COMMON FIXED POINT AND INVARIANT APPROXIMATION RESULTS IN CERTAIN METRIZABLE TOPOLOGICAL VECTOR SPACES

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We obtain common fixed point results for generalized I-nonexpansive R-subweakly commuting maps on nonstarshaped domain. As applications, we establish noncommutative versions of various best approximation results for this class of maps in certain metrizable topological vector spaces.

1. Introduction and preliminaries

Let X be a linear space. A p-norm on X is a real-valued function on X with $0 < p \leq 1$, satisfying the following conditions:

(i) $\|x\|_p \geq 0$ and $\|x\|_p = 0 \Leftrightarrow x = 0$,

(ii) $\|\alpha x\|_p = |\alpha|^p \|x\|_p$,

(iii) $\|x+y\|_p \leq \|x\|_p + \|y\|_p$

for all $x, y \in X$ and all scalars α. The pair $(X, \|,\|_p)$ is called a p-normed space. It is a metric linear space with a translation invariant metric d_p defined by $d_p(x,y) = \|x-y\|_p$ for all $x, y \in X$. If $p = 1$, we obtain the concept of the usual normed space. It is well-known that the topology of every Hausdorff locally bounded topological linear space is given by some p-norm, $0 < p \leq 1$ (see [9] and references therein). The spaces l_p and L_p, $0 < p \leq 1$ are p-normed spaces. A p-normed space is not necessarily a locally convex space. Recall that dual space X^* (the dual of X) separates points of X if for each nonzero $x \in X$, there exists $f \in X^*$ such that $f(x) \neq 0$. In this case the weak topology on X is well-defined and is Hausdorff. Notice that if X is not locally convex space, then X^* need not separate the points of X. For example, if $X = L_p[0,1]$, $0 < p < 1$, then $X^* = \{0\}$ ([12, pages 36 and 37]). However, there are some non-locally convex spaces X (such as the p-normed spaces l_p, $0 < p < 1$) whose dual X^* separates the points of X.

Let X be a metric linear space and M a nonempty subset of X. The set $P_M(u) = \{x \in M : d(x,u) = \operatorname{dist}(u,M)\}$ is called the set of best approximants to $u \in X$ out of M, where $\operatorname{dist}(u,M) = \inf\{d(y,u) : y \in M\}$. Let $f : M \to M$ be a mapping. A mapping $T : M \to M$

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is called an f-contraction if there exists $0 \le k < 1$ such that $d(Tx,Ty) \le k\ d(fx,fy)$ for any $x,y \in M$. If $k = 1$, then T is called f-nonexpansive. A mapping $T : M \to M$ is called condensing if for any bounded subset B of M with $\alpha(B) > 0$, $\alpha(T(B)) < \alpha(B)$, where $\alpha(B) = \inf\{r > 0 : B$ can be covered by a finite number of sets of diameter $\le r\}$. A mapping $T : M \to M$ is hemicompact if any sequence $\{x_n\}$ in M has a convergent subsequence whenever $d(x_n, Tx_n) \to 0$ as $n \to \infty$. The set of fixed points of T (resp. f) is denoted by $F(T)$ (resp. $F(f)$). A point $x \in M$ is a common fixed point of f and T if $x = fx = Tx$. The pair $\{f,T\}$ is called (1) commuting if $Tfx = fTx$ for all $x \in M$; (2) R-weakly commuting [16] if for all $x \in M$ there exists $R > 0$ such that $d(fTx,Tfx) \le Rd(fx,Tx)$. If $R = 1$, then the maps are called weakly commuting. The set M is called q-starshaped with $q \in M$ if the segment $[q,x] = \{(1-k)q + kx : 0 \le k \le 1\}$ joining q to x, is contained in M for all $x \in M$. Suppose that M is q-starshaped with $q \in F(f)$ and is both T- and f-invariant. Then T and f are called R-subweakly commuting on M (see [17]) if for all $x \in M$, there exists a real number $R > 0$ such that $d(fTx,Tfx) \le R\,\text{dist}(fx,[q,Tx])$. It is well-known that commuting maps are R-subweakly commuting maps and R-subweakly commuting maps are R-weakly commuting but not conversely in general (see [16, 17]).

A set M is said to have property (N) if [7, 11]

(i) $T : M \to M$,

(ii) $(1-k_n)q + k_nTx \in M$, for some $q \in M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1 and for each $x \in M$.

A mapping f is said to have property (C) on a set M with property (N) if $f((1-k_n)q + k_nTx) = (1-k_n)fq + k_nfTx$ for each $x \in M$ and $n \in N$.

We extend the concept of R-subweakly commuting maps to nonstarshaped domain in the following way (see [7]):

Let f and T be self-maps on the set M having property (N) with $q \in F(f)$. Then f and T are called R-subweakly commuting on M, provided for all $x \in M$, there exists a real number $R > 0$ such that $d(fTx,Tfx) \le Rd(fx,T_nx)$ where $T_nx = (1-k_n)q + k_nTx$, and the sequence $\{k_n\}$ is as in definition of property (N) of M. Each T-invariant q-starshaped set has property (N) but not conversely in general. Each affine map on a q-starshaped set M satisfies condition (C).

