FIXED POINT THEOREMS FOR A FAMILY OF HYBRID PAIRS OF MAPPINGS IN METRICALLY CONVEX SPACES

M. IMDAD AND LADLAY KHAN

Received 30 December 2004 and in revised form 24 March 2005

The present paper establishes some coincidence and common fixed point theorems for a sequence of hybrid-type nonself-mappings defined on a closed subset of a metrically convex metric space. Our results generalize some earlier results due to Khan et al. (2000), Itoh (1977), Khan (1981), Ahmad and Imdad (1992 and 1998), and several others. Some related results are also discussed.

1. Introduction

In recent years several fixed point theorems for hybrid pairs of mappings are proved and by now there exists considerable literature in this direction. To mention a few, one can cite Imdad and Ahmad [10], Pathak [19], Popa [20] and references cited therein. On the other hand Assad and Kirk [4] gave a sufficient condition enunciating fixed point of set-valued mappings enjoying specific boundary condition in metrically convex metric spaces. In the current years the work due to Assad and Kirk [4] has inspired extensive activities which includes Itoh [12], Khan [14], Ahmad and Imdad [1, 2], Imdad et al. [11] and some others.

Most recently, Huang and Cho [9] and Dhage et al. [6] proved some fixed point theorems for a sequence of set-valued mappings which generalize several results due to Itoh [12], Khan [14], Ahmad and Khan [3] and others. The purpose of this paper is to prove some coincidence and common fixed point theorems for a sequence of hybrid type nonself mappings satisfying certain contraction type condition which is essentially patterned after Khan et al. [15]. Our results either partially or completely generalize earlier results due to Khan et al. [15], Itoh [12], Khan [14], Ahmad and Imdad [1, 2], Ahmad and Khan [3] and several others.

2. Preliminaries

Before proving our results, we collect the relevant definitions and results for our future use.

Fixed Point Theory and Applications 2005:3 (2005) 281–294
DOI: 10.1155/FPTA.2005.281

Let (X,d) be a metric space. Then following Nadler [17], we recall

(i) $CB(X) = \{A : A$ is nonempty closed and bounded subset of $X\}$.

(ii) $C(X) = \{A : A$ is nonempty compact subset of $X\}$.

(iii) For nonempty subsets A, B of X and $x \in X$,

$$\begin{aligned} d(x,A) &= \inf\{d(x,a) : a \in A\}, \\ H(A,B) &= \max\left[\{\sup d(a,B) : a \in A\}, \{\sup d(A,b) : b \in B\}\right]. \end{aligned} \tag{2.1}$$

It is well known (cf. Kuratowski [16]) that $CB(X)$ is a metric space with the distance H which is known as Hausdorff-Pompeiu metric on X.

The following definitions and lemmas will be frequently used in the sequel.

Definition 2.1. Let K be a nonempty subset of a metric space (X,d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F,T) is said to be pointwise R-weakly commuting on K if for given $x \in K$ and $Tx \in K$, there exists some $R = R(x) > 0$ such that

$$d(Ty, FTx) \le R \cdot d(Tx, Fx) \quad \text{for each } y \in K \cap Fx. \tag{2.2}$$

Moreover, the pair (F,T) will be called R-weakly commuting on K if (2.2) holds for each $x \in K$, $Tx \in K$ with some $R > 0$.

If $R = 1$, we get the definition of weak commutativity of (F,T) on K due to Hadzic and Gajic [8]. For $K = X$ Definition 2.1 reduces to "pointwise R-weak commutativity and R-weak commutativity" for single valued self mappings due to Pant [18].

Definition 2.2 [7, 8]. Let K be a nonempty subset of a metric space (X,d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F,T) is said to be weakly commuting (cf. [7]) if for every $x, y \in K$ with $x \in Fy$ and $Ty \in K$, we have

$$d(Tx, FTy) \le d(Ty, Fy), \tag{2.3}$$

whereas the pair (F,T) is said to be compatible (cf. [8]) if for every sequence $\{x_n\} \subset K$, from the relation

$$\lim_{n\to\infty} d(Fx_n, Tx_n) = 0 \tag{2.4}$$

and $Tx_n \in K$ (for every $n \in N$) it follows that $\lim_{n\to\infty} d(Ty_n, FTx_n) = 0$, for every sequence $\{y_n\} \subset K$ such that $y_n \in Fx_n$, $n \in N$.

For hybrid pairs of self type mappings these definitions were introduced by Kaneko and Sessa [13].

Definition 2.3 [11]. Let K be a nonempty subset of a metric space (X,d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F,T) is said to be quasi-coincidentally commuting if for all coincidence points "x" of (T,F), $TFx \subset FTx$ whenever $Fx \subset K$ and $Tx \in K$ for all $x \in K$.

Definition 2.4 [11]. A mapping $T : K \to X$ is said to be coincidentally idempotent w.r.t mapping $F : K \to CB(X)$, if T is idempotent at the coincidence points of the pair (F,T).

