COMMON FIXED POINT THEOREMS FOR COMPATIBLE SELF-MAPS OF HAUSDORFF TOPOLOGICAL SPACES

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The concept of *proper orbits* of a map g is introduced and results of the following type are obtained. If a continuous self-map g of a Hausdorff topological space X has relatively compact proper orbits, then g has a fixed point. In fact, g has a common fixed point with every continuous self-map f of X which is nontrivially compatible with g. A collection of metric and semimetric space fixed point theorems follows as a consequence. Specifically, a theorem by Kirk regarding diminishing orbital diameters is generalized, and a fixed point theorem for maps with no recurrent points is proved.

1. Introduction

Let g be a mapping of a topological space X into itself. Let N denote the set of positive integers and $\omega = N \cup \{0\}$. For $x \in X$, $\mathbb{O}(x)$ is called the *orbit of g at x* and defined by $\mathbb{O}(x) = \{g^k(x) : k \in \omega\}$, where $g^o(x) = x$. Thus, if $n \in \omega$, the orbit of g at $g^n(x)$ is the set $\mathbb{O}(g^n(x)) = \{g^k(x) : k \in \omega \text{ and } k \geq n\}$. (Clearly, $\mathbb{O}(g^n(x)) \subset \mathbb{O}(x)$ for $n \in N$.) And if X has a metric or semimetric d, we will designate the *diameter* of a set $M \subset X$ by $\delta(M)$ which of course is defined $\delta(M) = \sup\{d(x,y) : x, y \in M\}$.

The purpose of this paper is to introduce the concept of *proper orbits* and to demonstrate its role in obtaining fixed points. (We use $\text{cl}(A)$ to denote the closure of the set A.)

Definition 1.1. Let g be a self-map of a topological space X and let $x \in X$. The orbit $\mathbb{O}(x)$ of g at x is *proper* if and only if $\mathbb{O}(x) = \{x\}$ or there exists $n = n_x \in N$ such that $\text{cl}(\mathbb{O}(g^n(x)))$ is a proper subset of $\text{cl}(\mathbb{O}(x))$. If $\mathbb{O}(x)$ is proper for each $x \in M \subset X$, we will say that g *has proper orbits on M*. If $M = X$, we say g has *proper orbits*.

The concept of proper orbits generalizes the concept of diminishing orbital diameters, which was introduced by Belluce and Kirk [1] in 1969. They introduced the concept of mappings with diminishing orbital diameters to obtain fixed point theorems for nonexpansive self-maps of metric spaces. A self-map g of a metric space X has *diminishing orbital diameters* if for each $x \in X$, $\delta(\mathbb{O}(x)) < \infty$, and whenever $\delta(\mathbb{O}(x)) > 0$, there exists $n = n_x \in N$ such that $\delta(\mathbb{O}(x)) > \delta(\mathbb{O}(g^n(x)))$. (If the given property holds for a specific x,

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we will say that $\mathbb{O}(x)$ has diminishing diameters.) Subsequently Kirk [13] (1969) extended the concept to more general mappings. In particular, he proved the following interesting result for a metric space M.

Theorem 1.2 (Kirk [13]). *Suppose M is compact and $g : M \to M$ is continuous with diminishing orbital diameters. Then for each $x \in M$, some subsequence $\{g^{n_k}(x)\}$ of the sequence $\{g^n(x)\}$ of iterates of x has a limit which is a fixed point of g.*

One purpose of this paper is to extend Theorem 1.2, and in the following manner. The underlying space will be a Hausdorff topological space. Instead of requiring that the space M be compact, we will require that the orbits be relatively compact. And we will replace the requirement that orbits have diminishing diameters by the demand that they be proper. Moreover, it will be shown that g has a common fixed point with every $f \in K_g$, a set we now define.

Definition 1.3. If g is a continuous self-map of a topological space X, K_g is the set of all continuous maps $f : X \to X$ such that $M = \{x \in X : fx = gx\} \neq \emptyset$ and $fgx = gfx$ for $x \in M$. ($K_g \neq \emptyset$, since $g \in K_g$.)