Example 1.1 [7]. Consider $X = R^2$ and $M = \{(0,y) : y \in [-1,1]\} \cup \{(1-1/(n+1),0) : n \in N\} \cup \{(1,0)\}$ with the metric induced by the norm $\|(a,b)\| = |a| + |b|$, $(a,b) \in R^2$. Define T on M as follows:

$$T(0,y) = (0,-y), \qquad T\left(1 - \frac{1}{n+1}, 0\right) = \left(0, 1 - \frac{1}{n+1}\right), \qquad T(1,0) = (0,1). \tag{1.1}$$

Clearly, M is not starshaped [11] but M has the property (N) for $q = (0,0)$ and $k_n = 1 - 1/(n+1)$. Define $I(0,y) = I(1-1/(n+1),0) = (0,0)$, $I(1,0) = (1,0)$. Then $\|TIx - ITx\| = 0$ or 1. Thus for all x in M, $\|TIx - ITx\| \le R\|k_nTx - Ix\|$ with each $R \ge 1$ and $q = (0,0) \in F(I)$. Thus I and T are R-subweakly commuting but not commuting on M.

The map $T : M \to X$ is said to be completely continuous if $\{x_n\}$ converges weakly to x implies that $\{Tx_n\}$ converges strongly to Tx.

In 1963, Meinardus [10] employed the Schauder fixed point theorem to prove a result regarding invariant approximation. In 1979, Singh [19] proved the following extension of "Meinardus" result.

Theorem 1.2. *Let T be a nonexpansive operator on a normed space X, M be a T-invariant subset of X and $u \in F(T)$. If $P_M(u)$ is nonempty compact and starshaped, then $P_M(u) \cap F(T) \neq \emptyset$.*

In 1988, Sahab et al. [13] established the following result which contains Theorem 1.2 and many others.

Theorem 1.3. *Let I and T be selfmaps of a normed space X with $u \in F(I) \cap F(T)$, $M \subset X$ with $T(\partial M) \subset M$, and $q \in F(I)$. If $P_M(u)$ is compact and q-starshaped, $I(P_M(u)) = P_M(u)$, I is continuous and linear on $P_M(u)$, I and T are commuting on $P_M(u)$ and T is I-nonexpansive on $P_M(u) \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(I) \neq \emptyset$.*

Let $D = P_M(u) \cap C_M^I(u)$, where $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$.

Theorem 1.4 [1, Theorem 3.2]. *Let I and T be selfmaps of a Banach space X with $u \in F(I) \cap F(T)$, $M \subset X$ with $T(\partial M \cap M) \subset M$. Suppose that D is closed and q-starshaped with $q \in F(I)$, $I(D) = D$, I is linear and continuous on D. If I and T are commuting on D and T is I-nonexpansive on $D \cup \{u\}$ with $\mathrm{cl}(T(D))$ compact, then $P_M(u) \cap F(T) \cap F(I) \neq \emptyset$.*

Recently, by introducing the concept of non-commuting maps to this area, Shahzad [14–18], Hussain and Khan [6] and Hussain et al. [7], further extended and improved the above mentioned results to non-commuting maps.

The aim of this paper is to prove new results extending and subsuming the above mentioned invariant approximation results. To do this, we establish a general common fixed point theorem for R-subweakly commuting generalized I-nonexpansive maps on nonstarshaped domain in the setting of locally bounded topological vector spaces, locally convex topological vector spaces and metric linear spaces. We apply a new theorem to derive some results on the existence of best approximations. Our results unify and extend the results of Al-Thagafi [1], Dotson [3], Guseman and Peters [4], Habiniak [5], Hussain and Khan [6], Hussain et al. [7], Khan and Khan [9], Sahab et al. [13], Shahzad [14–18], and Singh [19].

2. Common fixed point and approximation results

The following common fixed point result is a consequence of Theorem 1 of Berinde [2], which will be needed in the sequel.

Theorem 2.1. *Let M be a closed subset of a metric space (X,d) and T and f be R-weakly commuting self-maps of M such that $T(M) \subset f(M)$. Suppose there exists $k \in (0,1)$ such that*

$$d(Tx,Ty) \leq k \max\{d(fx,fy), d(Tx,fx), d(Ty,fy), d(Tx,fy), d(Ty,fx)\} \tag{2.1}$$

for all $x, y \in M$. If $\mathrm{cl}(T(M))$ is complete and T is continuous, then there is a unique point z in M such that $Tz = fz = z$.

We can prove now the following.