Definition 2.5 [4]. A metric space (X,d) is said to be metrically convex if for any $x,y \in X$ with $x \neq y$ there exists a point $z \in X$, $x \neq z \neq y$ such that

$$d(x,z) + d(z,y) = d(x,y). \tag{2.5}$$

LEMMA 2.6 [4]. *Let K be a nonempty closed subset of a metrically convex metric space (X,d). If $x \in K$ and $y \notin K$ then there exists a point $z \in \delta K$ (the boundary of K) such that $d(x,z) + d(z,y) = d(x,y)$.*

LEMMA 2.7 [17]. *Let $A,B \in CB(X)$ and $a \in A$, then for any positive number $q < 1$ there exists $b = b(a)$ in B such that $q \cdot d(a,b) \leq H(A,B)$.*

3. Main results

Our main result runs as follows.

THEOREM 3.1. *Let (X,d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S,T : K \to X$ satisfying*

(iv) $\delta K \subseteq SK \cap TK$, $F_i(K) \cap K \subseteq SK$, $F_j(K) \cap K \subseteq TK$,

(v) $Tx \in \delta K \Rightarrow F_i(x) \subseteq K$, $Sx \in \delta K \Rightarrow F_j(x) \subseteq K$, *and*

$$\begin{aligned} H(F_i(x),F_j(y)) \leq a \cdot \max\left\{\frac{1}{2}d(Tx,Sy), d(Tx,F_i(x)), d(Sy,F_j(y))\right\} \\ + b\{d(Tx,F_j(y)) + d(Sy,F_i(x))\}, \end{aligned} \tag{3.1}$$

where $i = 2n-1$, $j = 2n$, $(n \in N)$, $i \neq j$ for all $x,y \in K$ with $x \neq y$, $a,b \geq 0$, and $2b < a$, $2a + 3b < q < 1$,

(vi) *(F_i,T) and (F_j,S) are compatible pairs,*

(vii) *$\{F_n\}$, S and T are continuous on K.*

Then (F_i,T) as well as (F_j,S) has a point of coincidence.

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way.

Let $x \in \delta K$. Then (due to $\delta K \subseteq TK$) there exists a point $x_0 \in K$ such that $x = Tx_0$. From the implication $Tx \in \delta K$ which implies $F_1(x_0) \subseteq F_1(K) \cap K \subseteq SK$, let $x_1 \in K$ be such that $y_1 = Sx_1 \in F_1(x_0) \subseteq K$. Since $y_1 \in F_1(x_0)$, there exists a point $y_2 \in F_2(x_1)$ such that

$$q \cdot d(y_1,y_2) \leq H(F_1(x_0),F_2(x_1)). \tag{3.2}$$

Suppose $y_2 \in K$. Then $y_2 \in F_2(K) \cap K \subseteq TK$ implies that there exists a point $x_2 \in K$ such that $y_2 = Tx_2$. Otherwise, if $y_2 \notin K$, then there exists a point $p \in \delta K$ such that

$$d(Sx_1,p) + d(p,y_2) = d(Sx_1,y_2). \tag{3.3}$$

Since $p \in \delta K \subseteq TK$, there exists a point $x_2 \in K$ with $p = Tx_2$ so that

$$d(Sx_1, Tx_2) + d(Tx_2, y_2) = d(Sx_1, y_2). \tag{3.4}$$

Let $y_3 \in F_3(x_2)$ be such that $q \cdot d(y_2, y_3) \le H(F_2(x_1), F_3(x_2))$.

Thus, repeating the foregoing arguments, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(viii) $y_{2n} \in F_{2n}(x_{2n-1})$, $y_{2n+1} \in F_{2n+1}(x_{2n})$,

(ix) $y_{2n} \in K \Rightarrow y_{2n} = Tx_{2n}$ or $y_{2n} \notin K \Rightarrow Tx_{2n} \in \delta K$ and

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n}), \tag{3.5}$$

(x) $y_{2n+1} \in K \Rightarrow y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin K \Rightarrow Sx_{2n+1} \in \delta K$ and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}). \tag{3.6}$$

We denote

$$\begin{aligned} P_\circ &= \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}\}, \\ P_1 &= \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}\}, \\ Q_\circ &= \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}\}, \\ Q_1 &= \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}\}. \end{aligned} \tag{3.7}$$

One can note that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$ and $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$. □

Now, we distinguish the following three cases.

