MERGING OF DEGREE AND INDEX THEORY

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The topological approaches to find solutions of a coincidence equation $f_1(x) = f_2(x)$ can roughly be divided into degree and index theories. We describe how these methods can be combined. We are led to a concept of an extended degree theory for function triples which turns out to be natural in many respects. In particular, this approach is useful to find solutions of inclusion problems $F(x) \in \Phi(x)$. As a side result, we obtain a necessary condition for a compact AR to be a topological group.

1. Introduction

There are many situations where one would like to apply topological methods like degree theory for maps which act between *different* Banach spaces. Many such approaches have been studied in literature and they roughly divide into two classes as we explain now.

All these approaches have in common that they actually deal in a sense either with coincidence points or with fixed points of two functions: given two functions $f_1, f_2 : X \to Y$, the *coincidence points* on $A \subseteq X$ are the elements of the set

$$\operatorname{coin}_A(f_1, f_2) := \{x \in A \mid f_1(x) = f_2(x)\} = \{x \in A : x \in f_1^{-1}(f_2(x))\} \tag{1.1}$$

(we do not mention A if $A = X$). The *fixed points* on $B \subseteq Y$ are the elements of the image of $\operatorname{coin}(f_1, f_2)$ in B, that is, they form the set

$$\operatorname{fix}_B(f_1, f_2) := \{y \in B \mid \exists x : y = f_1(x) = f_2(x)\} = \{y \in B : y \in f_2(f_1^{-1}(y))\} \tag{1.2}$$

(we do not mention B if $B = Y$). There is a strong relation of this definition with the usual definition of fixed points of a (single or multivalued) *map*: the coincidence and

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fixed points of a *pair* (f_1, f_2) of functions corresponds to the usual notion of fixed points of the multivalued map $f_1^{-1} \circ f_2$ (with domain and codomain in X) and $f_2 \circ f_1^{-1}$ (with domain and codomain in Y), respectively.

The two classes of approaches can now be roughly described as follows: they define some sort of degree or index which homotopically or homologically counts either

(1) the cardinality of $\operatorname{coin}_\Omega(f_1, f_2)$ where $\Omega \subseteq X$ is open and $\operatorname{coin}_{\partial\Omega}(f_1, f_2) = \emptyset$ or

(2) the cardinality of $\operatorname{fix}_\Omega(f_1, f_2)$ where $\Omega \subseteq Y$ is open and $\operatorname{fix}_{\partial\Omega}(f_1, f_2) = \emptyset$.

To distinguish the two types of theories, we speak in the first case of a degree and in the second case of an index theory. Traditionally, these two cases are not strictly distinguished which is not surprising if one thinks of the classical Leray-Schauder case [44] that $f_1 = \mathrm{id}$, $f_2 = F$ is a compact map, and $X = Y$ is a Banach space: in this case $\operatorname{coin}(f_1, f_2) = \operatorname{fix}(f_1, f_2)$ is the (usual) fixed point set of the map F, that is, the set of zeros of $\mathrm{id} - F$. In general, one has always $\operatorname{coin}(f_1, f_2) \neq \emptyset$ if and only if $\operatorname{fix}(f_1, f_2) \neq \emptyset$, and so in many practical respects both approaches are equally good. Examples of degree theories in the above sense include the following.

(1) The Leray-Schauder degree when $f_1 = \mathrm{id}$ and f_2 is compact. This degree is generalized by

(2) the Mawhin coincidence degree [45] (see also [28, 53]) when f_1 is a Fredholm map of index 0 and f_2 is compact. This degree is generalized by

(3) the Nirenberg degree when f_1 is a Fredholm map of nonnegative index and f_2 is compact (in particular when $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ with $m \leq n$) [29, 48, 49]. This degree can also be extended for certain noncompact functions f_2; see, for example, [26, 27].

(4) A degree theory for nonlinear Fredholm maps of index 0 is currently being developed by Beneveri and Furi; see, for example, [9].

(5) Some important steps have been made in the development of a degree theory for nonlinear Fredholm maps of positive index [68].

(6) The Nussbaum-Sadovskiĭ degree [50, 51, 54] applies for condensing perturbations of the identity. See, for example, [1] for an introduction to that theory.

(7) The Skrypnik degree can be used when $Y = X^*$, f_1 is a uniformly monotone map, and f_2 is compact [57].

(8) The theory of 0-epi maps [25, 37] (which are also called *essential* maps [34]) applies for general maps f_1 and compact f_2. This theory was also extended for certain noncompact f_2 [58, 61].

The latter differs from the other ones in the sense that it is of a purely homotopic nature, that is, one could define it easily in terms of the homotopy class of f_2 (with respect to certain admissible homotopies). In contrast, the other degrees are reduced to the Brouwer degree (or extensions thereof) whose natural topological description is through homology theory. Thus, it should not be too surprising that we have an analogous situation as between homotopy and homology groups: while the theory of 0-epi maps is much simpler to define than the other degrees and can distinguish the homotopy classes "finer," the other degree theories are usually harder to define but easier to calculate, mainly because they satisfy the excision property which we will discuss later. In contrast, the theory of 0-epi maps does not satisfy this excision property. This is analogous to the situation that homology theory satisfies the excision axiom of Eilenberg-Steenrod but homotopy theory does not.

Examples of index theories include many sorts of fixed point theories of multivalued maps: if Φ is a multivalued map, let X be the graph of Φ and let f_1 and f_2 be the projections of X onto its components. Then $\mathrm{fix}(f_1, f_2)$ is precisely the fixed point set of Φ. Note that if X and Y are metric spaces and Φ is upper semicontinuous with compact acyclic (with respect to Čech cohomology with coefficients in a group G) values, then f_1 is a *G-Vietoris map*. By the latter we mean, by definition, that f_1 is continuous, proper (i.e., preimages of compact sets are compact), closed (which in metric spaces follows from properness), surjective and such that the fibres $f_1^{-1}(x)$ are acyclic with respect to Čech cohomology with coefficients in G. If additionally each value $\Phi(x)$ is an R_δ-set (i.e., the intersection of a decreasing sequence of nonempty compact contractible metric spaces), then the fibres $f_1^{-1}(x)$ are even R_δ-sets. Note that by continuity of the Čech cohomology functor R_δ-sets are automatically acyclic for each group G. We call *cell-like* a Vietoris map with R_δ-fibres. For cell-like maps in ANRs the graph of f_1^{-1} can be approximated by single-valued maps. The following corresponding index theories (in our above sense) are known.

(1) For a $\mathbb{Z}$-Vietoris map f_1 and a compact map f_2 one can define a $\mathbb{Z}$-valued index based on the fact that by the Vietoris theorem f_1 induces an isomorphism on the Čech cohomology groups; see [41, 62] (for $\mathbb{Q}$ instead of $\mathbb{Z}$ see also [43] or [12–14]). However, it is unknown whether this index is topologically invariant. For noncompact f_2 this index was studied in [40, 52, 67].

(2) For a $\mathbb{Q}$-Vietoris map f_1 and a compact map f_2 one can define a topologically invariant $\mathbb{Q}$-valued index by chain approximations [22, 55] (see also [32, Sections 50–53]). For noncompact f_2 this index was studied in [24, 65]. The relation with the index for $\mathbb{Z}$-Vietoris maps is unknown, and it is also unknown whether this index actually attains only values in $\mathbb{Z}$ (which is expected).

(3) For a cell-like map f_1 (and also for $\mathbb{Z}$-Vietoris maps when X and the fibres $f_1^{-1}(x)$ have (uniformly) finite covering dimension) and compact f_2, one can define a homotopically invariant $\mathbb{Z}$-valued index by a homotopic approximation argument [8, 41, 42]. For noncompact f_2; see [4, 33]. This index is the same as the previous two indices (i.e., for such particular maps f_1 the previous two index theories coincide and give a $\mathbb{Z}$-valued index); see [41, 62].

(4) The theory of coepi maps [62] is an analogue of the theory of 0-epi maps.

General schemes of how to extend an index defined for compact maps f_2 to rich classes of noncompact maps f_2 were proposed in [5, 6, 60].

It is the purpose of the current paper to sketch how a degree theory and an (homotopic approach to) index theory can be combined so that one can, for example, obtain results about the equation $F(x) \in \Phi(x)$ when Φ is a multivalued acyclic map and F belongs to a class for which a degree theory is known. For the case that F is a linear Fredholm map of nonnegative index, such a unifying theory was proposed in [42] (for the compact case) and in [26, 27] (for the noncompact case). However, our approach works whenever *some* degree theory for F is known. In particular, our theory applies also for the Skrypnik degree and even for the degree theory of 0-epi maps (without the excision property). More precisely, we will define a triple-degree for function triples (F,p,q) of maps $F: X \to Y$, $p: \Gamma \to X$, and $q: \Gamma \to Y$ where X, Y, and Γ are topological spaces. For $A \subseteq X$, we are interested in the set

$$\begin{aligned} \operatorname{COIN}_A(F,p,q) &:= \{x \in A \mid F(x) \in q(p^{-1}(x))\} \\ &= \{x \in A \mid \exists z : x = p(z),\, F(p(z)) = q(z)\}. \end{aligned} \tag{1.3}$$

Our assumptions on F are, roughly speaking, that there exists a degree defined for each pair (F,φ) with compact φ (we make this precise soon). For p we require a certain homotopic property. In the last section of the paper we verify this property only for Vietoris maps or cell-like maps p if X has finite dimension, but we are optimistic that much more general results exist which we leave to future research. Our triple-degree applies for each compact map q with $\operatorname{COIN}_{\partial\Omega}(F,p,q) = \varnothing$.

