, Theorem 1] and [13, Theorem 3.1]. Our proof of Theorem 3.1 depends on the KKM-Fan theorem and fixed point theorem whereas the proof of main results in [11, 13] depends on one person game theorems. If K is weakly compact, $f(x,y) = 0$, $\eta(x,y) = x - y$ and $A(x) = K$, for all $x, y \in K$, Theorem 3.1 collapses to [17, Theorem 3.1]. Of course, in this case the coercivity assumption (v) is omitted. The coercivity assumption is unavoidable if K is only closed and convex. We note that when $T \equiv 0$ and $S : K \to 2^K$ in Corollary 3.2, we obtain existence results for the vector quasi-equilibrium problem (2.6) and if we take $f \equiv 0$ and $\mathrm{cl}_X S(x) = K$ for each $x \in K$ in Theorem 3.1 we obtain existence results for the generalized vector quasi-variational-like inequality problem (2.5).

4. Applications

In this section we establish some existence results for generalized vector quasi variational inequalities and vector quasi optimization problems.

We need the following special case of Definition 2.4

Definition 4.1. Let $T : K \to 2^{L(X,Y)}$ be a multifunction and $g : K \to K$ be a mapping then T is called *weakly generalized C_x-pseudomonotone* in K if for all $x, y \in K$,

$$\exists u \in T(x), \quad \langle u, y - g(x)\rangle \notin -\mathrm{int}_Y C(x) \Longrightarrow \exists v \in T(y), \quad \langle v, y - g(x)\rangle \notin -\mathrm{int}_Y C(x). \tag{4.1}$$

Theorem 4.2. *Let $T : K \to 2^{L(X,Y)}$ be generalized upper hemicontinuous and weakly generalized C_x-pseudomonotone multifunction in K with nonempty compact values. Let $\eta : K \times K \to K$ be affine in the first argument and continuous in the second argument such that for each $x \in K$, $\eta(x,x) = 0$. Suppose that there is a nonempty weakly compact subset D of K and a subset D_o of a weakly compact convex subset of K such that for all $x \in K \setminus D$, there exists $z \in D_o \cap S(x)$,*

$$\langle T(x), \eta(z,x)\rangle \subset -\mathrm{int}_Y C(x). \tag{4.2}$$

Then the generalized vector quasi-variational-like inequality problem of finding $x^* \in K \cap \mathrm{cl}_X S(x^*)$ such that for each $x \in S(x^*)$ there exists $t^* \in T(x^*)$ such that

$$\langle t^*, \eta(x, x^*)\rangle \notin -\mathrm{int}_Y C(x^*), \tag{4.3}$$

has a solution.

Proof. If we take $f = 0$, then we see that all the assumptions of Theorem 3.1 holds and hence the proof follows. □

Theorem 4.3. *Let $f : K \times K \to Y$ be bifunction such that $f(x,x) \in C(x) \cap -C(x)$ and is C_x-convex in second argument. Suppose that for each $x, y \in K$ and $x_\lambda \in K$ such that $x_\lambda \xrightarrow{w} x$ (weak), there exists a subnet x_μ of x_λ and $s \in f(x,y) - C(x)$ such that $f(x_\mu, y) \xrightarrow{w} s$. Also assume that there is a nonempty weakly compact subset D of K and a subset D_o of a weakly* *compact convex subset of K such that for all* $x \in K \setminus D$, *there exists* $z \in D_o \cap S(x)$,

$$f(x,z) \in -\operatorname{int}_Y C(x). \tag{4.4}$$

Then the vector quasi-equilibrium problem of finding $x^* \in K \cap \operatorname{cl}_X S(x^*)$ *such that for each* $x \in S(x^*)$,

$$f(x^*,x) \notin -\operatorname{int}_Y C(x^*), \tag{4.5}$$

has a solution.

Proof. If we take $T = 0$ then proof directly follows from Theorem 3.1. □

Corollary 4.4. *If in Theorem 4.2 we take* $\eta(x,y) = x - g(y)$ *for all* $x, y \in K$, *where* $g : K \to K$ *be a mapping. Then the generalized vector quasi-variational inequality problem of finding* $x^* \in K \cap \operatorname{cl}_X S(x^*)$ *such that for each* $x \in S(x^*)$ *there exists* $t^* \in T(x^*)$ *such that*

$$\langle t^*, x - g(x^*) \rangle \notin -\operatorname{int}_Y C(x^*), \tag{4.6}$$

has a solution.

Corollary 4.5. *If in Theorem 4.3 we take* $f(x,y) = \phi(y) - \phi(x)$ *for all* $x, y \in K$, *where* $\phi : K \to Y$ *be a vector-valued function. Then the vector quasi-optimization problem of finding* $x^* \in K \cap \operatorname{cl}_X S(x^*)$ *such that for each* $x \in S(x^*)$,

$$\phi(x^*) - \phi(x) \notin -\operatorname{int}_Y C(x^*), \tag{4.7}$$

has a solution.

Remark 4.6. Ding [6] and Kim and Tan [14] respectively employed the scalarization technique and one person game theorems to prove the existence results for the generalized vector quasi-variational-like inequality problem and the generalized vector quasi-variational inequality problem whereas we have used KKM-Fan theorem and fixed point theorem in our results. The results of this section extends, generalizes and improves the corresponding results in [1, 2, 6, 8, 10, 12, 14, 16].

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