We will expand further on these concepts, but first note that the motivation behind our approach and our interest in the set K_g is the following theorem.

Theorem 1.4 [10]. *A continuous self-map g of $I = [0,1]$ has a common fixed point with every function in K_g if and only if every periodic point of g is a fixed point of g (i.e., $g^k(x) = x \Rightarrow g(x) = x$).*

To appreciate the significance of Theorem 1.4 in the present context, suppose there exist $x \in I$ and $k \in N$ such that $k > 1$, $g^k(x) = x$ and $g^i(x) \neq x$ for $0 < i < k$. Then it is clear that $\mathbb{O}(x) = \mathbb{O}(g^n(x)) \neq \{x\}$ for all $n \in N$. Thus, g does not have proper orbits. So, if the function g of Theorem 1.4 has proper orbits, then g has a common fixed point with each $f \in K_g$. We now show that this implication extends to very general settings.

2. Preliminaries

We first review background regarding semimetric spaces and compatible maps. Semimetric spaces give us a vehicle for extending metric space results. Moreover, since semimetric spaces are first countable and orbits are virtually sequences, semimetric spaces provide a natural and relatively unrestricted arena for examples and results involving orbits.

A semimetric on a set X is a function $d : X \times X \to [0,\infty)$ such that $d(x,y) = 0$ if and only if $x = y$, and $d(x,y) = d(y,x)$ for $x, y \in X$. For $p \in X$ and $\epsilon > 0$ we let $S(p,\epsilon) = \{x \in X : d(x,p) < \epsilon\}$. A semimetric space is a pair $(X;d)$ in which X is a Hausdorff topological space and d is a semimetric on X. The topology on X is the family $t(d) = \{U \subset X : p \in U \Rightarrow S(p,\epsilon) \subset U$ for some $\epsilon > 0\}$. We require that the topological interior of $S(p,\epsilon)$ be nonempty and contain p; that is, there exists $U \in t(d)$ such that $p \in U \subset S(p,\epsilon)$. Consequently, a sequence $\{x_n\}$ in X converges in $t(d)$ to $p \in X$, denoted $x_n \to p$, if and only if $d(x_n, p) \to 0$. And a function (map) $g : X \to X$ is continuous if and only if $gx_n \to gx$ whenever $x_n \to x$. See [5, 11] for further details on semimetric spaces.

Compatible maps were introduced by Jungck [8] in 1986 for metric spaces as a generalization of commuting maps. The concept proved useful in obtaining generalizations of established fixed point theorems. The definition for a semimetric space $(X;d)$ is formally the same and is as follows. Maps $f,g : X \to X$ are compatible if and only if $d(gfx_n, fgx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $fx_n, gx_n \to p$ for some $p \in X$. If f and g are compatible and do have a coincidence point, f and g are *nontrivially* compatible. An immediate consequence of the definition is that compatible maps commute at coincidence points, a property we highlight. So if f and g are continuous nontrivially compatible self-maps of X, then $f \in K_g$ and $g \in K_f$. In fact, if f and g are continuous and X is compact, then f and g are compatible if and only if $fx = gx$ implies that $fgx = gfx$. This result is proved in [9] for metric spaces but also holds for semimetric spaces (the triangle inequality is not used). (See [8, 9] for properties of compatible maps.)

Before continuing, we pay for the freedom granted by semimetric spaces. Let A be a subset of a semimetric space. Since we have no triangle inequality, it is not necessarily true—as the next example shows—that $\delta(A) = \delta(\mathrm{cl}(A))$. (Kirk's proof of Theorem 1.2 appealed to this "metric space" equality.)

Example 2.1. Let $X = [0,1]$ and define $d(x,y) = |x-y|$ if $x,y \in (0,1]$ and $d(b,0) = d(0,b) = 2b$ if $b \in [0,1]$. Then $\delta((0,1/2]) = 1/2$, whereas $\delta([0,1/2]) = \delta(\mathrm{cl}((0,1/2])) = 1$. Note that the topological space $(X,t(d))$ is metrizable but d is not a metric.