For $p = \mathrm{id}$ the triple-degree for (F,id,q) reduces to the given degree for the pair (F,q), and for $F = \mathrm{id}$ (with the Leray-Schauder degree) our triple-degree for (id,p,q) reduces essentially to the fixed point index for (p,q).

As remarked above, in this paper we are able to verify the hypothesis of our triple-degree essentially for the case that X has finite (inductive or covering) dimension. In particular, if F is, for example, a nonlinear Fredholm map of degree 0, then our method provides a degree for inclusions of the type

$$F(x) \in \Phi(x) \tag{1.4}$$

when Φ is an upper semicontinuous multivalued map such that $\Phi(x)$ is acyclic for each x and the range of Φ is contained in a finite-dimensional subspace Y_0. Indeed, one can restrict the considerations to the finite-dimensional set $X := F^{-1}(Y_0)$, and let p and q be the projections of the graph of Φ onto the components, then p is a Vietoris map and $\operatorname{COIN}_A(F,p,q)$ is the solution set of (1.4) on $A \subseteq X$. Hence, the degree in this paper is tailored for problem (1.4).

Note that inclusions of type (1.4) with a linear or a nonlinear Fredholm map of index 0 and usually convex values $\Phi(x)$ arise naturally, for example, in the weak formulation of boundary value problems of various partial differential equations $D(u) = f$ under multivalued boundary conditions $\partial u/\partial n \in g(u)$. For example, for the differential operator $D(u) = \Delta u - \lambda u$ the problem reduces to (1.4) with $F = \mathrm{id} - \lambda A$ with a symmetric compact operator A; see [23]. Multivalued boundary conditions for such equations are motivated by physical obstacles for the solution, for example, by unilateral membranes (in typical models arising in biochemistry).

Unfortunately, in the previous example, although the map Φ (and thus q) is usually compact, its range is usually not finite-dimensional. It seems therefore necessary to extend the triple-degree of this paper from the finite-dimensional setting at least to a degree for compact q, similarly as one gets the Leray-Schauder degree from the Brouwer degree. However, since the corresponding arguments are rather lengthy and require a slightly different setting, we postpone these considerations to a separate paper [63]. In fact, it will be even possible to extend the triple-degree even to noncompact maps q under certain hypotheses on measures of noncompactness as will be described in the forthcoming paper [64]. The current paper constitutes the "topological background" for these further extensions: in a sense, the finite-dimensional case is the most complicated one. However, although we verify the hypothesis for the index only in the finite-dimensional case, the definition of the index in this paper is not restricted to finite dimensions; it seems only that currently topological tools (from homotopy theory) are missing to employ this definition directly in natural infinite-dimensional situations (without using the reduction of [63]). Nevertheless, we also sketch some methods which might be directly applied for the infinite-dimensional case. As a side result of that discussion, we obtain a strange property of topological groups (Theorem 4.16) which might be of independent interest.

2. Definition and examples of degree theories

First, let us make precise what we mean by a degree theory.

Throughout this paper, let X and Y be fixed topological spaces, and let G be a commutative semigroup with neutral element 0 (we will later also consider the Boolean addition which forms not a group). Let $\mathbb{O}$ be a family of open subsets $\Omega \subseteq X$, and let $\mathcal{F}$ be a nonempty family of pairs (F,Ω) where $F : \operatorname{Dom} F \to Y$ with $\overline{\Omega} \subseteq \operatorname{Dom} F \subseteq X$. We require that for each $(F,\Omega) \in \mathcal{F}$ and each $\Omega_0 \subseteq \Omega$ with $\Omega_0 \in \mathbb{O}$ also $(F|_{\overline{\Omega_0}},\Omega_0) \in \mathcal{F}$.

The canonical situation one should have in mind is that Y is a Banach space, X is some normed space, $\mathbb{O}$ is the system of all open (or all open and bounded) subsets of X, and the functions F are from a certain class like, for example, compact perturbations of the identity. Note that we do not require that F is continuous (in fact, e.g., demicontinuity suffices for the Skrypnik degree).

We call a map with values in Y *compact* if its range is contained in a compact subset of Y.

Definition 2.1. Let $\mathcal{F}_0$ denote the system of all triples (F,φ,Ω) where $(F,\Omega) \in \mathcal{F}$ and $\varphi : \overline{\Omega} \to Y$ is continuous and compact and $\operatorname{coin}_{\partial\Omega}(F,\varphi) = \emptyset$.

$\mathcal{F}$ *provides a compact degree* $\deg : \mathcal{F}_0 \to G$ if deg has the following two properties.

(1) *Existence.* $\deg(F,\varphi,\Omega) \neq 0$ implies $\operatorname{coin}_{\Omega}(F,\varphi) \neq \emptyset$.

(2) *Homotopy invariance.* If $(F,\Omega) \in \mathcal{F}$ and $h : [0,1] \times \overline{\Omega} \to Y$ is continuous and compact and such that $(F,h(t,\cdot),\Omega) \in \mathcal{F}_0$ for each $t \in [0,1]$, then

$$\deg\big(F,h(0,\cdot),\Omega\big) = \deg\big(F,h(1,\cdot),\Omega\big). \tag{2.1}$$

A compact degree might or might not possess the following properties.

(3) *Restriction.* If $(F,\varphi,\Omega) \in \mathcal{F}_0$ and $\Omega_0 \in \mathbb{O}$ is contained in Ω with $\operatorname{coin}_{\Omega}(F,\varphi) \subseteq \Omega_0$, then

$$\deg(F,\varphi,\Omega) \neq 0 \Longrightarrow \deg\big(F,\varphi,\Omega_0\big) = \deg(F,\varphi,\Omega). \tag{2.2}$$

(4) *Excision.* Under the same assumptions as above,

$$\deg\big(F,\varphi,\Omega_0\big) = \deg(F,\varphi,\Omega). \tag{2.3}$$

(5) *Additivity.* If $(F,\varphi,\Omega) \in \mathcal{F}_0$ and $\Omega_1, \Omega_2 \in \mathbb{O}$ are disjoint with $\Omega = \Omega_1 \cup \Omega_2$, then

$$\deg(F,\varphi,\Omega) = \deg\left(F,\varphi,\Omega_1\right) + \deg\left(F,\varphi,\Omega_2\right). \tag{2.4}$$

Usually in literature, the additivity is combined with the excision property such that (2.4) is required also if $\Omega_1 \cup \Omega_2$ is only a subset of Ω containing $\operatorname{coin}_\Omega(F,\varphi)$. Of course, the excision property implies the restriction property. However, the excision property will in general not be satisfied if the degree is defined "only by homotopic methods," that is, in some straightforward way in terms of the homotopy class of (f_1, f_2). In fact, experience shows that if one wants to obtain a degree theory with the excision property, it seems that in some sense one has to apply (at least implicitly) homology theory for the definition. A deeper reason for this empiric observation is probably that homology groups satisfy the excision axiom of Eilenberg and Steenrod while homotopy groups in general do not. In Theorem 2.4 we give an example of a degree which is instead defined "by homotopic methods" and which fails to satisfy the excision property.

The simplest example of a degree with all the above properties is the Leray-Schauder degree. Recall that we mean by compactness of a map $f : \overline{\Omega} \to Y$ that $f(\overline{\Omega})$ is contained in a compact subset of Y. In particular, a completely continuous map f might fail to be compact if Ω is an *unbounded* subset of Banach space.

Theorem 2.2. *Let $X = Y$ be Banach spaces, let $G := \mathbb{Z}$, and let $\mathbb{O}$ be the system of all open subsets of X. Let $\mathcal{F}$ be the system of all pairs (F,Ω) where $\Omega \in \mathbb{O}$ and $F : \overline{\Omega} \to Y$ is such that* $\mathrm{id} - F$ *is continuous and compact. Then $\mathcal{F}$ provides a degree* $\deg_{\mathrm{LS}}$ *with all of the above properties such that the following holds.*

(8) *Normalization of* id. *If $F - \varphi = \mathrm{id} - c$ with $c \in \Omega$, then*

$$\deg_{\mathrm{LS}}(F,\varphi,\Omega) = 1. \tag{2.5}$$

This degree is uniquely determined by these properties. Moreover, it has then automatically the Borsuk normalization for each $(F,\varphi,\Omega) \in \mathcal{F}_0$.

(10) *Borsuk normalization. If $0 \in \Omega = -\Omega$ and $F - \varphi$ is odd, then*

$$\deg_{\mathrm{LS}}(F,\varphi,\Omega) \ \textit{is odd}. \tag{2.6}$$

Note that the well-known Leray-Schauder degree is concerned with a single map and not with a *pair* of maps. Therefore, some (easy) additional arguments are needed for the proof of Theorem 2.2, in particular for the uniqueness claim.