Case 1. If $(Tx_{2n}, Sx_{2n+1}) \in P_\circ \times Q_\circ$, then

$$\begin{aligned} &q \cdot d(Tx_{2n}, Sx_{2n+1}) \\ &\quad \le H(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})) \\ &\quad \le a \cdot \max\left\{\frac{1}{2}d(Tx_{2n}, Sx_{2n-1}), d(Tx_{2n}, F_{2n+1}(x_{2n})), d(Sx_{2n-1}, F_{2n}(x_{2n-1}))\right\} \\ &\qquad + b \cdot \{d(Tx_{2n}, F_{2n}(x_{2n-1})) + d(Sx_{2n-1}, F_{2n+1}(x_{2n}))\} \\ &\quad \le a \cdot \max\left\{\frac{1}{2}d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\right\} \\ &\qquad + b \cdot \{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\}, \end{aligned} \tag{3.8}$$

which in turn yields

$$d(Tx_{2n},Sx_{2n+1}) \leq \begin{cases} \left(\dfrac{a+b}{q-b}\right)d(Sx_{2n-1},Tx_{2n}), & \text{if } d(y_{2n-1},y_{2n}) \geq d(y_{2n+1},y_{2n}) \\ \left(\dfrac{b}{q-b-a}\right)d(Sx_{2n-1},Tx_{2n}), & \text{if } d(y_{2n-1},y_{2n}) \leq d(y_{2n+1},y_{2n}), \end{cases} \tag{3.9}$$

or

$$d(Tx_{2n},Sx_{2n+1}) \leq h \cdot d(Sx_{2n-1},Tx_{2n}), \tag{3.10}$$

where $h = \max\{((a+b)/(q-b)),(b/(q-b-a))\} < 1$, since $2a+3b < 1$.

Similarly if $(Sx_{2n-1},Tx_{2n}) \in Q_\circ \times P_\circ$, then

$$d(Sx_{2n-1},Tx_{2n}) \leq \begin{cases} \left(\dfrac{a+b}{q-b}\right)d(Sx_{2n-1},Tx_{2n-2}), & \text{if } d(y_{2n-2},y_{2n-1}) \geq d(y_{2n-1},y_{2n}) \\ \left(\dfrac{b}{q-b-a}\right)d(Sx_{2n-1},Tx_{2n-2}), & \text{if } d(y_{2n-2},y_{2n-1}) \leq d(y_{2n-1},y_{2n}), \end{cases} \tag{3.11}$$

or

$$d(Sx_{2n-1},Tx_{2n}) \leq h \cdot d(Sx_{2n-1},Tx_{2n-2}), \tag{3.12}$$

where $h = \max\{((a+b)/(q-b)),(b/(q-b-a))\} < 1$, since $2a+3b < 1$.

Case 2. If $(Tx_{2n},Sx_{2n+1}) \in P_\circ \times Q_1$, then

$$d(Tx_{2n},Sx_{2n+1}) + d(Sx_{2n+1},y_{2n+1}) = d(Tx_{2n},y_{2n+1}), \tag{3.13}$$

which in turn yields

$$d(Tx_{2n},Sx_{2n+1}) \leq d(Tx_{2n},y_{2n+1}) = d(y_{2n},y_{2n+1}), \tag{3.14}$$

and hence

$$q \cdot d(Tx_{2n},Sx_{2n+1}) \leq q \cdot d(y_{2n},y_{2n+1}) \leq H(F_{2n+1}(x_{2n}),F_{2n}(x_{2n-1})). \tag{3.15}$$

Now, proceeding as in Case 1, we have

$$d(Tx_{2n},Sx_{2n+1}) \leq \begin{cases} \left(\dfrac{a+b}{q-b}\right)d(Sx_{2n-1},Tx_{2n}), & \text{if } d(y_{2n-1},y_{2n}) \geq d(y_{2n+1},y_{2n}) \\ \left(\dfrac{b}{q-b-a}\right)d(Sx_{2n-1},Tx_{2n}), & \text{if } d(y_{2n-1},y_{2n}) \leq d(y_{2n+1},y_{2n}), \end{cases} \tag{3.16}$$

or

$$d(Tx_{2n},Sx_{2n+1}) \leq h \cdot d(Sx_{2n-1},Tx_{2n}). \tag{3.17}$$

In case $(Sx_{2n-1},Tx_{2n}) \in Q_1 \times P_\circ$, then as earlier, one also obtains

$$d(Sx_{2n-1},Tx_{2n}) \leq \begin{cases} \left(\dfrac{a+b}{q-b}\right)d(Sx_{2n-1},Tx_{2n-2}), & \text{if } d(y_{2n-2},y_{2n-1}) \geq d(y_{2n-1},y_{2n}) \\ \left(\dfrac{b}{q-b-a}\right)d(Sx_{2n-1},Tx_{2n-2}), & \text{if } d(y_{2n-2},y_{2n-1}) \leq d(y_{2n-1},y_{2n}), \end{cases} \tag{3.18}$$

or

$$d(Sx_{2n-1},Tx_{2n}) \leq h \cdot d(Sx_{2n-1},Tx_{2n-2}), \tag{3.19}$$

where $h = \max\{((a+b)/(q-b)),(b/(q-b-a))\} < 1$, since $2a+3b < 1$.