Theorem 2.2. *Let T, I be self-maps on a subset M of a p-normed space X. Assume that M has the property (N) with $q \in F(I)$, I satisfies the condition (C) and $M = I(M)$. Suppose that T and I are R-subweakly commuting and satisfy*

$$\begin{aligned}\|Tx - Ty\|_p \le \max\{&\|Ix - Iy\|_p, \operatorname{dist}(Ix,[Tx,q]), \operatorname{dist}(Iy,[Ty,q]),\\ &\operatorname{dist}(Ix,[Ty,q]), \operatorname{dist}(Iy,[Tx,q])\}\end{aligned} \tag{2.2}$$

for all $x, y \in M$. If T is continuous, then $F(T) \cap F(I) \neq \emptyset$, provided one of the following conditions holds:

(i) *M is closed, $\operatorname{cl}(T(M))$ is compact and I is continuous,*
(ii) *M is bounded and complete, T is hemicompact and I is continuous,*
(iii) *M is bounded and complete, T is condensing and I is continuous,*
(iv) *X is complete with separating dual X^*, M is weakly compact, T is completely continuous and I is continuous.*

Proof. Define T_n by $T_n x = (1 - k_n)q + k_n Tx$ for all $x \in M$ and fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1. Then, each T_n is a well-defined self-mapping of M as M has property (N) and for each n, $T_n(M) \subset M = I(M)$. Now the property (C) of I and the R-subweak commutativity of $\{T, I\}$ imply that

$$\begin{aligned}\|T_n Ix - IT_n x\|_p &= (k_n)^p \|TIx - ITx\|_p \le (k_n)^p R \operatorname{dist}(Ix,[Tx,q])\\ &\le (k_n)^p R \|T_n x - Ix\|_p\end{aligned} \tag{2.3}$$

for all $x \in M$. This implies that the pair $\{T_n, I\}$ is $(k_n)^p R$-weakly commuting for each n. Also by (2.2),

$$\begin{aligned}\|T_n x - T_n y\|_p &= (k_n)^p \|Tx - Ty\|_p\\ &\le (k_n)^p \max\{\|Ix - Iy\|_p, \operatorname{dist}(Ix,[Tx,q]), \operatorname{dist}(Iy,[Ty,q]),\\ &\qquad \operatorname{dist}(Ix,[Ty,q]), \operatorname{dist}(Iy,[Tx,q])\}\\ &\le (k_n)^p \max\Big\{\|Ix - Iy\|_p, \|Ix - T_n x\|_p, \|Iy - T_n y\|_p,\\ &\qquad \|Ix - T_n y\|_p, \|Iy - T_n x\|_p\Big\}\end{aligned} \tag{2.4}$$

for each $x, y \in M$.

(i) Since $\operatorname{cl} T(M)$ is compact, $\operatorname{cl}(T_n(M))$ is also compact. By Theorem 2.1, for each $n \ge 1$, there exists $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. The compactness of $\operatorname{cl} T(M)$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to y$ as $m \to \infty$. Then the definition of $T_m x_m$ implies $x_m \to y$, so by the continuity of T and I we have $y \in F(T) \cap F(I)$. Thus $F(T) \cap F(I) \neq \emptyset$.

(ii) As in (i) there exists $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. And M is bounded, so $x_n - Tx_n = (1-(k_n)^{-1})(x_n - q) \to 0$ as $n \to \infty$ and hence $d_p(x_n, Tx_n) \to 0$ as $n \to \infty$. The hemicompactness of T implies that $\{x_n\}$ has a subsequence $\{x_j\}$ which converges to some $z \in M$. By the continuity of T and I we have $z \in F(T) \cap F(I)$. Thus $F(T) \cap F(I) \neq \emptyset$.

(iii) Every condensing map on a complete bounded subset of a metric space is hemicompact. Hence the result follows from (ii).

(iv) As in (i) there exists $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. Since M is weakly compact, we can find a subsequence $\{x_m\}$ of $\{x_n\}$ in M converging weakly to $y \in M$ as $m \to \infty$. Since T is completely continuous, $Tx_m \to Ty$ as $m \to \infty$. Since $k_n \to 1$, $x_m = T_m x_m = k_m Tx_m + (1-k_m)q \to Ty$ as $m \to \infty$. Thus $Tx_m \to T^2 y$ as $m \to \infty$ and consequently $T^2 y = Ty$ implies that $Tw = w$, where $w = Ty$. Also, since $Ix_m = x_m \to Ty = w$, using the continuity of I and the uniqueness of the limit, we have $Iw = w$. Hence $F(T) \cap F(I) \neq \emptyset$. □

It is clear that each T-invariant q-starshaped set satisfies the property (N) and if I is affine, then I satisfies the condition (C) and $T_n(M) \subset I(M)$ provided $T(M) \subset I(M)$ and $q \in F(I)$.

