APPROXIMATING COMMON FIXED POINTS OF TWO ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Received 21 April 2005; Revised 13 July 2005; Accepted 18 July 2005

Suppose K is a nonempty closed convex subset of a real Banach space E. Let $S, T : K \to K$ be two asymptotically quasi-nonexpansive maps with sequences $\{u_n\}, \{v_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, and $F = F(S) \cap F(T) := \{x \in K : Sx = Tx = x\} \neq \emptyset$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in K$, $x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n[(1-\beta_n)x_n + \beta_n T^n x_n]$, $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$. It is proved that (a) $\{x_n\}$ converges strongly to some $x^* \in F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$; (b) if X is uniformly convex and if either T or S is compact, then $\{x_n\}$ converges strongly to some $x^* \in F$. Furthermore, if X is uniformly convex, either T or S is compact and $\{x_n\}$ is generated by $x_1 \in K$, $x_{n+1} = \alpha_n x_n + \beta_n S^n[\alpha_n' x_n + \beta_n' T^n x_n + \gamma_n' z_n'] + \gamma_n z_n$, $n \geq 1$, where $\{z_n\}$, $\{z_n'\}$ are bounded, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$, $\{\gamma_n'\}$ are real sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha_n' + \beta_n' + \gamma_n'$ and $\{\gamma_n\}$, $\{\gamma_n'\}$ are summable; it is established that the sequence $\{x_n\}$ (with error member terms) converges strongly to some $x^* \in F$.

1. Introduction

Let K be a nonempty subset of a real normed linear space E. Let T be a self-mapping of K. Then T is said to be *asymptotically nonexpansive* with sequence $\{v_n\} \subset [0,\infty)$ if $\lim_{n\to\infty} v_n = 0$ and

$$\|T^n x - T^n y\| \leq (1+v_n)\|x-y\| \tag{1.1}$$

for all $x, y \in K$ and $n \geq 1$; and is said to be *asymptotically quasi-nonexpansive* with sequence $\{v_n\} \subset [0,\infty)$ if $F(T) := \{x \in K : Tx = x\} \neq \emptyset$, $\lim_{n\to\infty} v_n = 0$ and

$$\|T^n x - x^*\| \leq (1+v_n)\|x-x^*\| \tag{1.2}$$

Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2006, Article ID 18909, Pages 1–10
DOI 10.1155/FPTA/2006/18909

for all $x \in K$, $x^* \in F(T)$ and $n \geq 1$. The mapping T is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$; and is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - x^*\| \leq \|x - x^*\|$ for all $x \in K$ and $x^* \in F(T)$. It is therefore clear that a nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive and an asymptotically nonexpansive mapping with a nonempty fixed point set is asymptotically quasi-nonexpansive. The converses do not hold in general. The mapping T is called *uniformly* (L,γ)*-Lipschitzian* if there exists a constant $L > 0$ and $\gamma > 0$ such that

$$||T^n x - T^n y|| \leq L\|x - y\|^{\gamma} \tag{1.3}$$

for all $x, y \in K$ and $n \geq 1$.

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [3] as an important generalization of the class of nonexpansive maps. They established that if K is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point. In [4], they extended this result to the broader class of uniformly $(L,1)$-Lipschitzian mappings with $L < \lambda$, where λ is sufficiently near 1.

Iterative techniques for approximating fixed points of nonexpansive mappings and their generalizations (asymptotically nonexpansive mappings, etc.) have been studied by a number of authors (see, e.g., [1, 12–15] and references cited therein), using the *Mann iteration method* (see, e.g., [7]) or the *Ishikawa-type iteration method* (see, e.g., [5]).

In 1973, Petryshyn and Williamson [8] established a necessary and sufficient condition for a Mann iterative sequence to converge strongly to a fixed point of a quasi-nonexpansive mapping. Subsequently, Ghosh and Debnath [2] extended Petryshyn and Williamson's results and obtained some necessary and sufficient conditions for an Ishikawa-type iterative sequence to converge to a fixed point of a quasi-nonexpansive mapping. Recently, in [9, 10], Qihou extended the results of Ghosh and Debnath to the more general asymptotically quasi-nonexpansive mappings. More precisely, he obtained the following result.

Theorem 1.1 [9, Theorem 1, page 2]. *Let K be a nonempty closed convex subset of a Banach space E and $T : K \to K$ an asymptotically quasi-nonexpansive mapping with sequence $\{v_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} v_n < \infty$, and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n\big[(1 - \beta_n)x_n + \beta_n T^n x_n\big], \quad n \geq 1, \tag{1.4}$$

converges strongly to some fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, here $d(y,C)$ denotes the distance of y to a set C, that is, $d(y,C) = \inf\{d(y,x) : x \in C\}$.