Case 3. If $(Tx_{2n},Sx_{2n+1}) \in P_1 \times Q_\circ$, then $Sx_{2n-1} = y_{2n-1}$. Proceeding as in Case 1, one gets

$$\begin{aligned} q \cdot d(Tx_{2n},Sx_{2n+1}) &= q \cdot d(Tx_{2n},y_{2n+1}) \leq q \cdot d(Tx_{2n},y_{2n}) + q \cdot d(y_{2n},y_{2n+1}) \\ &\leq q \cdot d(Sx_{2n-1},y_{2n}) + H(F_{2n+1}(x_{2n}),F_{2n}(x_{2n-1})) \\ &\leq q \cdot d(Sx_{2n-1},y_{2n}) + a \cdot \max\left\{\frac{1}{2}d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n})\right\} \\ &\quad + b\{d(y_{2n},y_{2n}) + d(y_{2n-1},y_{2n+1})\}, \end{aligned} \tag{3.20}$$

which in turn yields

$$d(Tx_{2n},Sx_{2n+1}) \leq \begin{cases} \left(\dfrac{q+b}{q-a-b}\right)d(Sx_{2n-1},y_{2n}), & \text{if } d(y_{2n-1},y_{2n}) \leq d(y_{2n+1},y_{2n}) \\ \left(\dfrac{q+a+b}{q-b}\right)d(Sx_{2n-1},y_{2n}), & \text{if } d(y_{2n-1},y_{2n}) \geq d(y_{2n+1},y_{2n}). \end{cases} \tag{3.21}$$

Now, proceeding as earlier, one also obtains

$$d(Sx_{2n-1},y_{2n}) \leq \begin{cases} \left(\dfrac{a+b}{q-b}\right) d(Sx_{2n-1},Tx_{2n-2}), & \text{if } d(y_{2n-2},y_{2n-1}) \geq d(y_{2n-1},y_{2n}) \\ \left(\dfrac{b}{q-a-b}\right) d(Sx_{2n-1},Tx_{2n-2}), & \text{if } d(y_{2n-2},y_{2n-1}) \leq d(y_{2n-1},y_{2n}). \end{cases} \tag{3.22}$$

Therefore combining above inequalities, we have

$$d(Tx_{2n},Sx_{2n+1}) \leq k \cdot d(Sx_{2n-1},Tx_{2n-2}), \tag{3.23}$$

where

$$\begin{aligned} k = \max\Bigg\{ &\left(\frac{a+b}{q-b}\right)\left(\frac{q+b}{q-a-b}\right), \left(\frac{a+b}{q-b}\right)\left(\frac{q+a+b}{q-b}\right), \\ &\left(\frac{b}{q-a-b}\right)\left(\frac{q+b}{q-a-b}\right), \left(\frac{b}{q-a-b}\right)\left(\frac{q+a+b}{q-b}\right)\Bigg\} < 1, \end{aligned} \tag{3.24}$$

since $2a+3b<1$.

To substantiate that, the inequality $2a+3b<q<1$ implies all foregoing inequalities, one may note that

$$2a+3b<q \Longrightarrow 2aq+3bq<q^2, \tag{3.25}$$

or

$$aq+ab+bq+b^2+aq+2bq-ab-b^2<q^2, \tag{3.26}$$

or

$$aq+ab+bq+b^2<q^2-aq-2bq+ab+b^2, \tag{3.27}$$

or

$$\left(\frac{a+b}{q-b}\right)\left(\frac{q+b}{q-a-b}\right)<1, \tag{3.28}$$

and

$$2a+3b<q \Longrightarrow a+3b<q, \tag{3.29}$$

or

$$aq+3bq<q^2 \Longrightarrow aq+bq+bq+bq<q^2, \tag{3.30}$$

or

$$bq+ab+b^2<q^2-bq-aq+ab-bq+b^2, \tag{3.31}$$

or

$$\left(\frac{b}{q-a-b}\right)\left(\frac{q+a+b}{q-b}\right)<1. \tag{3.32}$$

Similarly one can establish the other inequalities as well. Thus in all the cases, we have

$$d(Tx_{2n},Sx_{2n+1}) \le k\cdot\max\{d(Sx_{2n-1},Tx_{2n}),d(Tx_{2n-2},Sx_{2n-1})\} \tag{3.33}$$

whereas

$$d(Sx_{2n+1},Tx_{2n+2}) \le k\cdot\max\{d(Sx_{2n-1},Tx_{2n}),d(Tx_{2n},Sx_{2n+1})\}. \tag{3.34}$$