Corollary 2.3. *Let M be a closed q-starshaped subset of a p-normed space X, and T and I continuous self-maps of M. Suppose that I is affine with $q \in F(I)$, $T(M) \subset I(M)$ and $\operatorname{cl} T(M)$ is compact. If the pair $\{T,I\}$ is R-subweakly commuting and satisfy (2.2) for all $x, y \in M$, then $F(T) \cap F(I) \neq \emptyset$.*

Corollary 2.4 [18, Theorem 2.2]. *Let M be a closed q-starshaped subset of a normed space X, and T and I continuous self-maps of M. Suppose that I is affine with $q \in F(I)$, $T(M) \subset I(M)$ and $\operatorname{cl} T(M)$ is compact. If the pair $\{T,I\}$ is R-subweakly commuting and satisfy, for all $x, y \in M$,*

$$\|Tx - Ty\| \leq \max\Big\{\|Ix - Iy\|, \operatorname{dist}(Ix,[Tx,q]), \operatorname{dist}(Iy,[Ty,q]), \frac{1}{2}[\operatorname{dist}(Ix,[Ty,q]) + \operatorname{dist}(Iy,[Tx,q])]\Big\}, \tag{2.5}$$

then $F(T) \cap F(I) \neq \emptyset$.

The following corollary improves and generalizes [1, Theorem 2.2].

Corollary 2.5. *Let M be a nonempty closed and q-starshaped subset of a p-normed space X and I be continuous self-map of M. Suppose that I is affine with $q \in F(I)$, $T(M) \subset I(M)$ and $\operatorname{cl} T(M)$ is compact. If the pair $\{T,I\}$ is R-subweakly commuting and T is I-nonexpansive on M, then $F(T) \cap F(I) \neq \emptyset$.*

The following corollaries improve and generalize [3, Theorem 1] and [5, Theorem 4].

Corollary 2.6. *Let M be a nonempty closed and q-starshaped subset of a p-normed space X, T and I be continuous self-maps of M. Suppose that I is affine with $q \in F(I)$, $T(M) \subset I(M)$ and $\operatorname{cl} T(M)$ is compact. If the pair $\{T,I\}$ is commuting and T and I satisfy (2.2), then $F(T) \cap F(I) \neq \emptyset$.*

Corollary 2.7 [9, Theorem 2]. *Let M be a nonempty closed and q-starshaped subset of a p-normed space X. If T is nonexpansive self-map of M and* $\operatorname{cl} T(M)$ *is compact, then* $F(T) \neq \varnothing$.

We now derive some approximation results.

Let $D_M^{R,I}(u) = P_M(u) \cap G_M^{R,I}(u)$, where $G_M^{R,I}(u) = \{x \in M : \|Ix - u\|_p \le (2R+1)\operatorname{dist}(u,M)\}$.

The following result extends Theorem 2.3 of Shahzad [16] from the I-nonexpansiveness of T to a more general condition.

Theorem 2.8. *Let M be subset of a p-normed space X and $I,T : X \to X$ be mappings such that $u \in F(T) \cap F(I)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. If $I(D_M^{R,I}(u)) = D_M^{R,I}(u)$ and the pair $\{T,I\}$ is R-subweakly commuting and continuous on $D_M^{R,I}(u)$ and satisfy for all $x \in D_M^{R,I}(u) \cup \{u\}$,*

$$\|Tx - Ty\|_p \le \begin{cases} \|Ix - Iu\|_p & \text{if } y=u, \\ \max\{\|Ix - Iy\|_p, \operatorname{dist}(Ix,[q,Tx]), \operatorname{dist}(Iy,[q,Ty]), & \\ \qquad \operatorname{dist}(Ix,[q,Ty]), \operatorname{dist}(Iy,[q,Tx])\} & \text{if } y \in D_M^{R,I}(u), \end{cases} \tag{2.6}$$

then $D_M^{R,I}(u)$ is T-invariant. Suppose that $D_M^{R,I}(u)$ is closed and $\operatorname{cl}(T(D_M^{R,I}(u)))$ *is compact. If $D_M^{R,I}(u)$ has property (N) with $q \in F(I)$, and I satisfies property (C) on $D_M^{R,I}(u)$, then $P_M(u) \cap F(I) \cap F(T) \neq \varnothing$.*

Proof. Let $x \in D_M^{R,I}(u)$. Then, $x \in P_M(u)$ and hence $\|x - u\|_p = \operatorname{dist}(u,M)$. Note that for any $k \in (0,1)$,

$$\|ku + (1-k)x - u\|_p = (1-k)^p \|x - u\|_p < \operatorname{dist}(u,M). \tag{2.7}$$

It follows that the line segment $\{ku + (1-k)x : 0 < k < 1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, Tx must be in M. Also since $Ix \in P_M(u)$, $u \in F(T) \cap F(I)$ and T and I satisfy (2.6), we have

$$\|Tx - u\|_p = \|Tx - Tu\|_p \le \|Ix - Iu\|_p = \|Ix - u\|_p = \operatorname{dist}(u,M). \tag{2.8}$$

Thus $Tx \in P_M(u)$. From the R-subweak commutativity of the pair $\{T,I\}$ and (2.6), it follows that (see also proof of [16, Theorem 2.3]),

$$\begin{aligned} \|ITx - u\|_p &= \|ITx - TIx + TIx - Tu\|_p \le R\|Tx - Ix\|_p + \|I^2x - Iu\|_p \\ &= R\|Tx - u + u - Ix\|_p + \|I^2x - u\|_p \\ &\le R(\|Tx - u\|_p + \|Ix - u\|_p) + \|I^2x - u\|_p \\ &\le (2R+1)\operatorname{dist}(u,M). \end{aligned} \tag{2.9}$$