With $A = (0,1/2]$ and $B = [0,1/2]$, Example 2.1 also shows that in the context of semimetric spaces, even though $A \subset B$ and $\delta(A) < \delta(B)$, $\mathrm{cl}(A)$ may not be a proper subset of $\mathrm{cl}(B)$.

The above observations prompt the following definition.

Definition 2.2. Let $(X;d)$ be a semimetric space. A mapping $g : X \to X$ has orbits with *diminishing closure diameters* if and only if for each $x \in X$, $\delta(\mathrm{cl}(\mathbb{O}(x))) < \infty$, and whenever $\delta(\mathrm{cl}(\mathbb{O}(x))) > 0$, there exists $n \in N$ such that $\delta(\mathrm{cl}(\mathbb{O}(x))) > \delta(\mathrm{cl}(\mathbb{O}(g^n(x))))$.

Of course, if $(X;d)$ is a semimetric space and $g : X \to X$ has orbits with diminishing closure diameters (d.c.d.), then g has proper orbits. For suppose $\mathbb{O}(x) \neq \{x\}$. Then $\delta(\mathrm{cl}(\mathbb{O}(x))) > 0$. Since g has d.c.d., there is an $n \in N$ such that $\delta(\mathrm{cl}(\mathbb{O}(g^n(x)))) < \delta(\mathrm{cl}(\mathbb{O}(x)))$. But this implies that $\mathrm{cl}(\mathbb{O}(g^n(x)))$ is a proper subset of $\mathrm{cl}(\mathbb{O}(x))$, as desired.

Observe also that in metric spaces (X,d) Definition 2.2 reduces to that of diminishing orbital diameters (d.o.d.). Thus, if g has d.o.d. on (X,d), then g has proper orbits.

3. Results and examples

We now state and prove our results. The statements and proofs of the initial theorems and corollaries are completely topological in nature. The proof of our first theorem repeats an argument used by Schwartz in the proof of Proposition 1 on page 353 of [18].

Theorem 3.1. *Let X be a Hausdorff topological space and let $g : X \to X$ be continuous. If g has relatively compact proper orbits, then any nonempty g-invariant closed subset of X contains a fixed point of g. Specifically, the closure of each orbit $\mathbb{O}(x)$ has a fixed point of g.*

Proof. Let M be a nonempty g-invariant closed subset of X, and let $x \in M$. By hypothesis, $g^n(x) \in M$ for each $n \in N$ so that $\mathbb{O}(x) \subset M$. Then $\mathrm{cl}(\mathbb{O}(x)) \subset M$ since M is closed. Clearly, $g(\mathbb{O}(x)) \subset \mathbb{O}(x)$; therefore the continuity of g implies that $g(\mathrm{cl}(\mathbb{O}(x))) \subset \mathrm{cl}(g(\mathbb{O}(x))) \subset \mathrm{cl}(\mathbb{O}(x))$ $(\subset M)$. But $\mathrm{cl}(\mathbb{O}(x))$ is compact by hypothesis. Consequently, $\mathrm{cl}(\mathbb{O}(x))$ is a nonempty g-invariant compact subset of M. A Zorn's lemma argument produces a minimal nonempty g-invariant compact subset A of M.

Let $a \in A$. Since A is g-invariant and closed, $\mathrm{cl}(\mathbb{O}(a)) \subset A$. As above, $g(\mathrm{cl}(\mathbb{O}(a))) \subset \mathrm{cl}(O(a))$ $(\subset A)$. But then $\mathrm{cl}(\mathbb{O}(a))$ is a g-invariant nonempty compact subset of A and thus of M. By the minimality of A, $\mathrm{cl}(\mathbb{O}(a)) = A$. Consequently, $A = \mathrm{cl}(\mathbb{O}(g^n(a)))$ for $n \in \omega$, since a was an arbitrary element of A and $g^n(a) \in A$. Thus, $\mathrm{cl}(\mathbb{O}(a)) = \mathrm{cl}(\mathbb{O}(g^n(a)))$ for all $n \in N$, and therefore $\mathbb{O}(a) = \{a\}$ since $\mathbb{O}(a)$ is *proper*. We conclude, $g(a) = a$, and a is the desired fixed point of M.