Proof. To see the uniqueness, consider a fixed pair $(F,\Omega_1) \in \mathcal{F}$, and let $\mathcal{F}'$ denote the system of all $(F|_{\overline{\Omega}}, \Omega) \in \mathcal{F}$ with bounded open $\Omega \subseteq \Omega_1$. Let $\mathcal{F}_0'$ be the system of all pairs $(F - \varphi, \Omega)$ with $(F,\varphi,\Omega) \in \mathcal{F}_0$ and $(F,\Omega) \in \mathcal{F}'$. We define a map $\deg_0 : \mathcal{F}_0' \to \mathbb{Z}$ by

$$\deg_0(F - \varphi, \Omega) = \deg_{\mathrm{LS}}(F,\varphi,\Omega) \tag{2.7}$$

(this is well defined, because we keep F fixed in the definition of $\mathcal{F}'$). Then $\deg_0$ satisfies the basic axioms of the Leray-Schauder degree (with respect to 0), that is, the normalization, homotopy invariance, excision, and additivity, and so $\deg_0$ must be the Leray-Schauder degree; see, for example, [17]. It follows that $\deg_{\mathrm{LS}}$ is uniquely determined on $\mathcal{F}_0'$ and thus also on $\mathcal{F}$. To prove the existence, we let $\deg_0$ denote the Leray-Schauder degree (extended to unbounded sets Ω in the standard way by means of the excision property) and use (2.7) to define $\deg_{\mathrm{LS}}$. The required properties are easily verified, and the Borsuk normalization follows from Borsuk's famous odd map theorem for the Leray-Schauder degree. □

We remark that, at least concerning the existence part, the well-known extensions of the Leray-Schauder degree provide corresponding degrees also if $X = Y$ is a locally convex space or, more general, a so-called admissible space (in the sense of Klee); see, for example, [35]. Moreover, a degree also exists if F is only a condensing perturbation of the identity. In fact, it suffices that $\mathrm{id} - F$ is condensing on the countable subsets; see, for example, [59, 60]. We skip these well-known extensions.

Instead, we give now an example of a degree theory without the excision axiom. To this end, we recall the notion of 0-epi maps in a slightly generalized context.

Definition 2.3. Let X be a topological space, and let Y be a commutative topological group. Let $\Omega \subseteq X$ be open, and let $\varphi : \overline{\Omega} \to Y$. A map $F : \overline{\Omega} \to Y$ is called φ-epi (on Ω) if for each continuous compact perturbation $\psi : \overline{\Omega} \to Y$ with $\psi|_{\partial\Omega} = 0$ the equation $F(x) = \varphi(x) - \psi(x)$ has a solution $x \in \Omega$.

Since Y is a group, the map F is φ-epi if and only if $F - \varphi$ is 0-epi. The concept of 0-epi maps was introduced by M. Furi, M. Martelli, A. Vignoli, and independently by A. Granas. Therefore, we call the corresponding degree $\deg_{\mathrm{FMVG}}$.

Theorem 2.4. *Let $\mathbb{O}$ be the system of all open subsets $\Omega \subseteq X$. Let $G := \{0,1\}$ with the Boolean addition $(1 + 1 := 1)$, and let $\mathcal{F}$ be the system of all pairs (F,Ω) with $F : \overline{\Omega} \to Y$ and $\Omega \in \mathbb{O}$ such that one of the following holds:*

(1) $\overline{\Omega}$ is a T_4-space (e.g., normal), and F is continuous;

(2) $\overline{\Omega}$ is a T_{3a}-space (e.g., completely regular), and F is continuous and proper;

(3) $\overline{\Omega}$ is a T_{3a}-space, F is continuous, and $\partial\Omega$ is compact.

Then $\mathcal{F}$ provides a compact degree $\deg_{\mathrm{FMVG}}$, defined for $(F,\varphi,\Omega) \in \mathcal{F}_0$ by

$$\deg_{\mathrm{FMVG}}(F,\varphi,\Omega) := \begin{cases} 1 & \text{if } F \text{ is } \varphi\text{-epi on } \Omega, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

This degree $\deg_{\mathrm{FMVG}}$ has the restriction and additivity property, but it fails to satisfy the excision property even for the case that $X \subseteq \mathbb{R}$ contains an open interval and $Y := \mathbb{R}$.

Proof. The existence property is an immediate consequence of the definition of φ-epi maps (put $\psi := 0$). To see the homotopy invariance, let $h : [0,1] \times \overline{\Omega} \to Y$ be continuous and compact with $h(t,x) \neq F(x)$ for all $(t,x) \in [0,1] \times \partial\Omega$. We prove for each $t_0, t_1 \in [0,1]$ that the relation $\deg_{\mathrm{FMVG}}(F,h(t_0,\cdot),\Omega) = 1$ implies $\deg_{\mathrm{FMVG}}(F,h(t_1,\cdot),\Omega) = 1$. For a continuous compact perturbation $\psi:\overline{\Omega}\to Y$ with $\psi|_{\partial\Omega}=0$, put

$$C:=\pi_2\big(\{(t,x)\in[0,1]\times\overline{\Omega}: F(x)=h(t,x)-\psi(x)\}\big), \tag{2.9}$$

where π_2 denotes the projection onto the second component. Note that π_2 is closed, because $[0,1]$ is compact (see, e.g., [16, Proposition I.8.2]). Hence, C is closed. Moreover, if F is proper, then C is compact. Since $C\cap\partial\Omega=\varnothing$, we find by Urysohn's lemma (resp., by Lemma 2.5 below) a continuous function $\lambda:\overline{\Omega}\to[0,1]$ with $\lambda|_{\partial\Omega}=t_0$ and $\lambda|_C=t_1$. Then the map

$$\Psi(x):=h(t_0,x)-h(\lambda(x),x)+\psi(x) \tag{2.10}$$

is continuous and compact with $\Psi|_{\partial\Omega}=0$. Hence, if F is $h(t_0,\cdot)$-epi, we conclude that $F(x)=h(t_0,x)-\Psi(x)$ has a solution $x\in\Omega$ which thus satisfies

$$F(x)=h(\lambda(x),x)-\psi(x). \tag{2.11}$$

In particular, $x\in C$ and so $\lambda(x)=t_1$ which proves that $F(x)=h(t_1,x)-\psi(x)$, that is, F is $h(t_1,\cdot)$-epi, as required.

To see the restriction property, let $\deg_{\mathrm{FMVG}}(F,\varphi,\Omega)=1$, and let $\Omega_0\subseteq\Omega$ be open and contain $\mathrm{coin}_\Omega(F,\varphi)$. Given some continuous compact $\psi:\overline{\Omega}_0\to Y$ with $\psi|_{\partial\Omega_0}=0$, extend ψ to a continuous compact map on $\overline{\Omega}$ by putting it 0 outside Ω_0. Then $F(x)=\varphi(x)-\psi(x)$ has a solution $x\in\Omega$, and if $\psi(x)=0$, then $x\in\mathrm{coin}_\Omega(F,\varphi)\subseteq\Omega_0$. Hence, $x\in\Omega_0$, and so $\deg_{\mathrm{FMVG}}(F,\varphi,\Omega_0)=1$.

To prove the additivity, let $\Omega=\Omega_1\cup\Omega_2$ with disjoint open $\Omega_i\subseteq X$ $(i=1,2)$. Note that

$$\partial\Omega=(\overline{\Omega}_1\cup\overline{\Omega}_2)\setminus\Omega=(\overline{\Omega}_1\setminus\Omega)\cup(\overline{\Omega}_2\setminus\Omega)=\partial\Omega_1\cup\partial\Omega_2. \tag{2.12}$$

If $\deg_{\mathrm{FMVG}}(F,\varphi,\Omega_i)=0$ for $i=1$ and $i=2$, then we find continuous compact functions $\psi_i:\overline{\Omega}_i\to Y$ with $\psi_i|_{\partial\Omega_i}=0$ such that $F(x)=\varphi(x)+\psi_i(x)$ has no solution in Ω_i. By (2.12) we can define a continuous compact function by

$$\psi(x):=\begin{cases}\psi_i(x) & \text{if } x\in\Omega_i,\\ 0 & \text{if } x\in\partial\Omega,\end{cases} \tag{2.13}$$

and by construction $F(x)=\varphi(x)+\psi(x)$ has no solution in $\Omega_1\cup\Omega_2=\Omega$, that is, $\deg_{\mathrm{FMVG}}(F,\varphi,\Omega)=0$.

Conversely, if $\deg_{\mathrm{FMVG}}(F,\varphi,\Omega_i)=1$ for $i=1$ or $i=2$, then for each continuous compact function $\psi:\overline{\Omega}\to Y$ with $\psi|_{\partial\Omega}=0$, we have $\psi|_{\partial\Omega_i}=0$ by (2.12), and so $F(x)=\varphi(x)+\psi(x)$ has a solution $x\in\Omega_i\subseteq\Omega$ which implies $\deg_{\mathrm{FMVG}}(F,\varphi,\Omega)=1$.