Furthermore, in [11], Qihou also established sufficient conditions for the strong convergence of the Ishikawa-type iterative sequences with error member for a uniformly (L,γ)-Lipschitzian asymptotically nonexpansive self-mapping of a nonempty *compact convex* subset of a uniformly convex Banach space. In [6], Khan and Takahashi studied the problem of approximating common fixed points of two asymptotically nonexpansive mappings and obtained the following result.

Theorem 1.2 [6, Theorem 2, page 147]. *Let E be a uniformly convex Banach space and K a nonempty compact convex subset of E. Let $S,T : K \to K$ be two asymptotically nonexpansive mappings with sequence $\{k_n - 1\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and $F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n[(1-\beta_n)x_n + \beta_n T^n x_n], \quad n \geq 1 \tag{1.5}$$

converges strongly to some common fixed point of S and T.

The purpose of this paper is to establish:

(i) necessary and sufficient conditions for the convergence of the Ishikawa-type iterative sequences involving two asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings defined on a nonempty *closed convex* subset of a Banach space, and

(ii) a sufficient condition for the convergence of the Ishikawa-type iterative sequences involving two *uniformly continuous* asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings defined on a nonempty *closed convex* subset of a uniformly convex Banach space. Further, we establish, as corollaries, the cases with error member terms. Our results are significant generalizations of the corresponding results of Ghosh and Debnath [2], Petryshyn and Williamson [8], Qihou [9–11], and of Khan and Takahashi [6].

2. Preliminaries

In what follows, we will make use of the following lemmas.

Lemma 2.1 (see, e.g., [13]). *Let E be a uniformly convex Banach space and $\{\alpha_n\}$ a sequence in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in E such that $\limsup_{n\to\infty}\|x_n\| \leq r$, $\limsup_{n\to\infty}\|y_n\| \leq r$, and $\limsup_{n\to\infty}\|\alpha_n x_n + (1-\alpha_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.2 (see, e.g., [16]). *Let $p > 1$ and $R > 1$ be two fixed numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that $\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x-y\|)$ for all $x,y \in B_R(0) = \{x \in E : \|x\| \leq R\}$, and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.*

Lemma 2.3 (see, e.g., [14]). *Let $\{\lambda_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real numbers such that $\lambda_{n+1} \leq \lambda_n + \sigma_n$, $\forall n \geq 1$ and $\sum_{n=1}^{\infty}\sigma_n < \infty$. Then $\lim_{n\to\infty}\lambda_n$ exists. Moreover, if there exists a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_j} \to 0$ as $j \to \infty$, then $\lambda_n \to 0$ as $n \to \infty$.*

3. Main results

Let K be a nonempty closed convex subset of a real Banach space E. Let $S,T : K \to K$ be two asymptotically quasi-nonexpansive mappings. The following iteration scheme is studied:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n[(1-\beta_n)x_n + \beta_n T^n x_n], \tag{3.1}$$

with $x_1 \in K$, $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

Theorem 3.1. *Let E be a real Banach space and K a nonempty closed convex subset of E. Let $S,T : K \to K$ be two asymptotically quasi-nonexpansive mappings with sequences $\{u_n\},\{v_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, and $F = F(S) \cap F(T) := \{x \in K : Sx = Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (3.1). Then*

(1) *$\|x_{n+1} - x^*\| \leq (1+b_n)\|x_n - x^*\|$ for all $n \geq 1$, $x^* \in F$, and for some sequence $\{b_n\}$ of numbers with $\sum_{n=1}^{\infty} b_n < \infty$.*

(2) *There exists a constant $M > 0$ such that $\|x_{n+m} - x^*\| \leq M\|x_n - x^*\|$ for all $n,m \geq 1$ and $x^* \in F$.*

Proof. (1) Let $x^* \in F$ and $y_n = (1-\beta_n)x_n + \beta_n T^n x_n$. Then

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n S^n y_n - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n(1+u_n)\|y_n - x^*\|, \\ \|y_n - x^*\| &= \|(1-\beta_n)x_n + \beta_n T^n x_n - x^*\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n(1+v_n)\|x_n - x^*\| \\ &\leq (1+v_n)\|x_n - x^*\|. \end{aligned} \tag{3.2}$$

Using (3.2), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq [1 + \alpha_n(u_n + v_n) + \alpha_n u_n v_n]\|x_n - x^*\| \\ &\leq [1 + u_n + v_n + u_n v_n]\|x_n - x^*\| \\ &\leq (1+b_n)\|x_n - x^*\|, \end{aligned} \tag{3.3}$$

where $b_n = u_n + v_n + u_n v_n$ with $\sum_{n=1}^{\infty} b_n < \infty$.