Moreover, the above argument shows that for each $x \in X$, $\mathrm{cl}(\mathbb{O}(x))$ is a nonvoid g-invariant compact (and therefore, closed) subset of X. As such, $\mathrm{cl}(\mathbb{O}(x))$ contains a fixed point of g by the preceding result. □

Corollary 3.2. *A continuous self-map g of a Hausdorff topological space X has a fixed point if and only if there exist $x \in X$ such that* $\mathrm{cl}(\mathbb{O}(x))$ *is compact and g has proper orbits on* $\mathrm{cl}(\mathbb{O}(x))$.

Proof. If $g(x) = x$, then $\mathbb{O}(x) = \{x\} = \mathrm{cl}(\mathbb{O}(x))$. Thus $\mathrm{cl}(\mathbb{O}(x))$ is compact, and the only orbit of g on $\mathrm{cl}(\mathbb{O}(x))$, namely $\{x\}$, is proper. So the condition is necessary.

To see that the condition is sufficient, let $x \in X$ such that $\mathrm{cl}(\mathbb{O}(x))$ is compact and g has proper orbits on $\mathrm{cl}(\mathbb{O}(x))$. Since g is continuous, $\mathrm{cl}(\mathbb{O}(x))$ is g-invariant. Therefore, we apply Theorem 3.1 with $X = M = \mathrm{cl}(\mathbb{O}(x))$. Since g has relatively compact orbits on $\mathrm{cl}(\mathbb{O}(x))$ ($\mathrm{cl}(\mathbb{O}(x))$ is compact) and g has proper orbits on $\mathrm{cl}(\mathbb{O}(x))$ by hypothesis, the conclusion follows. □

Why would it not be sufficient in the above result to merely require that there exist a proper orbit $\mathbb{O}(x)$ of g having compact closure? Consider the following.

Example 3.3. Let $0 < a < 1$. Let X be the planar annulus defined in polar coordinates by $X = \{(r,\theta) : 0 < a \le r \le 1\}$.

Define $g : X \to X$ by $g((r,\theta)) = (r/2 + 1/2, \theta + \pi)$. Then for $r < 1$ and $n \in N$, $g^n((r,\theta)) = (r/2^n + \sum_{k=1}^{n}(1/2^k), \theta + n\pi)$.

Clearly, $g^{2n}((r,\theta)) \to (1,\theta)$ and $g^{2n+1}((r,\theta)) \to (1,\theta+\pi)$, and consequently, $\mathrm{cl}(\mathbb{O}((r,\theta))) = \mathbb{O}((r,\theta)) \cup \{(1,\theta),(1,\theta+\pi)\}$. Thus for all $r < 1$, $\mathbb{O}((r,\theta))$ is proper and has compact closure; but g does not have proper orbits on $\mathrm{cl}(\mathbb{O}((r,\theta)))$ since $\mathbb{O}((1,\theta)) = \{(1,\theta),(1,\theta+\pi)\}$ (i.e., g is periodic of period 2 on $r = 1$). And of course, g has no fixed points. Note also that the diameter of every orbit in X is 2, and therefore no orbit in X has d.o.d.

The following example satisfies the hypothesis of Corollary 3.2 and thus the mapping g of this example has a fixed point.

Example 3.4. Let $X = [-1,1]$ with the usual metric and let $g(x) = -x^{1/3}$ for $x \in X$. It is easy to verify that every point in $(-1,1)$ has a proper orbit. However, $x = 0$, the only fixed point of g, is the only point in X with proper orbits on the closure of its orbit $\{0\}$.