Now on the lines of Assad and Kirk [4], it can be shown by induction that for $n\ge 1$, we have

$$d(Tx_{2n},Sx_{2n+1})<k^n\cdot\delta, \qquad d(Sx_{2n+1},Tx_{2n+2})<k^{n+(1/2)}\cdot\delta \tag{3.35}$$

whereas

$$\delta=k^{-1/2}\max\{d(Tx_0,Sx_1),d(Sx_1,Tx_2)\}. \tag{3.36}$$

Thus the sequence $\{Tx_0,Sx_1,Tx_2,Sx_3,\dots,Sx_{2n-1},Tx_{2n},Sx_{2n+1},\dots\}$ is Cauchy and hence converges to the point z in X. Then as noted in [7] there exists at least one subsequence $\{Tx_{2n_k}\}$ or $\{Sx_{2n_k+1}\}$ which is contained in $P_\circ$ or $Q_\circ$ respectively. Suppose that the subsequence $\{Tx_{2n_k}\}$ contained in $P_\circ$ for each $k\in N$ converges to z. Using compatibility of (F_j,S), we have

$$\lim_{k\to\infty} d(Sx_{2n_k-1},F_j(x_{2n_k-1}))=0 \quad \text{for any even integer } j\in N, \tag{3.37}$$

which implies that $\lim_{k\to\infty} d(STx_{2n_k},F_j(Sx_{2n_k-1}))=0$.

Using the continuity of S and F_j, one obtains $Sz\in F_j(z)$, for any even integer $j\in N$. Similarly the continuity of T and F_i implies $Tz\in F_i(z)$, for any odd integer $i\in N$. Now

$$\begin{aligned}
q\cdot d(Tz,Sz) &\le H(F_i(z),F_j(z))\\
&\le a\cdot\max\left\{\frac{1}{2}d(Tz,Sz),d(Tz,F_i(z)),d(Sz,F_j(z))\right\}\\
&\quad +b\{d(Tz,F_j(z))+d(Sz,F_i(z))\}\\
&\le a\cdot\max\left\{\frac{1}{2}d(Tz,Sz),0,0\right\}+b\{d(Tz,Sz)+d(Tz,Sz)\}\\
&\le\left(\frac{a}{2}+2b\right)\cdot d(Tz,Sz),
\end{aligned} \tag{3.38}$$

yielding thereby $Tz = Sz$ which shows that z is a common coincidence point of the maps $\{F_n\}$, S and T.

Remark 3.2. By setting $F_i = F$ (for any odd integer $i \in N$) and $F_j = G$ (for any even integer $j \in N$) in Theorem 3.1, one deduces a rectified and sharpened form of a result due to Ahmad and Imdad [2].

Remark 3.3. By setting $F_i = F$ (for any odd integer $i \in N$), $F_j = G$ (for any even integer $j \in N$) and $S = T$ in Theorem 3.1, one deduces a rectified and improved version of a result due to Ahmad and Imdad [1].

In an attempt to prove Theorem 3.1 for pointwise R-weakly commuting mappings, we have the following.

Theorem 3.4. *Let (X,d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S,T : K \to X$ satisfying (3.1), (iv), (v) and (vii). Suppose that*

(xi) *(F_i,T) and (F_j,S) are pointwise R-weakly commuting pairs.*

Then (F_i,T) as well as (F_j,S) has a point of coincidence.

Proof. On the lines of the proof of Theorem 3.1, one can show that the sequence $\{Tx_{2n}\}$ converges to a point $z \in X$. Now we assume that there exists a subsequence $\{Tx_{2n_k}\}$ of $\{Tx_{2n}\}$ which is contained in $P_\circ$. Further subsequence $\{Tx_{2n_k}\}$ and $\{Sx_{2n_k+1}\}$ both converge to $z \in K$ as K is a closed subset of the complete metric space (X,d). Since $Tx_{2n_k} \in F_j(x_{2n_k-1})$ for any even integer $j \in N$ and $Sx_{2n_k-1} \in K$. Using pointwise R-weak commutativity of (F_j,S), we have

$$d\big(SF_j(x_{2n_k-1}),F_j(Sx_{2n_k-1})\big) \le R_1 \cdot d\big(F_j(x_{2n_k-1}),Sx_{2n_k-1}\big) \tag{3.39}$$

for any even integer $j \in N$ with some $R_1 > 0$. Also

$$d\big(SF_j(x_{2n_k-1}),F_j(z)\big) \le d\big(SF_j(x_{2n_k-1}),F_j(Sx_{2n_k-1})\big) + H\big(F_j(Sx_{2n_k-1}),F_j(z)\big). \tag{3.40}$$

Making $k \to \infty$ in (3.39) and (3.40) and using continuity of F_j as well as S, we get $d(Sz, F_j(z)) \le 0$ yielding thereby $Sz \in F_j(z)$ for any even integer $j \in N$.

Since $y_{2n_k+1} \in F_i(x_{2n_k})$ and $\{Tx_{2n_k}\} \in K$, pointwise R-weak commutativity of (F_i,T) implies

$$d\big(TF_i(x_{2n_k}),F_i(Tx_{2n_k})\big) \le R_2 \cdot d\big(F_i(x_{2n_k}),Tx_{2n_k}\big) \tag{3.41}$$

for any odd integer $i \in N$ with some $R_2 > 0$, besides

$$d\big(TF_i(x_{2n_k}),F_i(z)\big) \le d\big(TF_i(x_{2n_k}),F_i(Tx_{2n_k})\big) + H\big(F_i(Tx_{2n_k}),F_i(z)\big). \tag{3.42}$$

Therefore, as earlier the continuity of F_i as well as T implies $d(Tz,F_i(z)) \le 0$ giving thereby $Tz \in F_i(z)$ as $k \to \infty$.