(2) Notice that for any $n,m \geq 1$

$$\begin{aligned} \|x_{n+m} - x^*\| &\leq (1+b_{n+m-1})\|x_{n+m-1} - x^*\| \\ &\leq \exp(b_{n+m-1})\|x_{n+m-1} - x^*\| \\ &\leq \cdots \leq \exp\left(\sum_{k=n}^{n+m-1} b_k\right)\|x_n - x^*\|. \end{aligned} \tag{3.4}$$

Let $M = \exp(\sum_{k=1}^{\infty} b_k)$. Then $0 < M < \infty$ and

$$||x_{n+m} - x^*|| \le M||x_n - x^*||. \tag{3.5}$$

□

THEOREM 3.2. *Let E be a real Banach space and K a nonempty closed convex subset of E. Let $S,T : K \to K$ be two asymptotically quasi-nonexpansive mappings (S and T need not be continuous) with sequences $\{u_n\},\{v_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, and $F = F(S) \cap F(T) := \{x \in K : Sx = Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (3.1). Then $\{x_n\}$ converges strongly to some common fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n,F) = 0$.*

Proof. It suffices that we only prove the sufficiency. By Theorem 3.1, we have $\|x_{n+1} - x^*\| \le (1+b_n)\|x_n - x^*\|$ for all $n \ge 1$ and $x^* \in F$. Therefore, $d(x_{n+1},F) \le (1+b_n)d(x_n,F)$. Since $\sum_{n=1}^{\infty} b_n < \infty$ and $\liminf_{n\to\infty} d(x_n,F) = 0$, it follows by Lemma 2.3 that $\lim_{n\to\infty} d(x_n, F) = 0$. Next we will show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n,F) = 0$, given any $\epsilon > 0$, there exists a natural number n_0 such that $d(x_n,F) < \epsilon/4M$ for all $n \ge n_0$. Here $M > 0$ is the constant in Theorem 3.1(2). So we can find $w^* \in F$ such that $\|x_{n_0} - w^*\| \le \epsilon/3M$. Using Theorem 3.1(2), we have for all $n \ge n_0$ and $m \ge 1$ that

$$\begin{aligned} ||x_{n+m} - x_n|| &\le ||x_{n+m} - w^*|| + ||x_n - w^*|| \\ &\le M||x_{n_0} - w^*|| + M||x_{n_0} - w^*|| \\ &= 2M||x_{n_0} - w^*|| < \epsilon. \end{aligned} \tag{3.6}$$

This implies that $\{x_n\}$ is a Cauchy sequence and so is convergent, since X is complete. Let $\lim_{n\to\infty} x_n = y^*$. Then $y^* \in K$. It remains to show that $y^* \in F$. Let $\hat{\epsilon} > 0$ be given. Then there exists a natural number n_1 such that $\|x_n - y^*\| < \hat{\epsilon}/2\max\{2+u_1, 2+v_1\}$ for all $n \ge n_1$. Since $\lim_{n\to\infty} d(x_n,F) = 0$, there exists a natural number $n_2 \ge n_1$ such that for all $n \ge n_2$ we have $d(x_n,F) < \hat{\epsilon}/3\max\{2+u_1,2+v_1\}$ and in particular we have $d(x_{n_2},F) < \hat{\epsilon}/3\max\{2+u_1,2+v_1\}$. Therefore, there exists $z^* \in F$ such that $\|x_{n_2} - z^*\| \le \hat{\epsilon}/2\max\{2+u_1,2+v_1\}$. Consequently we have

$$\begin{aligned} \|Sy^* - y^*\| &= ||Sy^* - z^* + z^* - x_{n_2} + x_{n_2} - y^*|| \\ &\le \|Sy^* - z^*\| + ||z^* - x_{n_2}|| + ||x_{n_2} - y^*|| \\ &\le (1+u_1)\|y^* - z^*\| + ||z^* - x_{n_2}|| + ||x_{n_2} - y^*|| \\ &\le (2+u_1)||y^* - x_{n_2}|| + (2+u_1)||x_{n_2} - z^*|| \\ &< (2+u_1)\frac{\hat{\epsilon}}{2\max\{2+u_1,2+v_1\}} + (2+u_1)\frac{\hat{\epsilon}}{2\max\{2+u_1,2+v_1\}} \\ &\le \hat{\epsilon}. \end{aligned} \tag{3.7}$$