We now apply Theorem 3.1 to obtain information regarding the set K_g, of Definition 1.3.

Corollary 3.5. *Let g be a continuous self-map of a Hausdorff topological space X. If g has relatively compact proper orbits, then g has a common fixed point with each* $f \in K_g$.

Proof. Let $f \in K_g$, and let $M = \{x \in X : fx = gx\}$. Then $M \neq \emptyset$ by definition of K_g. If $x \in M$, then $fx = gx$. Since $f \in K_g$, $f^2x = fgx = gfx = g^2x$. But $f^2x = gfx$ implies that $fx \in M$, and $fgx = g^2x$ implies that $gx \in M$. Thus $f(M), g(M) \subset M$.

It is well known that M is closed. Thus, M is a nonempty g-invariant closed subset of X. By Theorem 3.1, M has a fixed point p of g. By definition of M, $f(p) = g(p) = p$, and p is the promised common fixed point. □

For purposes of reference we now combine Theorem 3.1 and Corollary 3.5.

Theorem 3.6. *Let g be a continuous self-map of a Hausdorff topological space X. If g has relatively compact proper orbits, then g has a common fixed point with each function* $f \in K_g$. *Moreover, each nonempty g-invariant closed subset of X has a fixed point of g. Specifically, the closure of each orbit* $\mathbb{O}(x)$ *contains a fixed point of g.*

If a Hausdorff topological space X is compact, then all orbits of a self-map of X are relatively compact. Consequently, Theorem 3.6 yields the following result which stresses the role of proper orbits.

Corollary 3.7. *Any continuous self-map g of a compact Hausdorff topological space with proper orbits has a fixed point. In fact, the closure of each orbit contains a fixed point of g. Moreover, g has a common fixed point with each* $f \in K_g$.

If g is a self-map of a topological space X, a point $x \in X$ is called a *recurrent point* if and only if x is a limit point (accumulation point) of $\mathbb{O}(x)$. And x is a *nontrivial periodic point* if and only if $g^kx = x$ for some $k \in N$ but $gx \neq x$.

Theorem 3.8. *Let g be a continuous self-map of a Hausdorff topological space X. If g has no recurrent or nontrivial periodic points, then g has proper orbits.*

Proof. We show $\mathbb{O}(x)$ is proper. If $x = gx$, then we are done. So let $x \neq gx$. Since x is not a recurrent point, there exists a neighborhood $N(x)$ of x such that $N(x) \cap (\mathbb{O}(x) \setminus \{x\}) = \emptyset$. By hypothesis x is not a periodic point, so $\mathbb{O}(x) \setminus \{x\} = \mathbb{O}(gx)$. Thus $N(x) \cap \mathbb{O}(gx) = \emptyset$, that is, $x \notin \text{cl}(\mathbb{O}(gx))$. Hence $\text{cl}(\mathbb{O}(gx)) \neq \text{cl}(\mathbb{O}(x))$ which means $\mathbb{O}(x)$ is proper. □

Corollary 3.7 and Theorem 3.8 yield the following.

Corollary 3.9. *Any continuous self-map of a compact Hausdorff topological space which has no recurrent or nontrivial periodic points has a fixed point.*

Of course, if in Corollary 3.9 g has nontrivial periodic points, g may have no fixed points. Define g by $g(x) = 1 - x$ for $x \in X = [0, 1/4] \cup [3/4, 1]$. Then g is a continuous self-map of a compact space X of period 2 which has neither recurrent points nor fixed points.

As an application of Corollary 3.9, consider the following.

Example 3.10. Let $C = \{(r,\theta) : \theta \in R\}$ be a circle in polar coordinates of radius $r > 0$ and let g be a rational rotation of C. Thus, $g((r,\theta)) = (r,\theta+\phi)$ for some rational $\phi \in (0,2\pi)$. Then every point of C is a recurrent point of g. To see this, first note that $g^n((r,\theta)) = (r,\theta+n\phi)$ for $n \in N$ and that $(r,\theta+n\phi) = (r,\theta+2k\pi)$ for no $n,k \in N$ since ϕ is rational. Thus, g has no fixed or periodic points so that g has a recurrent point in C by Corollary 3.9. We leave it to the reader to show that therefore every point of C is a recurrent point of g. Consequently, if A is a rational rotation of the plane about the origin, then every point of $R^2 - \{(0,0)\}$ is a recurrent point of A.