Thus $Tx \in G_M^{R,I}(u)$. Consequently, $T(D_M^{R,I}(u)) \subset D_M^{R,I}(u) = I(D_M^{R,I}(u))$. Now Theorem 2.2(i) guarantees that, $P_M(u) \cap F(I) \cap F(T) \neq \varnothing$. □

Remarks 2.9. (1) If $p = 1$ and M is q-starshaped with $q \in F(I)$, $T(M) \subset I(M)$ and I is linear on $D_M^{R,I}(u)$ in Theorem 2.8, we obtain the conclusion of a recent result [18, Theorem 2.5] for the more general inequality (2.6).

(2) Let $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$. Then $I(P_M(u)) \subset P_M(u)$ implies $P_M(u) \subset C_M^I(u) \subset G_M^{R,I}(u)$ and hence $D_M^{R,I}(u) = P_M(u)$. Consequently, Theorem 2.8 remains valid when $D_M^{R,I}(u) = P_M(u)$. Hence we obtain the following result which contains properly Theorems 1.2 and 1.3 and improves and extends Theorem 8 of [5], Theorem 4 in [9], and Theorem 6 in [14, 15].

Corollary 2.10. *Let M be subset of a p-normed space X and let $I, T : X \to X$ be mappings such that $u \in F(T) \cap F(I)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Assume that $I(P_M(u)) = P_M(u)$ and the pair $\{T, I\}$ is R-subweakly commuting and continuous on $P_M(u)$ and satisfy for all $x \in P_M(u) \cup \{u\}$,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|Ix - Iu\|_p & \text{if } y = u, \\ \max\{\|Ix - Iy\|_p, \operatorname{dist}(Ix, [q, Tx]), \operatorname{dist}(Iy, [q, Ty]), & \\ \quad \operatorname{dist}(Ix, [q, Ty]), \operatorname{dist}(Iy, [q, Tx])\} & \text{if } y \in P_M(u). \end{cases} \tag{2.10}$$

Suppose that $P_M(u)$ is closed, q-starshaped with $q \in F(I)$, I is affine and $\operatorname{cl}(T(P_M(u)))$ is compact. Then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Let $D = P_M(u) \cap C_M^I(u)$, where $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$.

The following result contains Theorem 1.4 and many others.

Theorem 2.11. *Let M be subset of a p-normed space X and $I, T : X \to X$ be mappings such that $u \in F(T) \cap F(I)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. If $I(D) = D$ and the pair $\{T, I\}$ is commuting and continuous on D and satisfy for all $x \in D \cup \{u\}$,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|Ix - Iu\|_p & \text{if } y = u, \\ \max\{\|Ix - Iy\|_p, \operatorname{dist}(Ix, [q, Tx]), \operatorname{dist}(Iy, [q, Ty]), & \\ \quad \operatorname{dist}(Ix, [q, Ty]), \operatorname{dist}(Iy, [q, Tx])\} & \text{if } y \in D, \end{cases} \tag{2.11}$$

then D is T-invariant. Suppose that D is closed and $\operatorname{cl}(T(D))$ is compact. If D has property (N) with $q \in F(I)$, and I satisfies property (C) on D, then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Proof. Let $x \in D$, then proceeding as in the proof of Theorem 2.8, we obtain $Tx \in P_M(u)$. Moreover, since T commutes with I on D and T satisfies (2.11),

$$\|ITx - u\|_p = \|TIx - Tu\|_p \leq \|I^2x - Iu\|_p = \|I^2x - u\|_p = \operatorname{dist}(u, M). \tag{2.12}$$

Thus $ITx \in P_M(u)$ and so $Tx \in C_M^I(u)$. Hence $Tx \in D$. Consequently, $T(D) \subset D = I(D)$. Now Theorem 2.2(i) guarantees that $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$. □

In the following result we obtain a non-locally convex space analogue of [6, Theorem 3.3] for nonstarshaped set D.

Theorem 2.12. *Let M be subset of a p-normed space X and $I,T : X \to X$ be mappings such that $u \in F(T) \cap F(I)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. If $I(D) = D$ and the pair $\{T,I\}$ is R-subweakly commuting and continuous on D and, for all $x \in D \cup \{u\}$, satisfies the following inequality,*

$$\|Tx - Ty\|_p \le \begin{cases} \|Ix - Iu\|_p & \text{if } y = u, \\ \max\{\|Ix - Iy\|_p, \operatorname{dist}(Ix,[q,Tx]), \operatorname{dist}(Iy,[q,Ty]), \\ \qquad \operatorname{dist}(Ix,[q,Ty]), \operatorname{dist}(Iy,[q,Tx])\} & \text{if } y \in D, \end{cases} \tag{2.13}$$

and I is nonexpansive on $P_M(u) \cup \{u\}$, then D is T-invariant. Suppose that D is closed, has property (N) with $q \in F(I)$, $\operatorname{cl}(T(D))$ is compact and I satisfies property (C) on D. Then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Proof. Let $x \in D$, then proceeding as in the proof of Theorem 2.8, we obtain $Tx \in P_M(u)$. Moreover, since I is nonexpansive on $P_M(u) \cup \{u\}$ and T satisfies (2.13), we obtain

$$\|ITx - u\|_p \le \|Tx - Tu\|_p \le \|Ix - Iu\|_p = \operatorname{dist}(u,M). \tag{2.14}$$