This implies that $y^* \in F(S)$. Similarly, $y^* \in F(T)$. Hence $y^* \in F$. This completes the proof. □

Theorem 3.3. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E. Let $S,T : K \to K$ be two uniformly continuous asymptotically quasi-nonexpansive mappings with sequences $\{u_n\},\{v_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, and $F = F(S) \cap F(T) := \{x \in K : Sx = Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (3.1). Then*

$$\lim_{n\to\infty} ||x_n - T^n x_n|| = 0 = \lim_{n\to\infty} ||x_n - S^n x_n||. \tag{3.8}$$

Proof. Let $x^* \in F$. Then, by Theorem 3.1(1) and Lemma 2.3, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Let $\lim_{n\to\infty} \|x_n - x^*\| = r$. If $r = 0$, then by the continuity of S and T the conclusion follows. Now suppose $r > 0$. We claim

$$\lim_{n\to\infty} ||S^n x_n - x_n|| = 0 = \lim_{n\to\infty} ||T^n x_n - x_n||. \tag{3.9}$$

Set $y_n = (1-\beta_n)x_n + \beta_n T^n x_n$. Since $\{x_n\}$ is bounded, there exists $R > 0$ such that $x_n - x^*, y_n - x^* \in B_R(0)$ for all $n \geq 1$. Using Lemma 2.2, we have that

$$\begin{aligned} ||y_n - x^*||^2 &= ||(1-\beta_n)x_n + \beta_n T^n x_n - x^*||^2 \\ &\leq \beta_n ||T^n x_n - x^*||^2 + (1-\beta_n)||x_n - x^*||^2 - W_2(\beta_n) g(||T^n x_n - x_n||) \\ &\leq \beta_n (1+v_n)^2 ||x_n - x^*||^2 + (1-\beta_n)||x_n - x^*||^2 \leq (1+v_n)^2 ||x_n - x^*||^2. \end{aligned} \tag{3.10}$$

From Lemma 2.2, it follows that

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||(1-\alpha_n)x_n + \alpha_n S^n y_n - x^*||^2 \\ &\leq (1-\alpha_n)||x_n - x^*||^2 + \alpha_n (1+u_n)^2 ||y_n - x^*||^2 \\ &\quad - W_2(\alpha_n) g(||S^n y_n - x_n||) \\ &\leq (1-\alpha_n)||x_n - x^*||^2 + \alpha_n (1+u_n)^2 (1+v_n)^2 ||x_n - x^*||^2 \\ &\quad - W_2(\alpha_n) g(||S^n y_n - x_n||) \\ &\leq ||x_n - x^*||^2 + c_n R^2 - W_2(\alpha_n) g(||S^n y_n - x_n||), \end{aligned} \tag{3.11}$$

where $c_n = (1-\epsilon)[2(u_n + v_n) + (u_n^2 + 4u_n v_n + v_n^2) + 2(u_n v_n^2 + u_n^2 v_n) + v_n^2 v_n^2]$. Observe that $W_2(\alpha_n) \geq \epsilon^2$ and $\sum_{n=1}^{\infty} c_n < \infty$. Now (3.11) implies that

$$\epsilon^2 \sum_{n=1}^{\infty} g(||S^n y_n - x_n||) < ||x_1 - x^*||^2 + R^2 \sum_{n=1}^{\infty} c_n < \infty. \tag{3.12}$$

Therefore, we have $\lim_{n\to\infty} g(\|S^n y_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0, it follows that

$$\lim_{n\to\infty} ||S^n y_n - x_n|| = 0. \tag{3.13}$$

Since S is asymptotically quasi-nonexpansive, we can get that

$$||x_n - x^*|| \leq ||x_n - S^n y_n|| + (1 + u_n)||y_n - x^*||, \tag{3.14}$$

from which we deduce that $r \leq \liminf_{n\to\infty} \| y_n - x^* \|$. On the other hand, we have

$$\begin{aligned} ||y_n - x^*|| &\leq ||(1-\beta_n)x_n + \beta_n T^n x_n - x^*|| \\ &= ||(1-\beta_n)(x_n - x^*) + \beta_n(T^n x_n - x^*)|| \\ &\leq (1-\beta_n)||x_n - x^*|| + (1+v_n)\beta_n||x_n - x^*|| \\ &= ||x_n - x^*|| + \beta_n v_n||x_n - x^*|| \leq (1+v_n)||x_n - x^*||, \end{aligned} \tag{3.15}$$

which implies $\limsup_{n\to\infty} \| y_n - x^* \| \leq r$. Therefore, $\lim_{n\to\infty} \| y_n - x^* \| = r$ and so

$$\lim_{n\to\infty} ||\beta_n(T^n x_n - x^*) + (1-\beta_n)(x_n - x^*)|| = r. \tag{3.16}$$