A self-map g of a space X is compact [4] if and only if X has a compact subset Y and $g(X) \subset Y$.

Corollary 3.11. *Let g be continuous self-map of a Hausdorff topological space with proper orbits. If g is compact, then g has a fixed point; indeed, g has a common fixed point with each $f \in K_g$.*

Proof. Since g is compact there exists a compact set $Y \subset X$ such that $g(X) \subset Y$. Then $g(Y) \subset g(X) \subset Y$, and therefore Y is g invariant and compact. Since g has proper orbits, the restriction of g to Y satisfies the hypothesis of Corollary 3.7. □

Corollary 3.12. *Let g be a continuous self-map of a first countable Hausdorff topological space X. If g has relatively compact proper orbits, then g has a common fixed point with each $f \in K_g$. Moreover, for each $x \in X$, there is a subsequence of $\{g^n(x)\}$ which converges to a fixed point of g.*

Proof. Corollary 3.12 is an immediate consequence of Theorem 3.6, except for possibly the concluding statement. This last statement follows by noting that by Theorem 3.6, for each $x \in X$, $\mathrm{cl}(\mathbb{O}(x))$ contains a fixed point p of g. Since X is first countable, there is a sequence in $\mathbb{O}(x)$ which converges to p. This sequence can be chosen so as to be a legitimate subsequence of $\{g^n(x)\}$. □

Since any semimetric space is Hausdorff and first countable we have:

Corollary 3.13. *Let $(X;d)$ be a semimetric space and let $g : X \to X$ be continuous. If g has relatively compact orbits with proper orbits or diminishing closure diameters, then g has a common fixed point with each $f \in K_g$. Moreover, for each $x \in X$, some subsequence of $\{g^n(x)\}$ converges to a fixed point of g.*

As noted above, in metric spaces diminishing orbital diameters (d.o.d.) yield orbits with diminishing closure diameters (d.c.d.) and therefore, proper orbits. Thus, Corollary 3.13 is valid for metric spaces (X,d) with d.c.d. replaced by d.o.d.

The next example shows that common fixed points of g and the members of K_g in Corollary 3.13 need not be unique, even if $X = [0,1]$ with the usual metric. It also suggests that K_g can be large.

Example 3.14. Let $X = [0,1]$ and $d(x,y) = |x-y|$. Define $f(x) = 2x - x^2$ and $g(x) = x^2$. The common fixed points of f and g are 0 and 1 and are their only coincidence points. In general $fg \neq gf$, but since f and g clearly commute at 0 and 1, f and g are nontrivially compatible. If $x \in [0,1)$, $g^n(x) \downarrow 0$, and $g^n(1) = 1$ for all n so g has proper orbits. Note also that any continuous function $f : X \to X$ such that $f(x) > x^2$ or $f(x) < x^2$ for $x \in (0,1)$ is a member of K_g.

The following example shows us that Corollary 3.13 does generalize Kirk's Theorem 1.2 since $(X;d)$ is a noncompact semimetric space and g has proper orbits but no d.o.d. except at $x = 0$.

Example 3.15. Let $X = [0,1]$ and if $x, y \neq 0$, define d by $d(x,y) = 0$ if $x = y$ and 1 if $x \neq y$. And $d(0,x) = d(x,0) = x$ for $x \in X$. Define $g : X \to X$ by $g(x) = x/2$. Then $g^n(x) = x/2^n \downarrow 0$ for $x \in X$. Since $\mathbb{O}(x) = \{x/2^k : k \in \omega\}$, $\mathbb{O}(x)$ is clearly proper for all x. In fact, $\mathrm{cl}(\mathbb{O}(g(x)))$ is a proper subset of $\mathrm{cl}(\mathbb{O}(x))$ for all $x \neq 0$. But $\delta(\mathbb{O}(g^n(x))) = \delta(\mathrm{cl}(\mathbb{O}(g^n(x)))) = 1$ for all $n \in \omega$ and $x \neq 0$; that is, $\mathbb{O}(x)$, has diminishing closure diameters or d.o.d. only at 0.