Let now $X\subseteq\mathbb{R}$ contain an interval $[a,b]$ with $a<b$, and let $Y:=\mathbb{R}$. Let $\Omega:=(a,b)$, fix some $c\in(a,b)$, and put $\Omega_1:=(a,c)$ and $\Omega_2:=(c,b)$. Let $F:\overline{\Omega}\to\mathbb{R}$ be continuous with $\operatorname{sgn}F(a)=-\operatorname{sgn}F(c)=\operatorname{sgn}F(b)\neq0$, and let $\varphi:=0$. Although clearly $\deg_{\mathrm{FMVG}}(F,\varphi,\Omega)=0$, the intermediate value theorem implies that $\deg_{\mathrm{FMVG}}(F,\varphi,\Omega_i)=1$ $(i=1,2)$. In particular, on $\Omega_0:=\Omega_1\cup\Omega_2$, we have $\deg_{\mathrm{FMVG}}(F,\varphi,\Omega_0)=1$ which shows that the excision property fails. □

Lemma 2.5. *If X_0 is a T_{3a}-space, and $A,B \subseteq X_0$ are closed and disjoint and either A or B is compact, then there is a continuous function $f : X_0 \to [0,1]$ with $f|_A = 0$ and $f|_B = 1$.*

Proof. We may assume that B is compact. Then there are finitely many continuous functions $f_1,\dots,f_n : X_0 \to [0,1]$ with $f_i|_A = 0$ $(i = 1,\dots,n)$ such that $f_0(x) := \max\{f_i(x) : i = 1,\dots,n\} > 1/2$ for each $x \in B$. Then $f(x) := \min\{1, 2f_0(x)\}$ is the required function. □

Remarks 2.6. The degree of Theorem 2.4 satisfies

$$\deg_{\mathrm{FMVG}}(F,\varphi,\Omega) = \begin{cases} 1 & \text{if there is a connected component} \\ & \Omega_0 \text{ of } \Omega \text{ with } \deg_{\mathrm{LS}}(F,\varphi,\Omega_0) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.14}$$

where $\deg_{\mathrm{LS}}$ denotes the degree of Theorem 2.2, provided that the latter makes sense (i.e., provided that $X = Y$ is a Banach space and $\mathrm{id} - F$ is compact). In particular, if Ω is connected, then

$$\deg_{\mathrm{FMVG}}(F,\varphi,\Omega) = \left| \operatorname{sgn}\left(\deg_{\mathrm{LS}}(F,\varphi,\Omega)\right) \right|. \tag{2.15}$$

The above claim is a special case of the main result of [30] where it is also shown that this holds even if $\mathrm{id} - F$ is not compact but strictly condensing. Note, however, that the degree of Theorem 2.4 is defined for *all* maps F and also if $X \neq Y$.

We turn now to a *homologic* definition of a degree when $X \neq Y$: the Skrypnik degree. In the following, let X be a real Banach space, and $Y := X^*$ its dual space (with the usual pairing $\langle y,x\rangle := y(x)$). Let $\Omega \subseteq X$ be open and bounded.

Definition 2.7. A function $F : \overline{\Omega} \to X^*$ is called a *Skrypnik map* if the following holds:

(1) $F(\overline{\Omega})$ is bounded;

(2) F is demicontinuous, that is, $\overline{\Omega} \ni x_n \to x$ implies $F(x_n) \rightharpoonup F(x)$;

(3) the relations $\overline{\Omega} \ni x_n \rightharpoonup x$ and

$$\limsup_{n\to\infty} \langle F(x_n), x_n - x\rangle \leq 0 \tag{2.16}$$

imply that $(x_n)_n$ has a convergent subsequence.

A function $H : [0,1] \times \overline{\Omega} \to X^*$ is called a *Skrypnik homotopy* if $H(t,\cdot)$ is a Skrypnik map for each $t \in [0,1]$ and if in addition H is demicontinuous and the relations $\overline{\Omega} \ni x_n \rightharpoonup x$, $t_n \in [0,1]$, and

$$\langle H(t_n,x_n), x_n - x\rangle \longrightarrow 0 \tag{2.17}$$

imply that $(x_n)_n$ has a convergent subsequence.

Remarks 2.8. In the last property of Definition 2.7, we can actually conclude that $x_n \to x$ because each subsequence of x_n contains by assumption a further subsequence which converges to x.

Example 2.9. Let $H : [0,1] \times \overline{\Omega} \to X^*$ be demicontinuous and let $H(\{t\} \times \overline{\Omega})$ be bounded for each $t \in [0,1]$. Suppose that H has an extension $\widetilde{H} : [0,1] \times \overline{\operatorname{conv}}\,\Omega \to X^*$, where $\widetilde{H}(\cdot,x)$ is continuous for each $x \in \overline{\operatorname{conv}}\,\Omega$, such that $\widetilde{H}$ is monotone in the strict sense that there is a nondecreasing function $\beta : [0,\infty) \to [0,\infty)$ with $\beta(r) > 0$ for $r > 0$ such that

$$\langle \widetilde{H}(t,x) - \widetilde{H}(t,y), x - y \rangle \geq \beta(\|x - y\|), \quad x \in \overline{\Omega},\ y \in \overline{\operatorname{conv}}\,\Omega,\ t \in [0,1]. \tag{2.18}$$

Then H is a Skrypnik homotopy. An analogous result holds of course for Skrypnik maps.

Indeed, let $\overline{\Omega} \ni x_n \rightharpoonup x$ and $t_n \in [0,1]$ satisfy

$$\limsup_{n\to\infty} \langle H(t_n,x_n), x_n - x \rangle \leq 0. \tag{2.19}$$

Then $x \in \overline{\operatorname{conv}}\,\Omega$, and $\widetilde{H}([0,1] \times \{x\})$ is compact. A straightforward argument thus implies in view of $x_n \rightharpoonup x$ that $\langle \widetilde{H}(t_n,x), x_n - x \rangle \to 0$, and so we find for each $\varepsilon > 0$ that

$$\begin{aligned} \beta(\|x_n - x\|) &\leq \langle \widetilde{H}(t_n,x_n) - \widetilde{H}(t_n,x), x_n - x \rangle \\ &= \langle H(t_n,x_n), x_n - x \rangle + \langle \widetilde{H}(t_n,x), x_n - x \rangle < \beta(\varepsilon) \end{aligned} \tag{2.20}$$

for all sufficiently large n, which by the monotonicity of β implies $\|x_n - x\| < \varepsilon$. Hence, $x_n \to x$.

Lemma 2.10. (1) *If $F : \overline{\Omega} \to X^*$ is a Skrypnik map and $\varphi : \overline{\Omega} \to X^*$ is compact and demicontinuous, then $F - \varphi$ is also a Skrypnik map.*

(2) *If $H : [0,1] \times \overline{\Omega} \to X^*$ is a Skrypnik homotopy and $h : [0,1] \times \overline{\Omega} \to X^*$ is compact and demicontinuous, then $H - h$ is also a Skrypnik homotopy.*

Proof. Let $\overline{\Omega} \ni x_n \rightharpoonup x$. Since $\varphi(x_n)$ is contained in a compact set, this implies $\langle \varphi(x_n), x_n - x \rangle \to 0$. Hence,

$$\limsup_{n\to\infty} \langle F(x_n) - \varphi(x_n), x_n - x \rangle = \limsup_{n\to\infty} \langle F(x_n), x_n - x \rangle, \tag{2.21}$$

which implies the first claim. The proof of the second claim is similar. □

Since we could not find a reference for the additivity and excision property of the Skrypnik degree in literature, we prove the following result in some detail.

Theorem 2.11. *Let X be a real separable reflexive Banach space, and $\mathbb{O}$ the system of all bounded open subsets of X. Let $\mathcal{F}$ be the set of all pairs (F,Ω) where $\Omega \in \mathbb{O}$ and $F : \overline{\Omega} \to Y = X^*$ is a Skrypnik map. Then $\mathcal{F}$ provides a degree $\deg_{Skrypnik} : \mathcal{F}_0 \to G = \mathbb{Z}$ which satisfies the excision and additivity property. Moreover, for each $(F,\varphi,\Omega) \in \mathcal{F}_0$ the following holds.*

(8) *Invariance under Skrypnik homotopies. If $H : [0,1] \times \overline{\Omega} \to X^*$ is a Skrypnik homotopy and $h : [0,1] \times \overline{\Omega} \to X^*$ is continuous and compact with $\operatorname{coin}_{\partial\Omega}(H(t,\cdot), h(t,\cdot), \Omega) = \emptyset$ for each $t \in [0,1]$, then $(H(t,\cdot), h(t,\cdot), \Omega) \in \mathcal{F}_0$ and*

$$\deg_{Skrypnik}(H(t,\cdot), h(t,\cdot), \Omega) \ \text{ is independent of } t \in [0,1]. \tag{2.22}$$

(9) *Normalization of monotone maps. If* $\langle F(x)-\varphi(x),x\rangle \geq 0$ *for all* $x\in\overline{\Omega}$ *and* $0\in\Omega$*, then*

$$\deg_{Skrypnik}(F,\varphi,\Omega)=1. \tag{2.23}$$

(10) *Borsuk normalization on balls. If* $\Omega=\{x\in X:\|x\|<r\}$ *and* $F-\varphi$ *is odd, then*

$$\deg_{Skrypnik}(F,\varphi,\Omega) \text{ is odd.} \tag{2.24}$$

Proof. Note that Lemma 2.10 implies in particular that for each $(F,\varphi,\Omega)\in\mathcal{F}_0$ the map $F-\varphi$ is a Skrypnik map on $\overline{\Omega}$. Hence, we can define

$$\deg_{\text{Skrypnik}}(F,\varphi,\Omega):=\mathrm{d}_{\text{Skrypnik}}(F-\varphi,\Omega), \tag{2.25}$$

where $\mathrm{d}_{\text{Skrypnik}}$ denotes the Skrypnik degree [57]. The existence, normalization, and Borsuk normalization follow immediately from [57, Theorems 1.3.3, 1.3.4, and 1.3.5], respectively. The invariance under Skrypnik homotopies follows from [57, Theorem 1.3.1] in view of Lemma 2.10. Since for each Skrypnik map F the map $H(t,\cdot):=F$ is a Skrypnik homotopy, the homotopy invariance with respect to the third argument is a special case.