Thus $ITx \in P_M(u)$ and so $Tx \in C_M^I(u)$. Hence $Tx \in D$. Consequently, $T(D) \subset D = I(D)$. Now Theorem 2.2(i) guarantees that $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$. □

Remark 2.13. Notice that approximation results similar to Theorems 2.8, 2.11, and 2.12 can be obtained, using Theorem 2.2(ii), (iii), and (iv).

Example 2.14. Let $X = R$ and $M = \{0, 1, 1 - 1/(n+1) : n \in N\}$ be endowed with usual metric. Define $T1 = 0$ and $T0 = T(1 - 1/(n+1)) = 1$ for all $n \in N$. Clearly, M is not starshaped but M has the property (N) for $q = 0$ and $k_n = 1 - 1/(n+1)$, $n \in N$. Let $Ix = x$ for all $x \in M$. Now I and T satisfy (2.2) together with all other conditions of Theorem 2.2(i) except the condition that T is continuous. Note that $F(I) \cap F(T) = \emptyset$.

Example 2.15. Let $X = R^2$ be endowed with the p-norm $\|,\|_p$ defined by $\|(a,b)\|_p = |a|^p + |b|^p$, $(a,b) \in R^2$.

(1) Let $M = A \cup B$, where $A = \{(a,b) \in X : 0 \le a \le 1, 0 \le b \le 4\}$ and $B = \{(a,b) \in X : 2 \le a \le 3, 0 \le b \le 4\}$. Define $T : M \to M$ by

$$T(a,b) = \begin{cases} (2,b) & \text{if } (a,b) \in A, \\ (1,b) & \text{if } (a,b) \in B \end{cases} \tag{2.15}$$

and $I(x) = x$, for all $x \in M$. All of the conditions of Theorem 2.2(i) are satisfied except that M has property (N), that is, $(1 - k_n)q + k_n T(M)$ is not contained in M for any choice of $q \in M$ and k_n. Note $F(I) \cap F(T) = \emptyset$.

(2) If $M = \{(a,b) \in X : 0 \leq a < \infty, 0 \leq b \leq 1\}$ and $T : M \to M$ is defined by

$$T(a,b) = (a+1,b), \quad (a,b) \in M. \tag{2.16}$$

Define $I(x) = x$, for all $x \in M$. All of the conditions of Theorem 2.2(i) are satisfied except that M is compact. Note $F(I) \cap F(T) = \emptyset$. Notice that M, being convex and T-invariant, has the property (N) for any choice of q and $\{k_n\}$.

(3) If $M = \{(a,b) \in X : 0 < a < 1, 0 < b < 1\}$ and $T, I : M \to M$ are defined by $T(a,b) = (a/2, b/3)$, and $I(x) = x$ for all $x \in M$. All of the conditions of Theorem 2.2(i) are satisfied except the fact that M is closed. However $F(I) \cap F(T) = \emptyset$.

Example 2.16. Let $X = R$ and $M = [0,1]$ be endowed with the usual metric. Define $T(x) = 0$ and $I(x) = 1 - x$ for each $x \in M$. All of the conditions of Theorem 2.2(i) are satisfied except the condition that the pair $\{I, T\}$ is R-subweakly commuting. Note $F(I) \cap F(T) = \emptyset$.

3. Further results

All results of the paper (Theorem 2.2–Remark 2.13) remain valid in the setup of a metrizable locally convex topological vector space(tvs) (X,d) where d is translation invariant and $d(\alpha x, \alpha y) \leq \alpha d(x,y)$, for each α with $0 < \alpha < 1$ and $x, y \in X$ (recall that d_p is translation invariant and satisfies $d_p(\alpha x, \alpha y) \leq \alpha^p d_p(x,y)$ for any scalar $\alpha \geq 0$). Consequently, Theorem 2.2 (i)-(ii) and Theorem 3.3 (i)-(ii) due to Hussain and Khan [6] and Theorem 3.5 (i)-(ii) & (v), (ix)-(x) and Theorem 4.2 (i)-(ii) & (v), (ix)-(x) due to Hussain et al. [7] are extended to a class of maps satisfying a more general inequality.