If we assume that there exists a subsequence $\{Sx_{2n_k+1}\}$ contained in $Q_\circ$, then analogous arguments establish the earlier conclusions. This concludes the proof. □

In the next theorem, we utilize the closedness of TK and SK to replace the continuity requirements besides minimizing the commutativity requirements to merely coincidence points.

Theorem 3.5. *Let (X,d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S,T : K \to X$ satisfying (3.1), (iv) and (v). Suppose that*

(xii) *TK and SK are closed subspaces of X. Then*

($\star$) *(F_i,T) has a point of coincidence,*

($\star\star$) *(F_j,S) has a point of coincidence.*

Moreover, (F_i,T) has a common fixed point if T is quasi-coincidentally commuting and coincidentally idempotent w.r.t F_i whereas (F_j,S) has a common fixed point provided S is quasi-coincidentally commuting and coincidentally idempotent w.r.t F_j.

Proof. On the lines of Theorem 3.1, one assumes that there exists a subsequence $\{Tx_{2n_k}\}$ which is contained in $P_\circ$ and TK as well as SK are closed subspaces of X. Since $\{Tx_{2n_k}\}$ is Cauchy in TK, it converges to a point $u \in TK$. Let $v \in T^{-1}u$, then $Tv = u$. Since $\{Sx_{2n_k+1}\}$ is a subsequence of Cauchy sequence, $\{Sx_{2n_k+1}\}$ converges to u as well. Using (3.1), one can write

$$\begin{aligned} &q \cdot d(F_i(v), Tx_{2n_k}) \\ &\quad \le H(F_i(v), F_j(x_{2n_k-1})) \\ &\quad \le a \cdot \max\left\{\frac{1}{2}d(Tv, Sx_{2n_k-1}), d(Sx_{2n_k-1}, F_j(x_{2n_k-1})), d(Tv, F_i(v))\right\} \\ &\qquad + b\{d(Tv, F_j(x_{2n_k-1})) + d(Sx_{2n_k-1}, F_i(v))\}, \end{aligned} \tag{3.43}$$

which on letting $k \to \infty$, reduces to

$$\begin{aligned} q \cdot d(F_i(v), u) &\le a \cdot \max\{0, d(u, F_i(v)), 0\} + b\{0 + d(F_i(v), u)\} \\ &\le (a+b) \cdot d(u, F_i(v)), \end{aligned} \tag{3.44}$$

yielding thereby $u \in F_i(v)$ which implies that $u = Tv \in F_i(v)$ as $F_i(v)$ is closed.

Since Cauchy sequence $\{Tx_{2n}\}$ converges to $u \in K$ and $u \in F_i(v)$, $u \in F_i(K) \cap K \subseteq SK$, there exists $w \in K$ such that $Sw = u$. Again using (3.1), one gets

$$\begin{aligned} q \cdot d(Sw, F_j(w)) = q \cdot d(Tv, F_j(w)) &\le H(F_i(v), F_j(w)) \\ &\le a \cdot \max\left\{\frac{1}{2}d(Tv, Sw), d(Tv, F_i(v)), d(Sw, F_j(w))\right\} \\ &\quad + b\{d(Tv, F_j(w)) + d(Sw, F_i(v))\} \\ &\le (a+b) \cdot d(Sw, F_j(w)), \end{aligned} \tag{3.45}$$

implying thereby $Sw \in F_j(w)$, that is w is a coincidence point of (S, F_j).

If one assumes that there exists a subsequence $\{Sx_{2n_k+1}\}$ contained in $Q_\circ$ with TK as well as SK are closed subspaces of X, then noting that $\{Sx_{2n_k+1}\}$ is Cauchy in SK, the foregoing arguments establish that $Tv \in F_i(v)$ and $Sw \in F_j(w)$.

Since v is a coincidence point of (F_i, T) therefore using quasi-coincidentally commuting property of (F_i, T) and coincidentally idempotent property of T w.r.t F_i, one can have

$$Tv \in F_i(v), \quad u = Tv \Longrightarrow Tu = TTv = Tv = u, \tag{3.46}$$

therefore $u = Tu = TTv \in TF_i(v) \subset F_i(Tv) = F_i(u)$ which shows that u is the common fixed point of (F_i, T). Similarly using the quasi-coincidentally commuting property of (F_j, S) and coincidentally idempotent property of S w.r.t F_j, one can show that (F_j, S) has a common fixed point as well.