To prove the excision property and the additivity, we have to recall how the Skrypnik degree is constructed: let $e_n\in X$ $(n=1,2,\dots)$ be linearly independent and have a dense span. Let $X_n:=\operatorname{span}\{e_1,\dots,e_n\}$, and for a Skrypnik map $F:\overline{\Omega}\to X^*$ define $\Omega_n:=\Omega\cap X_n$ and $F_n:\overline{\Omega}_n\to X_n$ by

$$F_n(x):=\sum_{k=1}^{n}\langle F(x),e_n\rangle\,e_n. \tag{2.26}$$

If $0\notin F(\partial\Omega)$, then for sufficiently large n the Brouwer degree $\mathrm{d}_{\text{Brouwer}}(F_n,\Omega_n)$ (with respect to 0) is defined and independent of n [57, Theorem 1.1.1]. Moreover, this number is independent of the particular choice of e_n; see [57, Theorem 1.1.2]. The Skrypnik degree $\mathrm{d}_{\text{Skrypnik}}(F,\Omega)$ denotes this common number.

We prove the excision and additivity simultaneously. Let $(F,\varphi,\Omega)\in\mathcal{F}_0$ be given, and let $\Omega^1,\Omega^2\subseteq\Omega^0:=\Omega$ be open and disjoint with $\operatorname{coin}_\Omega(F,\varphi)\subseteq\Omega^1\cup\Omega^2$. We have to prove that

$$\deg_{\text{Skrypnik}}(F,\varphi,\Omega^0)=\deg_{\text{Skrypnik}}(F,\varphi,\Omega^1)+\deg_{\text{Skrypnik}}(F,\varphi,\Omega^2). \tag{2.27}$$

Since the definition of $\deg_{\text{Skrypnik}}$ implies that $(F-\varphi,0,\Omega^i)\in\mathcal{F}_0$ and

$$\deg_{\text{Skrypnik}}(F,\varphi,\Omega^i)=\deg_{\text{Skrypnik}}(F-\varphi,0,\Omega^i)\quad(i=0,1,2), \tag{2.28}$$

it is no loss of generality to assume $\varphi=0$. With X_n as above, let $\Omega_n^i:=\Omega^i\cap X_n$ $(i=0,1,2)$. We have to prove that, for sufficiently large n,

$$\mathrm{d}_{\text{Brouwer}}(F_n,\Omega_n^0)=\mathrm{d}_{\text{Brouwer}}(F_n,\Omega_n^1)+\mathrm{d}_{\text{Brouwer}}(F_n,\Omega_n^2). \tag{2.29}$$

By the excision and additivity of the Brouwer degree, it suffices to show that for all sufficiently large n

$$\operatorname{coin}_{\Omega_n^0}(F_n,0)\subseteq\Omega_n^1\cup\Omega_n^2. \tag{2.30}$$

Assume by contradiction that this is not true, that is, there is a sequence $x_n \in \Omega_n^0$ with $F_n(x_n) = 0$ such that $x_n \notin \Omega_n^1 \cup \Omega_n^2$ for infinitely many n, say for all $n \in \{n_1, n_2, \dots\}$ where $n_j \to \infty$. Since X is reflexive and $x_n \in \Omega$ is bounded, we may assume that $y_j := x_{n_j} \rightharpoonup x$. Then we have for all n that

$$\begin{gathered} y_n \in (X_n \cap \Omega) \setminus (\Omega^1 \cup \Omega^2), \\ \langle F(y_n), e_k \rangle = 0 \quad (k = 1, \dots, n). \end{gathered} \tag{2.31}$$

The latter implies

$$\langle F(y_n), z \rangle = 0 \quad \forall z \in X_n. \tag{2.32}$$

By our choice of e_n, we find a sequence $z_n \in X_n$ with $z_n \to x$. Since $y_n \in X_n$, two applications of (2.32) show that

$$\langle F(y_n), y_n - x \rangle = -\langle F(y_n), x \rangle = \langle F(y_n), z_n - x \rangle. \tag{2.33}$$

Since $F(y_n) \in F(\Omega)$ is bounded and $z_n \to x$, the last term tends to 0 as $n \to \infty$. Since F is a Skrypnik map, it follows that there is a subsequence $y_{n_k} \to x$. In particular, we have $x \in \overline{\Omega}$. The demicontinuity of F and (2.32) imply for each $z \in X_n$ that $0 = \langle F(y_{n_k}), z \rangle \to \langle F(x), z \rangle$, and so $\langle F(x), z \rangle = 0$ $(z \in X_n)$. It follows that $\langle F(x), \cdot \rangle$ vanishes on a dense subspace and thus on X, that is, $F(x) = 0$. This proves that $x \in \operatorname{coin}_{\overline{\Omega}}(F, 0)$. In view of $(F, 0, \Omega) \in \mathcal{F}_0$, we thus have $x \in \operatorname{coin}_{\Omega}(F, 0) \subseteq \Omega^1 \cup \Omega^2$. This is not possible, because $y_n \to x$ and $y_n \notin \Omega^1 \cup \Omega^2$. This contradiction shows (2.30), and the excision and additivity properties are proved. □

The final example we mention concerns the Mawhin coincidence degree [46, 47].

Theorem 2.12. *Let X and Y be Banach spaces, let $G := \mathbb{Z}$, and let $\mathbb{O}$ be the system of all bounded open subsets of X. Let $\mathcal{F}$ be the system of all pairs (F, Ω) where $\Omega \in \mathbb{O}$ and $F : \overline{\Omega} \to Y$ is a linear Fredholm map of index 0. Then $\mathcal{F}$ provides a compact degree $\deg_{Mawhin} : \mathcal{F}_0 \to G$ with all properties of Definition 2.1 such that the following holds for each $(F, \varphi, \Omega) \in \mathcal{F}_0$.*

(6) *Borsuk normalization. If $0 \in \Omega = -\Omega$ and φ is odd, then*

$$\deg_{Mawhin}(F, \varphi, \Omega) \ \textit{is odd.} \tag{2.34}$$

A simple proof of Theorem 2.12 can be found in [53]. The Borsuk normalization follows immediately from the definition of the degree given in [53] and the Borsuk normalization of the Leray-Schauder degree (note that all linear maps are odd).

Theorem 2.12 is the first example where the degree does not only depend on $(F - \varphi, \Omega)$ but on the actual splitting of the map $F - \varphi$ into the two functions. However, the absolute value $|\deg_{\text{Mawhin}}(F, \varphi, \Omega)|$ only depends on $F - \varphi$; see the remarks in [53].

It is possible to generalize the degree of Theorem 2.12 to the case when F is a linear Fredholm map of positive index k. In this case, one lets G be the kth stable homotopy group of the sphere (for $k = 0$, one obtains nothing new: $G \cong \mathbb{Z}$). However, the definitions are rather cumbersome, and a corresponding theorem cannot easily be formulated, because this degree lacks any "natural" normalization property. For this reason, we just refer to [26, 27].

3. Definition of the triple-degree

For a moment, we fix $(F,\Omega) \in \mathcal{F}$. Let Γ be some topological space, and let $p : \Gamma \to X$. We require that for each continuous compact q the multivalued map $q \circ p^{-1}$ is in the following sense homotopic to a single-valued map φ.

Definition 3.1. Let $M \subseteq \overline{\Omega}$. The map p is called an (F,M)-*compact-homotopy-surjection on* $A \subseteq M$ if $p(\Gamma) \supseteq M$ and the following holds.

For each continuous compact map $q : p^{-1}(M) \to Y$ with $\mathrm{COIN}_A(F,p,q) = \varnothing$ there is a continuous map $\varphi : M \to Y$ and a continuous compact map $h : [0,1] \times p^{-1}(M) \to Y$ with $h(0,\cdot) = q$ and $h(1,\cdot) = \varphi \circ p$ (on $p^{-1}(M)$) such that

$$\mathrm{COIN}_A\left(F,p,h(t,\cdot)\right) = \varnothing \quad (0 \le t \le 1), \tag{3.1}$$

that is, such that $F(x) \notin h(t,p^{-1}(x))$ for all $(t,x) \in [0,1] \times A$.

Since $p(\Gamma) \supseteq M = \operatorname{Dom}\varphi$ and $\varphi \circ p = h(1,\cdot)$, the map φ is automatically compact and satisfies $\mathrm{coin}_A(F,\varphi) = \varnothing$.

The technical definition above has a simple interpretation (explaining the name) when we assume that p is continuous. Denote for a moment by $[M \to Y]_{F,A}$ and $[p^{-1}(M), Y]_{F,p,A}$ the families of homotopy classes of continuous compact maps $\varphi : M \to Y$ or $q : p^{-1}(M) \to Y$ satisfying $\mathrm{coin}_A(F,\varphi) = \varnothing$ or $\mathrm{COIN}_A(F,p,q) = \varnothing$, respectively, with respect to the family of all those compact homotopies h such that $\mathrm{coin}_A(F,h(t,\cdot)) = \varnothing$ or $\mathrm{COIN}_A(F,p,h(t,\cdot)) = \varnothing$ for all $t \in [0,1]$. If p is continuous, then it induces canonically (by composition) a map $[M \to Y]_{F,A} \to [p^{-1}(M), Y]_{F,p,A}$. This map is onto if and only if p is an (F,M)-compact-homotopy-surjection. If this map is one-to-one, we call p an (F,M)-compact-homotopy-surjection on A. In other words the following definition holds.

Definition 3.2. Let $M \subseteq \overline{\Omega}$. The map p is called an (F,M)-*compact-homotopy-injection on* $A \subseteq M$ if each two continuous compact maps $\varphi, \widetilde{\varphi} : M \to Y$ with

$$\mathrm{coin}_A(F,\varphi) = \mathrm{coin}_A(F,\widetilde{\varphi}) = \varnothing \tag{3.2}$$

for which a continuous compact map $h : [0,1] \times p^{-1}(M) \to Y$ with (3.1), $h(0,\cdot) = \varphi \circ p$, and $h(1,\cdot) = \widetilde{\varphi} \circ p$ exists, are homotopic in the following sense.