Since $\limsup_{n\to\infty} \| T^n x_n - x^* \| \leq r$, it follows from Lemma 2.1 that

$$\lim_{n\to\infty} ||T^n x_n - x_n|| = 0. \tag{3.17}$$

Also, we have

$$||S^n x_n - x_n|| \leq ||S^n x_n - S^n y_n|| + ||S^n y_n - x_n||. \tag{3.18}$$

Since S is uniformly continuous and $\|x_n - y_n\| \to 0$ as $n \to \infty$, it follows from (3.18) that $\lim_{n\to\infty} \|S^n x_n - x_n\| = 0$. This completes the proof. □

Theorem 3.4. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E. Let $S, T : K \to K$ be two uniformly continuous asymptotically quasi-nonexpansive mappings with sequences $\{u_n\}, \{v_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, and $F = F(S) \cap F(T) := \{x \in K : Sx = Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (3.1). Assume, in addition, that either T or S is compact. Then $\{x_n\}$ converges strongly to some common fixed point of S and T.*

Proof. By Theorem 3.3, we have

$$\lim_{n\to\infty} ||S^n x_n - x_n|| = 0 = \lim_{n\to\infty} ||T^n x_n - x_n|| \tag{3.19}$$

and also

$$\lim_{n\to\infty} ||x_n - S^n y_n|| = 0. \tag{3.20}$$

If T is compact, then there exists a subsequence $\{T^{n_k} x_{n_k}\}$ of $\{T^n x_n\}$ such that $T^{n_k} x_{n_k} \to x^*$ as $k \to \infty$ for some $x^* \in K$ and so $T^{n_k+1} x_{n_k} \to Tx^*$ as $k \to \infty$. From (3.19), we have $x_{n_k} \to x^*$ as $k \to \infty$. Also $S^{n_k} y_{n_k} \to x^*$ as $k \to \infty$ by (3.20). Since $\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k} - S^{n_k} y_{n_k}\|$, it follows that $x_{n_k+1} \to x^*$ as $k \to \infty$. Again, from (3.20), we have $S^{n_k+1} y_{n_k+1} \to x^*$. Next we show that $x^* \in F$. Notice that

$$\begin{aligned} \|x^* - Tx^*\| \le\, & \|x^* - x_{n_k+1}\| + \|x_{n_k+1} - T^{n_k+1}x_{n_k+1}\| \\ & + \|T^{n_k+1}x_{n_k+1} - T^{n_k+1}x_{n_k}\| + \|T^{n_k+1}x_{n_k} - Tx^*\|. \end{aligned} \tag{3.21}$$

Since T is uniformly continuous, taking the limit as $k \to \infty$ and using (3.19), we obtain that $x^* = Tx^*$ and so $x^* \in F(T)$. Notice also that

$$\begin{aligned} \|x^* - Sx^*\| \le\, & \|x^* - x_{n_k+1}\| + \|x_{n_k+1} - S^{n_k+1}x_{n_k+1}\| \\ & + \|S^{n_k+1}x_{n_k+1} - S^{n_k+1}x_{n_k}\| + \|S^{n_k+1}x_{n_k} - Sx^*\|. \end{aligned} \tag{3.22}$$

Letting $k \to \infty$, we also have that $x^* = Sx^*$ and so $x^* \in F(S)$. Thus $x^* \in F$. Hence, by Lemma 2.3, $x_n \to x^* \in F$ since $\lim_{n\to\infty} \|x_n - x^*\|$ exists. If S is compact, then essentially the same arguments as above give the conclusion. This completes the proof. □

Corollary 3.5. *Let E be a real uniformly convex Banach space and K a nonempty compact convex subset of E. Let $S, T : K \to K$ be two continuous asymptotically quasi-nonexpansive mappings with sequences $\{u_n\}, \{v_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, and $F = F(S) \cap F(T) := \{x \in K : Sx = Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (3.1). Then $\{x_n\}$ converges strongly to some common fixed point of S and T.*

Corollary 3.6. *Let E be a real uniformly convex Banach space and K a nonempty compact convex subset of E. Let $T : K \to K$ be a continuous asymptotically quasi-nonexpansive mapping with sequence $\{v_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} v_n < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n\big[(1-\beta_n)x_n + \beta_n T^n x_n\big] \tag{3.23}$$

with $n \ge 1$. Then $\{x_n\}$ converges strongly to some fixed point of T.