From Corollary 2.3, we have the following result which extends [18, Theorem 2.2];

Corollary 3.1. *Let M be a closed q-starshaped subset of a metrizable locally convex space (X,d) where d is translation invariant and $d(\alpha x, \alpha y) \leq \alpha d(x,y)$, for each α with $0 < \alpha < 1$ and $x, y \in X$. Suppose that T and I are continuous self-maps of M, I is affine with $q \in F(I)$, $T(M) \subset I(M)$ and $\operatorname{cl} T(M)$ is compact. If the pair $\{T, I\}$ is R-subweakly commuting and satisfy for all $x, y \in M$,*

$$\begin{aligned} d(Tx,Ty) \leq \max\{&d(Ix,Iy), \operatorname{dist}(Ix,[Tx,q]), \operatorname{dist}(Iy,[Ty,q]), \\ &\operatorname{dist}(Ix,[Ty,q]), \operatorname{dist}(Iy,[Tx,q])\}, \end{aligned} \tag{3.1}$$

then $F(T) \cap F(I) \neq \emptyset$.

We define $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$ and denote by $\mathfrak{I}_0$ the class of closed convex subsets of X containing 0. For $M \in \mathfrak{I}_0$, we define $M_u = \{x \in M : \|x\| \leq 2\|u\|\}$. It is clear that $P_M(u) \subset M_u \in \mathfrak{I}_0$.

Following result includes [1, Theorem 4.1] and [5, Theorem 8] and provides an analogue of [18, Theorem 2.8] in the setting of metrizable locally convex space and contractive condition involved is more general.

Theorem 3.2. *Let X be as in Corollary 3.1, and T be a self-mapping of X with $u \in F(T)$, $M \in \mathfrak{I}_0$ such that $T(M) \subset M$. Suppose that $\operatorname{cl} T(M)$ is compact, T is continuous on M and* *satisfies for all* $x \in M \cup \{u\}$,

$$d(Tx,Ty) \le \begin{cases} d(x,u) & \text{if } y = u, \\ \max\{d(x,y), \operatorname{dist}(x,[0,Tx]), \operatorname{dist}(y,[0,Ty]), & \\ \qquad \operatorname{dist}(x,[0,Ty]), \operatorname{dist}(y,[0,Tx])\} & \text{if } y \in M, \end{cases} \tag{3.2}$$

then

(i) $P_M(u)$ *is nonempty, closed, and convex,*

(ii) $T(P_M(u)) \subset P_M(u)$,

(iii) $P_M(u) \cap F(T) \neq \emptyset$.

Proof. (i) Let $r = \operatorname{dist}(u,M)$. Then there is a minimizing sequence $\{y_n\}$ in M such that $\lim_n d(u,y_n) = r$. As $\operatorname{cl} T(M)$ is compact so $\{Ty_n\}$ has a convergent subsequence $\{Ty_m\}$ with $\lim Ty_m = x_0$ (say) in M. Now by (3.2)

$$r \le d(x_0,u) = \lim d(Ty_m,u) \le \lim d(y_m,u) = \lim d(y_n,u) = r. \tag{3.3}$$

Hence $x_0 \in P_M(u)$. Thus $P_M(u)$ is nonempty closed and convex.

(ii) Let $z \in P_M(u)$. Then $d(Tz,u) = d(Tz,Tu) \le d(z,u) = \operatorname{dist}(u,M)$. This implies that $Tz \in P_M(u)$ and so $T(P_M(u)) \subset P_M(u)$.

(iii) As $\operatorname{cl} T(P_M(u)) \subset \operatorname{cl} T(M)$, so $\operatorname{cl} T(P_M(u))$ is compact. Thus by Corollary 3.1, $P_M(u) \cap F(T) \neq \emptyset$. □

Theorem 3.3. *Let X be as in Theorem 3.2 and I and T be self-mappings of X with $u \in F(I) \cap F(T)$ and $M \in \Im_0$ such that $T(M_u) \subset I(M) \subset M$. Suppose that I is affine and continuous on M, $d(Ix,u) \le d(x,u)$ for all $x \in M$,* $\operatorname{cl} I(M)$ *is compact and I satisfies for all $x,y \in M$,*

$$d(Ix,Iy) \le \max\{d(x,y), \operatorname{dist}(x,[0,Ix]), \operatorname{dist}(y,[0,Iy]), \operatorname{dist}(x,[0,Iy]), \operatorname{dist}(y,[0,Ix])\}. \tag{3.4}$$