By setting $S = T = I_K$ in Theorem 3.5, we deduce the following corollary for a sequence of set-valued mappings which is a partially sharpened form of Theorem 2.2 due to Ćirić and Ume [5] as our contraction condition (below) is more general than the condition employed in Ćirić and Ume [5] but Theorem 2.2 due to Ćirić and Ume [5] cannot be derived completely from Theorem 3.5 as $2a + 3b < 1$ does not imply $3a + 3b + ab < 1$. Note that if $a = b$ and $b = c$ then $a + 2b + 3c + ac < 1$ reduces to $3a + 3b + ab < 1$.

Corollary 3.6. *Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ satisfying:*
(xiii) *$x \in \delta K \Rightarrow F_n(x) \subseteq K$, and*

$$\begin{aligned} H(F_i(x), F_j(y)) \le a \cdot \max\left\{\frac{1}{2}d(x,y), d(x,F_i(x)), d(y,F_j(y))\right\} \\ + b\{d(x,F_j(y)) + d(y,F_i(x))\} \end{aligned} \tag{3.47}$$

for all $x, y \in K$ with $x \neq y$, $i \neq j$, $a, b \geq 0$ such that $2a + 3b < 1$, then $\{F_n\}$ has a common fixed point.

Remark 3.7. Theorem 3.5 remains true if we substitute closedness of "TK and SK" with closedness of "$F_i(K)$ and $F_j(K)$."

Remark 3.8. By setting $S = T = I_K$ in Theorem 3.5, one deduces an extension of a result due to Khan et al. [15] to a sequence of multi-valued mappings.

Remark 3.9. By setting $F_n = F$ (for all $n \in N$) and $S = T = I_K$ in Theorem 3.5, one deduces a multi-valued version of a result due to Khan et al. [15].

Remark 3.10. By setting $F_i = F$ (for any odd integer $i \in N$), $F_j = G$ (for any even integer $j \in N$) and $S = T = I_K$ in Theorem 3.5, one deduces a sharpened and generalized form of a result due to Khan [14].

Finally, we prove a theorem when "closedness of K" is replaced by "compactness of K." □

Theorem 3.11. *Let (X,d) be a complete metrically convex metric space and K a nonempty compact subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $T : K \to X$ satisfying*
(xiv) $\delta K \subseteq TK$, $(F_i(K) \cup F_j(K)) \cap K \subseteq TK$
(xv) $Tx \in \delta K \Rightarrow F_i(x) \cup F_j(x) \subseteq K$ *with*

$$H(F_i(x),F_j(y)) < M(x,y) \tag{3.48}$$

when $M(x,y) > 0$, for all $x,y \in K$ where

$$\begin{aligned} M(x,y) = a \cdot \max &\left\{\frac{1}{2}d(Tx,Ty), d(Tx,F_i(x)), d(Ty,F_j(y))\right\} \\ &+ b\{d(Tx,F_j(y)) + d(Ty,F_i(x))\} \end{aligned} \tag{3.49}$$

for all $x,y \in X$ with $x \neq y$, where a, b are non-negative reals such that $2a+3b \leq q \leq 1$.

If T is compatible with $\{F_n\}$ $(n \in N)$ along with $\{F_n\}$ and T are continuous on K, then $\{F_n\}$ and T have a common point of coincidence.

Proof. We assert that $M(x,y) = 0$ for some $x,y \in K$. Otherwise $M(x,y) \neq 0$, for any $x,y \in K$ implies that

$$f(x,y) = \frac{H(F_i(x),F_j(y))}{M(x,y)} \tag{3.50}$$

is continuous and satisfies $f(x,y) < 1$ for all $(x,y) \in K \times K$. Since $K \times K$ is compact, there exists $(u,v) \in K \times K$ such that $f(x,y) \leq f(u,v) = c < 1$ for $x,y \in K$ which in turn yields $H(F_i(x),F_j(y)) \leq c \cdot M(x,y)$ for $x,y \in K$ and $0 < c < 1$. Therefore using (3.49), one obtains

$$\max\left\{\frac{ca+cb}{q-cb}, \frac{cb}{q-ca-cb}\right\} < 1. \tag{3.51}$$

Now by Theorem 3.1 (with restriction $S = T$), we get $Tz \in F_i(z) \cap F_j(z)$ for some $z \in K$ and one concludes $M(z,z) = 0$, contradicting the facts that $M(x,y) > 0$. Therefore $M(x,y) = 0$ for some $x,y \in K$ which implies $Tx \in F_i(x)$ for any odd integer $i \in N$ and $Tx = Ty \in F_j(y)$ for any even integer $j \in N$. If $M(x,x) = 0$ then $Tx \in F_j(x)$ for any even integer $j \in N$ and if $M(x,x) \neq 0$ then using (3.49), one infers that $d(Tx,F_j(x)) \leq 0$ yielding thereby $Tx \in F_j(x)$ for any even integer $j \in N$. Similarly in either of the cases $M(y,y) = 0$ or $M(y,y) > 0$ one concludes that $Ty \in F_i(y)$ for any odd integer $i \in N$. Thus we have shown that $\{F_n\}$ and T have a common point of coincidence.