There is a continuous compact map $H : [0,1] \times M \to Y$ with $H(0,\cdot) = \varphi$ and $H(1,\cdot) = \widetilde{\varphi}$ such that

$$\mathrm{coin}_A\left(F,H(t,\cdot)\right) = \varnothing \quad (0 \le t \le 1). \tag{3.3}$$

If p is also an (F,M)-compact-homotopy-surjection on A, we call p an (F,M)-*compact-homotopy-bijection on* A.

Definition 3.3. By $\mathcal{T}$, we denote the class of all triples (F,p,Ω) where $(F,\Omega) \in \mathcal{F}$ and on each closed $A \subseteq \overline{\Omega}$ with $A \supseteq \partial\Omega$, the map p is an $(F,\overline{\Omega})$-compact-homotopy-bijection.

By $\mathcal{T}_0$, we denote the class of all (F,p,q,Ω) where $(F,p,\Omega)\in\mathcal{T}$ and q is a continuous compact function $q:p^{-1}(\overline{\Omega})\to Y$ (q might also be defined on the larger set Γ), and $\mathrm{COIN}_{\partial\Omega}(F,p,q)=\emptyset$.

Now we are in a position to define the triple-degree for the class $\mathcal{T}_0$.

Theorem 3.4. *Let $\mathcal{F}$ provide a compact degree $\deg:\mathcal{F}_0\to G$. Then there is a unique triple-degree* DEG *which associates to each $(F,p,q,\Omega)\in\mathcal{T}_0$ an element of G which depends only on F, Ω, and on the restrictions of p and q to $p^{-1}(\overline{\Omega})$, such that the following properties hold for each $(F,p,q,\Omega)\in\mathcal{T}_0$.*

(1) *Normalization. If $(F,\varphi,\Omega)\in\mathcal{F}_0$ and $\varphi\circ p=q$, then*

$$\mathrm{DEG}(F,p,q,\Omega)=\deg(F,\varphi,\Omega). \tag{3.4}$$

(2) *Existence.* $\mathrm{DEG}(F,p,q,\Omega)\neq 0$ *implies* $\mathrm{COIN}_{\Omega}(F,p,q)\neq\emptyset$.

(3) *Homotopy invariance in the third argument. If h is a continuous compact function $h:[0,1]\times p^{-1}(\overline{\Omega})\to Y$ and $(F,p,h(t,\cdot),\Omega)\in\mathcal{T}_0$ for each $t\in[0,1]$, then*

$$\mathrm{DEG}\big(F,p,h(t,\cdot),\Omega\big)\ \text{ is independent of } t\in[0,1]. \tag{3.5}$$

If deg *satisfies in addition the restriction, excision, respectively, additivity property, then* DEG *automatically satisfies the corresponding properties.*

(4) *Restriction. If $(F,p,q,\Omega)\in\mathcal{T}_0$ and $\Omega_0\in\mathcal{O}$ is contained in Ω with $\mathrm{COIN}_{\Omega}(F,p,q)\subseteq\Omega_0$, then $(F,p,q,\Omega_0)\in\mathcal{T}_0$, and*

$$\mathrm{DEG}(F,p,q,\Omega)\neq 0\Longrightarrow\mathrm{DEG}\big(F,p,q,\Omega_0\big)=\mathrm{DEG}(F,p,q,\Omega). \tag{3.6}$$

(5) *Excision. Under the same assumptions as above on (F,p,q,Ω) and Ω_0, it holds that $(F,p,q,\Omega_0)\in\mathcal{T}_0$, and*

$$\mathrm{DEG}\big(F,p,q,\Omega_0\big)=\mathrm{DEG}(F,p,q,\Omega). \tag{3.7}$$

(6) *Additivity. If $(F,p,q,\Omega)\in\mathcal{T}_0$ and $\Omega_1,\Omega_2\in\mathcal{O}$ are disjoint with $\Omega=\Omega_1\cup\Omega_2$, then $(F,p,q,\Omega_i)\in\mathcal{T}_0$, and*

$$\mathrm{DEG}(F,p,q,\Omega)=\mathrm{DEG}\big(F,p,q,\Omega_1\big)+\mathrm{DEG}\big(F,p,q,\Omega_2\big). \tag{3.8}$$

Proof. To see that $\mathrm{DEG}(F,p,q,\Omega)$ is uniquely determined, we need only the normalization and homotopy invariance. In fact, let φ and h be as in Definition 3.1 with $A:=\partial\Omega$. The homotopy invariance in the third argument implies that we must have

$$\mathrm{DEG}(F,p,q,\Omega)=\mathrm{DEG}\big(F,p,h(1,\cdot),\Omega\big), \tag{3.9}$$

and since $\varphi\circ p=h(1,\cdot)$, the normalization property implies

$$\mathrm{DEG}\big(F,p,h(1,\cdot),\Omega\big)=\deg(F,\varphi,\Omega). \tag{3.10}$$

Hence, the only way to define a degree with the above properties is by putting

$$\mathrm{DEG}(F,p,q,\Omega) := \deg(F,\varphi,\Omega). \tag{3.11}$$

Let us show that this is well defined, that is, independent of the particular choice of φ. Thus, assume that $\widetilde{\varphi}$ is another map as in Definition 3.1 with $A := \partial\Omega$. By Definition 3.2, we find then a continuous compact map $H : [0,1] \times \overline{\Omega} \to Y$ with $H(0,\cdot) = \varphi$ and $H(1,\cdot) = \widetilde{\varphi}$ such that $(F,H(t,\cdot),\Omega) \in \mathcal{F}_0$ for each $t \in [0,1]$. The homotopy invariance of deg thus implies

$$\deg(F,\varphi,\Omega) = \deg(F,\widetilde{\varphi},\Omega), \tag{3.12}$$

and so (3.11) is well defined.

Now we verify the claimed properties of $\mathrm{DEG}(F,p,q,\Omega)$. The normalization property and the homotopy invariance in the third argument are immediate consequences of our definition (for the homotopy invariance just concatenate the given homotopy with the homotopy of our definition). To see the existence property, assume that $\mathrm{COIN}(F,p,q,\Omega) = \emptyset$ and apply Definition 3.1 with $A := \overline{\Omega}$ to find some φ with (3.11) and $\mathrm{coin}_\Omega(F,\varphi) = \emptyset$. Since the latter implies $\deg(F,\varphi,\Omega) = 0$, we must also have $\mathrm{DEG}(F,p,q,\Omega) = 0$ by (3.11).

To prove the restriction, respectively, excision property, apply Definition 3.1 with $A := \overline{\Omega} \setminus \Omega_0$. For the corresponding map φ, we have then simultaneously (3.11),

$$\mathrm{DEG}\left(F,p,q,\Omega_0\right) = \deg\left(F,\varphi,\Omega_0\right), \tag{3.13}$$

and $\mathrm{coin}_\Omega(F,\varphi) \subseteq \Omega_0$. Hence, the restriction, respectively, excision property of DEG follows from the corresponding property of deg. The proof of the additivity is analogous. $\square$

One should think of $\mathrm{DEG}(F,p,q,\Omega)$ as a "count" of the number of coincidences of F and the multivalued map $\Phi := q \circ p^{-1}$. From this point of view, one would like that DEG is homotopy invariant not only in the third argument but also under homotopies Φ such that p varies. We will formulate (and prove) such a property even in the more general situation when also F varies during the homotopy in the following sense.

Definition 3.5. For $\Omega \in \mathcal{O}$, a (not necessarily continuous) map $H : [0,1] \times \overline{\Omega} \to Y$ is called a deg-*admissible homotopy* if $(H(t,\cdot),\Omega) \in \mathcal{F}$ $(0 \le t \le 1)$ and if for each continuous compact map $h : [0,1] \times \overline{\Omega} \to Y$ with $\mathrm{coin}_{[0,1]\times\partial\Omega}(H,h) = \emptyset$ the value

$$\deg\left(H(t,\cdot),h(t,\cdot),\Omega\right) \tag{3.14}$$

is independent of $t \in [0,1]$.

Example 3.6. If $(F,\Omega) \in \mathcal{F}$, then $H(t,\cdot) := F$ $(0 \le t \le 1)$ is a deg-admissible homotopy for every degree deg (by the homotopy invariance of deg).

For some H and Ω as above, consider a topological space Γ and continuous maps $P : \Gamma \to [0,1] \times X$ and $Q : \Gamma \to Y$.

Definition 3.7. Assume that $P(\Gamma) \supseteq [0,1] \times \overline{\Omega}$ and that there are a continuous map $\varphi : [0,1] \times \overline{\Omega} \to Y$ and a continuous compact map $h : [0,1] \times P^{-1}([0,1] \times \overline{\Omega}) \to Y$ with $h(0, z) = Q(z)$ and $h(1,z) = \varphi(P(z))$ for all $z \in P^{-1}([0,1] \times \overline{\Omega})$ and such that

$$\mathrm{COIN}_{[0,1]\times\partial\Omega}\left(H,P,h(t,\cdot)\right) = \emptyset \quad (0 \le t \le 1). \tag{3.15}$$

Put $\Gamma_t := P^{-1}(\{t\} \times \overline{\Omega})$ and let $Q_t : \Gamma_t \to Y$ denote the restriction of Q to Γ_t. Define $P_t : \Gamma_t \to X$ by the relation $P(z) = (t, P_t(z))$ and call the map

$$T(t) := \left(H(t,\cdot), P_t, Q_t, \Omega\right) \quad (t \in [0,1]) \tag{3.16}$$

a *homotopy in* $\mathcal{T}_0$ if $T(t) \in \mathcal{T}_0$ for each $t \in [0,1]$.