Corollary 3.16. *Any continuous self-map g of a compact metric space with d.o.d. has a common fixed point with each $f \in K_g$. Moreover, for each $x \in X$ some subsequence $\{g^{k_n}(x)\}$ of the sequence $\{g^n(x)\}$ has a limit which is a fixed point of g.*

The need for compatibility of functions to ensure the existence of a common fixed point is demonstrated in the next simple example.

Example 3.17. Let $X = \{1,2,3,4\}$ with the usual metric. Let $f(1) = 2$, $f(2) = 2$, $f(3) = 1$, and $f(4) = 3$. Also let $g(1) = 2$, $g(2) = 4$, $g(3) = 3$, $g(4) = 3$. Then each of f and g has d.o.d., each is continuous, and f and g have a coincidence point. In fact, $f(1) = 2 = g(1)$. But $gf(1) = 4$ whereas $fg(1) = 2$; that is, f and g are not compatible and have no common fixed point.

Of course, proper orbits also play an essential role.

Example 3.18. Let $X = [0,1]$ with the usual metric and define $f, g : X \to X$ by $g(x) = 1 - x$ and $f(x) = 1 - x^2$ for $x \in X$. Then $f \in K_g$ since f and g coincide only at 0 and 1, and commute at 0 and at 1. But neither f nor g has proper orbits since both have periodic points of period 2 at 0 and at 1. And f and g have no common fixed point.

Since any closed and bounded subset of R^n is compact, and since the orbits of a map with d.o.d. are bounded by definition, we have the following somewhat amazing consequence of Corollary 3.13.

Corollary 3.19. *If g is a continuous self-map of R^n with diminishing orbital diameters, then g has a fixed point. In fact, g has a common fixed point with each $f \in K_g$. Moreover, each orbit $\mathbb{O}(x)$ has a sequence which converges to a fixed point of g.*

4. Retrospect

Compatible maps were introduced in [8] as a generalization of commuting maps, which had served as a vehicle for various generalizations of the Banach Contraction Theorem (see [7], and, e.g., [3]). Compatible maps, in turn, proved productive in producing common fixed points and further generalizations of Banach's theorem: [2, 6, 8, 9, 10, 11, 12, 14, 15, 16, 18, 19, 20]. On the other hand, the concept of proper orbits now provides new results, and a means of extending a bevy of known results. We conclude with such an example.

Suppose g is any continuous self-map of complete metric space X such that $\{g^n(x)\}$ converges for each $x \in X$. Then the orbit $\mathbb{O}(x)$ is proper and relatively compact for each x, and consequently, g has a common fixed point with each $f \in K_g$. With this in mind, consider a result presented at the Third World Congress of Nonlinear Analysis.

Theorem 4.1 (Rhoades [17]). *Let (X,d) be a complete metric space, g a weakly contractive map. Then g has a unique fixed point p in X.*

The map $g : X \to X$ is weakly contractive if and only if $d(gx,gy) < d(x,y) - \psi(d(x,y))$ for $x,y \in X$ and $\psi : [0,\infty) \to [0,\infty)$ is a continuous, nondecreasing function such that $\psi(0) = 0$, ψ is positive on $(0,\infty)$ and $\lim_{t\to\infty}\psi(t) = \infty$. Since ψ is nonnegative, g is nonexpansive, hence continuous. Moreover, in the proof it is shown that $g^n(x) \to p$ for each $x \in X$. By the above comments we can therefore augment Theorem 4.1 with the statement, *p is also the common fixed point of g and each $f \in K_g$.*

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