If the pair $\{T,I\}$ is R-subweakly commuting and T is continuous on M_u and satisfy for all $x,y \in M_u \cup \{u\}$, and $q \in F(I)$,

$$d(Tx,Ty) \le \begin{cases} d(Ix,Iu) & \text{if } y = u, \\ \max\{d(Ix,Iy), \operatorname{dist}(Ix,[q,Tx]), \operatorname{dist}(Iy,[q,Ty]), & \\ \qquad \operatorname{dist}(Ix,[q,Ty]), \operatorname{dist}(Iy,[q,Tx])\} & \text{if } y \in M_u, \end{cases} \tag{3.5}$$

then

(i) $P_M(u)$ *is nonempty, closed, and convex,*

(ii) $T(P_M(u)) \subset I(P_M(u)) \subset P_M(u)$,

(iii) $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Proof. From Theorem 3.2, we obtain (i). Also we have $I(P_M(u)) \subset P_M(u)$. Let $y \in TP_M(u)$. Since $T(M_u) \subset I(M)$ and $P_M(u) \subset M_u$, there exist $z \in P_M(u)$ and $x \in M$ such that $y = Tz = Ix$. By (3.5), we have

$$d(Ix,u) = d(Tz,Tu) \leq d(Iz,Iu) \leq d(z,u) = \operatorname{dist}(u,M). \tag{3.6}$$

Hence $x \in C_M^I(u) = P_M(u)$ and so (ii) holds.

(iii) Theorem 3.2 guarantees that $P_M(u) \cap F(I) \neq \emptyset$. Thus there exists $q \in P_M(u)$ such that $q \in F(I)$. Hence the conclusion follows from Corollary 3.1. □

Following corollary provides the conclusions of [1, Theorem 4.2(a)] and [17, Theorem 2.3], to the setup of metrizable locally convex space.

Corollary 3.4. *Let X be as above and I, T be self-mappings of X with $u \in F(I) \cap F(T)$ and $M \in \Im_0$ such that $T(M_u) \subset I(M) \subset M$. Suppose that I is affine and continuous on M, $d(Ix,u) \leq d(x,u)$ for all $x \in M$, $\operatorname{cl} I(M)$ is compact and I is nonexpansive on M. If the pair $\{T,I\}$ is R-subweakly commuting on M_u and T is I-nonexpansive on $M_u \cup \{u\}$, then*

(i) *$P_M(u)$ is nonempty, closed and convex,*
(ii) *$T(P_M(u)) \subset I(P_M(u)) \subset P_M(u)$,*
(iii) *$P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.*

Let (X,d) be a metric linear space with translation invariant metric d. We say that the metric d is strictly monotone [4], if $x \neq 0$ and $0 < t < 1$ imply $d(0,tx) < d(0,x)$. Each p-norm generates a translation invariant metric, which is strictly monotone [4].

Following the arguments of Jungck [8, Theorem 3.2] and using Theorem 2.1 instead of Theorem 3.1 of Jungck [8], we obtain,

Theorem 3.5. *Let T and f be continuous self-maps of a compact metric space (X,d) with $T(X) \subset f(X)$. If T and f are R-weakly commuting self-maps of X such that*

$$d(Tx,Ty) < \max\{d(fx,fy), d(Tx,fx), d(Ty,fy), d(Tx,fy), d(Ty,fx)\} \tag{3.7}$$

when right hand side is positive, then there is a unique point z in X such that $Tz = fz = z$.

Using Theorem 3.5, we establish common fixed point generalization of Theorem 1 of Dotson [3], and Theorem 2 of Guseman and Peters [4].

Theorem 3.6. *Let T, I be self-maps on a compact subset M of a metric linear space (X,d) with translation invariant and strictly monotone metric d. Assume that M has the property (N) with $q \in F(I)$, I satisfies the condition (C) and $M = I(M)$. Suppose that T and I are R-subweakly commuting and satisfy*

$$\begin{aligned} d(Tx,Ty) \leq \max\{&d(Ix,Iy), \operatorname{dist}(Ix,[Tx,q]), \operatorname{dist}(Iy,[Ty,q]), \\ &\operatorname{dist}(Ix,[Ty,q]), \operatorname{dist}(Iy,[Tx,q])\} \end{aligned} \tag{3.8}$$

for all $x, y \in M$. If T and I are continuous, then $F(T) \cap F(I) \neq \emptyset$.

Proof. Proof is similar to Theorem 2.2(i), instead of applying Theorem 2.1, we apply Theorem 3.5. □

Similarly, all other results of Section 2 (Corollary 2.3–Theorem 2.12) hold in the setting of metric linear space (X,d) with translation invariant and strictly monotone metric d provided we replace closedness of M and compactness of $\mathrm{cl}\,T(M)$ by compactness of M and using Theorem 3.6 instead of Theorem 2.2(i). Consequently, metric linear space versions of Corollary 2.3–Corollary 2.7 improve and extend Theorem 2 and the Corollary in [4].

A metric space (X,d) is said to be S-space [20], if there exists an x_0 in X such that for every $t \in (0,1)$ there is a d-contractive self-mapping f_t of X for which the inequality $d(f_t(x),x) \leq (1-t)d(x_0,x)$ holds for every x in X. As an application of Theorem 3.5 and [20, Theorem 1], we obtain the following extension of Theorems B, K, Z and C in [2] and Theorem 3 of [20] to generalized nonexpansive mappings.

Theorem 3.7. *Let (X,d) be a compact S-space and $T: X \to X$ satisfies for all $x, y \in X$,*

$$d(Tx,Ty) \leq \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \tag{3.9}$$

Then T has a fixed point.

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