Note that, under the above assumptions on h, the map $Q|_{P^{-1}([0,1]\times\overline{\Omega})}$ is automatically continuous, compact and satisfies

$$\mathrm{COIN}_{[0,1]\times\partial\Omega}(H,P,Q) = \emptyset. \tag{3.17}$$

Conversely, if $Q|_{P^{-1}([0,1]\times\overline{\Omega})}$ is continuous and compact and satisfies (3.17), then maps φ and h as required in Definition 3.7 automatically exist if P is an $(H, [0,1] \times \overline{\Omega})$-compact-homotopy-surjection on $[0,1] \times \partial\Omega$.

Theorem 3.8 (invariance under homotopies in $\mathcal{T}_0$). *If $T(t) = (H(t,\cdot), P_t, Q_t, \Omega)$ is a homotopy in $\mathcal{T}_0$, then* $\mathrm{DEG}(T(t))$ *is independent of $t \in [0,1]$.*

Proof. Let Γ_t, h, and φ be as in Definition 3.7, and let h_t denote the restriction of h to Γ_t. Then we have $h_t(0,\cdot) = Q_t$, $h_t(1,\cdot) = \varphi(t, P_t(\cdot))$, and

$$\mathrm{COIN}_{\partial\Omega}\left(H(t,\cdot), P_t, h_t(s,\cdot)\right) = \emptyset \quad (0 \le s \le 1). \tag{3.18}$$

Hence, the same argument as in the beginning of the proof of Theorem 3.4 shows that we must have

$$\mathrm{DEG}\left(H(t,\cdot), P_t, Q_t, \Omega\right) = \deg\left(h(t,\cdot), \varphi(t,\cdot), \Omega\right). \tag{3.19}$$

Since the assumptions imply that φ is compact and

$$\mathrm{coin}_{[0,1]\times\partial\Omega}(H,\varphi) = \emptyset, \tag{3.20}$$

and since H is deg-admissible, it follows that the right-hand side of (3.19) is independent of $t \in [0,1]$. □

The above definition of homotopy in $\mathcal{T}_0$ is only satisfactory if it is additionally allowed to identify certain pairs (p,q). Otherwise, for example, $(F,p,q,\Omega) \in \mathcal{T}_0$ could never be homotopic to itself.

For example, if $H(t,\cdot) = F$, $\Gamma := [0,1] \times \widetilde{\Gamma}$ with some space $\widetilde{\Gamma}$, and $P(t,z) := (t, p(z))$, one is tempted to say that the homotopy $(H(t,\cdot), P_t, Q_t, \Omega)$ corresponds to a homotopy in the third argument $(F, p, h(t,\cdot), \Omega)$ in a canonical way. However, this is only true if we are allowed to identify (P_t, Q_t) in a canonical way with $(p, h(t,\cdot))$ by identifying the space $\Gamma_t = \{t\} \times \widetilde{\Gamma}$ with $\widetilde{\Gamma}$.

We have not proved yet that we get the same triple-degree under such a canonical identification, although this is a natural expectation. However, this claim is not completely obvious, because one cannot expect that the triple-degree depends in general only on F, Ω, and the multivalued map $q \circ p^{-1}$. In general, the triple-degree will also depend on the particular decomposition (p,q) of the last map; see, for example, [42, Example 4.14]. Nevertheless, under a special identification of the space Γ with another space $\widetilde{\Gamma}$ the triple-degree does not change as we will prove. Actually, this is not only true for a special identification but even under any continuous (not necessarily injective) embedding of Γ into a (not necessarily closed) subspace of $\widetilde{\Gamma}$ (or vice versa). More general, the following equivalence relation is appropriate in this context.

Definition 3.9. $(F, p_0, q_0, \Omega) \in \mathcal{T}_0$ is *embedded into* $(F, p_1, q_1, \Omega) \in \mathcal{T}_0$ if there is a continuous map $J : p_0^{-1}(\overline{\Omega}) \to p_1^{-1}(\overline{\Omega})$ such that $p_0(z) = p_1(J(z))$ and $q_0(z) = q_1(J(z))$ for all $z \in p_0^{-1}(\overline{\Omega})$.

$T \in \mathcal{T}_0$ is *equivalent* to $\widetilde{T} \in \mathcal{T}_0$ (in symbols $T \sim \widetilde{T}$) if there are finitely many $T_1, \ldots, T_n \in \mathcal{T}_0$ with $T_1 = T$ and $T_n = \widetilde{T}$ such that for each $i = 1, \ldots, n-1$ either T_i is embedded into T_{i+1} or T_{i+1} is embedded into T_i (or both; the choice may depend on i).

Clearly, each $T \in \mathcal{T}_0$ embeds into itself with $J = \mathrm{id}$, and $\sim$ is by construction an equivalence relation.

Theorem 3.10 (invariance under equivalence). *If* $(F, p, q, \Omega) \sim (F, \widetilde{p}, \widetilde{q}, \Omega)$ *then*

$$\mathrm{DEG}(F, p, q, \Omega) = \mathrm{DEG}(F, \widetilde{p}, \widetilde{q}, \Omega). \tag{3.21}$$

Proof. It suffices to prove that if $(F, p_0, q_0, \Omega) \in \mathcal{T}_0$ is embedded into (F, p_1, q_1, Ω) then they have the same degree. Choose φ and h as in Definition 3.1 with $(F, p, q, \Omega) := (F, p_1, q_1, \Omega)$ and $A := \partial\Omega$. Then, as in the proof of Theorem 3.4, we must have

$$\mathrm{DEG}\,(F, p_1, q_1, \Omega) = \deg(F, \varphi, \Omega). \tag{3.22}$$

Put $H(t,\cdot) := h(t, J(\cdot))$ and note that $H(0,z) = q_0(z)$ and $H(1,z) = \varphi(p_0(z))$ for all $z \in p_0^{-1}(\overline{\Omega})$ and

$$\mathrm{COIN}_{\partial\Omega}\,(F, p_0, H(t,\cdot)) \subseteq \mathrm{COIN}_{\partial\Omega}\,(F, p, h(t,\cdot)) = \emptyset. \tag{3.23}$$

Consequently, H witnesses that φ corresponds also to (F, p_0, q_0, Ω) in the sense of Definition 3.1 which implies by the same argument as before that

$$\mathrm{DEG}\,(F, p_0, q_0, \Omega) = \deg(F, \varphi, \Omega). \tag{3.24}$$

Hence, $\mathrm{DEG}(F, p_0, q_0, \Omega) = \mathrm{DEG}(F, p_1, q_1, \Omega)$, as required. $\square$

Actually, the results in this section hold for a slightly larger class than $\mathcal{T}$, respectively, $\mathcal{T}_0$.

Remarks 3.11. Essentially, all results in this section hold true if we weaken in Definition 3.3 the requirement that p is an $(F,\overline{\Omega})$-compact-homotopy-bijection on *each* closed $A \subseteq \overline{\Omega}$ and require instead only that p is an $(F,\overline{\Omega})$-compact-homotopy-injection on $\partial\Omega$ and an $(F,\overline{\Omega})$-compact-homotopy-surjection on each A with $\partial\Omega \subseteq A \subseteq \overline{\Omega}$.

The only difference for this modified definition of $\mathcal{T}_0$ is that for the restriction, excision, and additivity property of DEG, we must then *require* that $(F,p,q,\Omega_i) \in \mathcal{T}_0$ and cannot conclude this from the fact that $(F,p,q,\Omega) \in \mathcal{T}_0$.

Remarks 3.12. Remark 3.11 remains even correct if we drop also the requirement that p is an $(F,\overline{\Omega})$-compact-homotopy-*injection* on $\partial\Omega$ but propose instead the following weaker assumption.

If φ and $\widetilde{\varphi}$ are two maps as in Definition 3.2 (with $A := \partial\Omega$), then the relation (3.12) holds for the degree deg under consideration.

Remarks 3.13. Also the assumption that p is an $(F,\overline{\Omega})$-compact-homotopy-*surjection* on *each* $A \subseteq \overline{\Omega}$ can be relaxed: except for the restriction, excision, and additivity property of DEG, all results in this section (including Remark 3.12) remain correct if we require in Remark 3.11 only for the two sets $A := \partial\Omega$ and $A := \overline{\Omega}$ that p is an $(F,\overline{\Omega})$-compact-homotopy-surjection on A.

4. Examples of (F,M)-compact-homotopy-bijections

Currently, there are no general methods known which allow to prove that a map is an $(F,\overline{\Omega})$-compact-homotopy-surjection/injection. Some related results which are known do not give *compact* homotopies, and they apply only in the case when F is a constant map. We want to use these results and thus have to get rid of these restrictions. We are first concerned with the compactness question. To this end, we require in addition that the maps of Definition 3.1 assume their values in a set $K \subseteq Y$ (with the intention that we choose later a set K with a compact closure). The following definition is analogous to Definitions 3.1 and 3.2 if $D := M$ and $K := Y$, only with the difference that we do not require any compactness of the maps.

Definition 4.1. Let M be a topological space, and $F : M \to Y$. Let Γ be an arbitrary topological space, let $p : \Gamma \to M$ be continuous, $K \subseteq Y$, and $A, D \subseteq M$.