Remarks. (1) Corollary 3.5 extends Theorem 1.2 to the more general class of mappings considered in this paper. It is worth noting that Theorem 1.2 is proved for two asymptotically nonexpansive mappings having the same sequence $\{u_n\}$ (here $u_n = k_n - 1$). However, in our results S and T have separate sequences $\{u_n\}$ and $\{v_n\}$, respectively.

(2) Theorem 3.2 contains as special cases Theorem 1.1, the main result of Qihou [9], together with [9, Corollaries 1 and 2], which are themselves extensions of the results of Ghosh and Debnath [2] and Petryshyn and Williamson [8].

(3) Theorem 3.7 and Corollary 3.8 below are easily provable since the sequences $\{\gamma_n\}$, $\{\gamma'_n\}$ in $[0,1]$ are assumed summable and the sequences $\{z_n\}$, $\{z'_n\}$ in K are bounded. Usually, once a convergence result has been established for an iteration scheme *without errors*, such as (1.4) or (3.1), it is not always difficult to establish the corresponding result for the case *with errors* such as the main theorem of [11] or Theorem 3.7 and Corollary 3.8 below, once $\{\gamma_n\}$, $\{\gamma'_n\}$ are assumed summable and the sequences of error terms are bounded.

Theorem 3.7. *Let E, K, S, T and F be as in Theorem 3.4. Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, *and* $\{\gamma'_n\}$ *be sequences in* $[0,1]$ *with* $\alpha_n+\beta_n+\gamma_n=\alpha'_n+\beta'_n+\gamma'_n=1$ *for all* $n\geq 1$. *From arbitrary* $x_1\in K$, *define the sequence* $\{x_n\}$ *by*

$$\begin{aligned} x_{n+1} &= \alpha_n x_n+\beta_n S^n y_n+\gamma_n z_n,\\ y_n &= \alpha'_n x_n+\beta'_n T^n x_n+\gamma'_n z'_n, \end{aligned} \tag{3.24}$$

where $\{z_n\}$ *and* $\{z'_n\}$ *are bounded sequences in K. Suppose* (i) for some $\epsilon\in(0,1)$, $\beta_n+\gamma_n\in[\epsilon,1-\epsilon]$ and $\beta'_n+\gamma'_n\in[\epsilon,1-\epsilon]$ for all $n\geq 1$, and (ii) $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\gamma'_n<\infty$. Then $\{x_n\}$ converges strongly to some common fixed point of S and T.

Corollary 3.8. *Let E, K, S, T and F be as in Corollary 3.5. Let* $\{x_n\}$ *be defined as in Theorem 3.7 and let the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, *and* $\{\gamma'_n\}$ *satisfy the same conditions as in Theorem 3.7. Then* $\{x_n\}$ *converges strongly to some common fixed point of S and T.*

Remarks. (4) Corollary 3.8 extends the results of Qihou [11] to the more general class of *continuous* asymptotically quasi-nonexpansive mappings on a compact convex subset of a uniformly convex Banach space.

(5) In Theorems 3.3, 3.4 and Corollaries 3.5, 3.6, the prototypes for the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ are $\alpha_n=1/2=\beta_n$ for all $n\geq 1$. In this case $\epsilon=1/4$ satisfies the conditions given therein.

(6) In Theorem 3.7 and Corollary 3.8, the prototypes for the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, and $\{\gamma'_n\}$ are $\alpha_n=3/4-1/(n+1)^2=\alpha'_n$; $\beta_n=1/4=\beta'_n$; $\gamma_n=1/(n+1)^2=\gamma'_n$ for all $n\geq 1$. In this case, $\epsilon=1/4$ satisfies the conditions given therein.

(7) Theorems 3.1, 3.2, 3.3, 3.4, 3.7 and Corollaries 3.5, 3.6, 3.8 remain true for the subclass of asymptotically nonexpansive mappings.

Acknowledgments

The authors thank the referees for their useful comments. This work was done while the authors were visiting the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. They would like to thank the Centre for hospitality and financial support.

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