By setting $F_i = F$ (for every odd integer $i \in N$), $F_j = G$ (for every even integer $j \in N$) and $T = I_K$ in Theorem 3.11, we deduce the following corollary for a pair of set-valued mappings which is a partial generalization of Theorem 2.3 of Ćirić and Ume [5] due to the reasons already stated in respect of Corollary 3.6. □

Corollary 3.12. *Let (X,d) be a complete metrically convex metric space and K a nonempty compact subset of X. Let $F,G : K \to CB(X)$ satisfying:*
(xvi) $x \in \delta K \Rightarrow F(x) \cup G(x) \subseteq K$ *and*

$$H(Fx,Gy) \leq a \cdot \max\left\{\frac{1}{2}d(x,y), d(x,Fx), d(y,Gy)\right\} + b\{d(x,Gy)+d(y,Fx)\} \tag{3.52}$$

for all $x,y \in K$ with $x \neq y$, $a,b \geq 0$ such that $2a+3b \leq 1$, then there exists $z \in K$ such that $z \in Fz \cap Gz$.

While proving Theorem 3.11 the following question remains unresolved: *Does Theorem 3.11 hold for $\{F_n\}$, S and T instead of $\{F_n\}$ and T?*

Acknowledgments

The authors are grateful to both the learned referees for their critical reading of the entire manuscript and suggesting many improvements. The first author is also thankful to University Grants Commission in India for financial assistance (Project no. F.30-246/2004(SR)).

References

[1] A. Ahmad and M. Imdad, *On common fixed point of mappings and multivalued mappings*, Rad. Mat. **8** (1992), no. 1, 147–158.

[2] ______, *Some common fixed point theorems for mappings and multi-valued mappings*, J. Math. Anal. Appl. **218** (1998), no. 2, 546–560.

[3] A. Ahmad and A. R. Khan, *Some common fixed point theorems for non-self-hybrid contractions*, J. Math. Anal. Appl. **213** (1997), no. 1, 275–286.

[4] N. A. Assad and W. A. Kirk, *Fixed point theorems for set-valued mappings of contractive type*, Pacific J. Math. **43** (1972), no. 3, 553–562.

[5] Lj. B. Ćirić and J. S. Ume, *On an extension of a theorem of Rhoades*, Rev. Roumaine Math. Pures Appl. **49** (2004), no. 2, 103–112.

[6] B. C. Dhage, U. P. Dolhare, and A. Petruşel, *Some common fixed point theorems for sequences of nonself multivalued operators in metrically convex metric spaces*, Fixed Point Theory **4** (2003), no. 2, 143–158.

[7] O. Hadžić, *On coincidence points in convex metric spaces*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **19** (1986), no. 2, 233–240.

[8] O. Hadžić and Lj. Gajić, *Coincidence points for set-valued mappings in convex metric spaces*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **16** (1986), no. 1, 13–25.

[9] N. J. Huang and Y. J. Cho, *Common fixed point theorems for a sequence of set-valued mappings*, J. Korean Math. Soc. **4** (1997), no. 1, 1–10.

[10] M. Imdad and A. Ahmad, *On common fixed point of mappings and set-valued mappings with some weak conditions of commutativity*, Publ. Math. Debrecen **44** (1994), no. 1-2, 105–114.

[11] M. Imdad, A. Ahmad, and S. Kumar, *On nonlinear nonself hybrid contractions*, Rad. Mat. **10** (2001), no. 2, 233–244.

[12] S. Itoh, *Multivalued generalized contractions and fixed point theorems*, Comment. Math. Univ. Carolinae **18** (1977), no. 2, 247–258.

[13] H. Kaneko and S. Sessa, *Fixed point theorems for compatible multi-valued and single-valued mappings*, Int. J. Math. Math. Sci. **12** (1989), no. 2, 257–262.

[14] M. S. Khan, *Common fixed point theorems for multivalued mappings*, Pacific J. Math. **95** (1981), no. 2, 337–347.
[15] M. S. Khan, H. K. Pathak, and M. D. Khan, *Some fixed point theorems in metrically convex spaces*, Georgian Math. J. **7** (2000), no. 3, 523–530.
[16] K. Kuratowski, *Topology. Vol. I*, Academic Press, New York, 1966.
[17] S. B. Nadler Jr., *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), no. 2, 475–488.
[18] R. P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl. **188** (1994), no. 2, 436–440.
[19] H. K. Pathak, *Fixed point theorems for weak compatible multi-valued and single-valued mappings*, Acta Math. Hungar. **67** (1995), no. 1-2, 69–78.
[20] V. Popa, *Coincidence and fixed points theorems for noncontinuous hybrid contractions*, Nonlinear Anal. Forum **7** (2002), no. 2, 153–158.

M. Imdad: Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
E-mail address: mhimdad@yahoo.co.in

Ladlay Khan: Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
E-mail address: k_ladlay@yahoo.com