(1) The map p is an (F,M,D,K)-*homotopy-surjection on* A if $p(\Gamma) \supseteq D$, and if for each continuous map $q : p^{-1}(M) \to Y$ with values in K and $\mathrm{COIN}_A(F,p,q) = \emptyset$ the following holds. There are two continuous maps: $\varphi : D \to K$ and $h : [0,1] \times p^{-1}(D) \to K$ such that $h(0,\cdot) = q$ and $h(1,\cdot) = \varphi \circ p$ (on $p^{-1}(D)$) and

$$\mathrm{COIN}_{A\cap D}(F,p,h(t,\cdot)) = \emptyset \quad (0 \le t \le 1). \tag{4.1}$$

(2) The map p is an (F,M,D,K)-*homotopy-injection on* A if each two continuous maps $\varphi, \widetilde{\varphi} : D \to K$ with

$$\mathrm{coin}_{A\cap D}(F,\varphi) = \mathrm{coin}_{A\cap D}(F,\widetilde{\varphi}) = \emptyset, \tag{4.2}$$

for which a continuous map $h : [0,1] \times p^{-1}(D) \to K$ with (4.1), $h(0,\cdot) = \varphi \circ p$, and $h(1,\cdot) = \widetilde{\varphi} \circ p$ exists, are homotopic in the following sense.

There is a continuous map $H : [0,1] \times \overline{F^{-1}(K)} \to K$ such that $H(0,\cdot) = \varphi$, $H(1,\cdot) = \widetilde{\varphi}$, and

$$\operatorname{coin}_{A \cap F^{-1}(K)}(F, H(t,\cdot)) = \varnothing \quad (0 \le t \le 1). \tag{4.3}$$

(3) The map p is an (F,M,D,K)-*homotopy-bijection on* A, if both of the above properties are satisfied.

Definition 4.2. A subset $K \subseteq Y$ is called an *extensor set* for a space M if, for each closed $A \subseteq M$, each continuous compact map $f : A \to Y$ with $f(A) \subseteq K$ has an extension to a continuous compact map $f : M \to Y$ with $f(M) \subseteq K$.

Note that the definition depends on the enclosing space Y, because we require only that the range of f is contained in a compact subset of Y, not necessarily in a compact subset of K.

Proposition 4.3. *Assume that Y is a retract of a locally convex (Hausdorff) space Z with the property that the closed convex hull of each compact subset of Y is compact. Then each retract of Y is an extensor set for each metric space M.*

Proof. Given a continuous compact $f : A \to K$, choose a convex compact $C \subseteq Z$ with $f(A) \subseteq C$. By Dugundji's extension theorem [18], we can extend f to a continuous map $f : \Gamma \to C$. By composing two retractions, we find a retraction $\rho : Z \to K$. Then $\rho \circ f : \Gamma \to K$ is a continuous extension of $f|_A$ and has its range in the compact set $\rho(C \cap Z_0) \subseteq K$. □

We note that the proof of Dugundji's extension theorem used in Proposition 4.3 makes essential use of the general axiom of choice (if we do not require any separability assumptions).

If Z is not complete, then the assumption on the compact convex hull in Proposition 4.3 is rather restrictive for Y. We can drop this requirement if we consider metrizable retracts, and in this case, we can also assume that M is a T_4-space.

Recall that a metric absolute (neighborhood) retract (denoted by AR resp., by ANR) is a metrizable space which is homeomorphic to a (neighborhood) retract of a locally convex space. Using Dugundji's extension theorem and the Arens-Eells embedding theorem [7], one can show that it is equivalent to require that K is homeomorphic to a (neighborhood) retract of a convex subset of a locally convex space. See [15] or [36] for the general theory of ARs and ANRs.

Proposition 4.4. *Let $K \subseteq Y$ be a closed metric AR. Assume that either K is compact or Y is a metric AR. Then K is an extensor set for each T_4-space M.*

Proof. We assume first that K is a compact AR. Then K is (up to a homeomorphism) a retract of the Hilbert cube H. Assume that ρ is a retraction of H onto K. By a variant of the Tietze extension theorem, each continuous map $f : A \to K \subseteq H$ with a closed set $A \subseteq M$ has an extension to a continuous map $f : M \to H$. Then $\rho \circ f : M \to K$ is the required extension of f.

Assume now that Y is an AR (and that K is closed but not necessarily compact). By the above cited Arens-Eells embedding theorem [7], we may assume that Y is a closed subset of a normed space Z. Let $A \subseteq M$ be closed, and let $f : A \to K$ be continuous and such that $f(A)$ is contained in a compact subset of $Y \subseteq Z$. By [31] we find some compact $C \subseteq Z$ which contains $f(A)$ and is an AR. Since the claim holds for a compact AR, as we have proved above, we can extend f to a continuous function $f_0 : M \to C$. Since Y is an AR, we can extend the identity map on $C \cap Y$ to a continuous map $J : C \to Y$. Let ρ be a retraction of Y onto K. Then $\rho \circ J \circ f_0$ is a continuous extension of f and its values are contained in the compact set $(\rho \circ J)(C) \subseteq K$. □

Since we use a definition of AR spaces which is not based on their extension properties, the proof of Proposition 4.4 makes use of the axiom of choice in the form of Dugundji's extension theorem. However, if K is separable, the countable axiom of choice suffices for the proof of this theorem in the form needed for Proposition 4.4; see [61]. Dugundji's extension theorem is also needed for the following result.

Proposition 4.5. *Let K be a metric AR contained in Y and not necessarily closed. If K is contained in a compact subset of Y, then it is an extensor set for each metrizable space M.*

Proof. The claim is an immediate consequence of the well-known fact that for each AR K, each metric space M, and each closed $A \subseteq M$, each continuous map $f : A \to K$ has an extension to a continuous map $f : X \to K$; see, for example, [32, Theorem 1.9]. □

We have seen that the class of extensor sets is rather large. Now we can formulate the result which explains why Definition 4.1 is useful.

Proposition 4.6. *Let $M \subseteq X$ and $F : M \to Y$. Let $p : \Gamma \to X$ be continuous and $A \subseteq M$. Assume that for each compact set $K_0 \subseteq Y$ there is a set $K \subseteq Y$ with $K_0 \subseteq K$ and a closed set $D \subseteq M$ with $D \supseteq F^{-1}(K) \cap A$ such that the following holds:*

(1) *either K is contained in a compact subset of Y or both of the sets D and $p^{-1}(D)$ are compact;*
(2) *either $D = M$ or $K \subseteq Y$ is an extensor set for M and for $[0,1] \times p^{-1}(M)$;*
(3) *p is an (F,M,D,K)-homotopy-surjection (resp., injection) on A.*

Then p is an (F,M)-compact-homotopy-surjection (resp., injection) on A.

Proof. We prove first the "surjection" part. Let $q : p^{-1}(M) \to Y$ be continuous with values in a compact set K_0. Choose $K \supseteq K_0$ as in the hypothesis. Choose h and φ as in Definition 4.1. Note that h and φ either take their values in the compact set containing K or are defined on a compact set. In both cases, h and φ are compact maps. We can extend φ to a continuous compact map $\varphi : M \to Y$ with values in K. For $z \in p^{-1}(M)$, we extend h by putting $h(0,z) := q(z)$ and $h(1,z) := \varphi(p(z))$. Then h is defined on a closed subset of $[0,1] \times p^{-1}(M)$. Moreover, h is continuous (by the glueing lemma), compact and assumes its values in K. Hence, we can extend h to a continuous compact map $h : [0,1] \times p^{-1}(M) \to Y$ with values in K. Since h assumes its values in K, we have

$$\mathrm{COIN}_A(F,p,h(t,\cdot)) = \mathrm{COIN}_{A\cap D}(F,p,h(t,\cdot)) = \emptyset. \tag{4.4}$$

Hence, p is an (F,M)-compact-homotopy-surjection.

Now we prove the "injection" part of the claim. Thus, let φ, $\widetilde{\varphi}$, and h be as in Definition 3.2, and let K_0 be a compact set which contains all values of these maps. Choose $K \supseteq K_0$ as in the hypothesis. Then the restrictions $\varphi|_D$, $\widetilde{\varphi}|_D$, and $h|_{[0,1]\times p^{-1}(D)}$ satisfy the properties of Definition 4.1, and so we find a continuous map H as in Definition 4.1 for the restrictions $\varphi|_D$ and $\widetilde{\varphi}|_D$. By similar arguments as above, we see that H is compact and that we can extend H to a continuous compact map $H : [0,1] \times M \to Y$ with values in K such that additionally $H(0,\cdot) = \varphi$ and $H(1,\cdot) = \widetilde{\varphi}$. Since H assumes its values in K, we have

$$\operatorname{coin}_A\left(F, H(t,\cdot)\right) = \operatorname{coin}_{A\cap D}\left(F, H(t,\cdot)\right) = \varnothing \quad (0 \le t \le 1). \tag{4.5}$$

Hence, p is an (F,M)-compact-homotopy-injection. □

Roughly speaking, Definition 4.1 allows us to get rid of the compactness requirements for the homotopies in Definitions 3.1 and 3.2. Unfortunately, it is not so easy to "replace" F by a constant map: the latter would allow a direct approach by homotopy theory to Definition 4.1, because one just has to look for appropriate homotopies in the space $K \setminus \{y_0\}$ (where y_0 denotes the constant value of F). The only way that we know to treat nonconstant maps F is to find an appropriate family of homeomorphisms of K as given in the next result.

Proposition 4.7. *Let M be a topological space and let $p : \Gamma \to M$ be continuous. Let $D \subseteq M$, $K_0, K_1 \subseteq Y$, and $F_0, F_1 : M \to Y$. Assume that there is a continuous map $\Phi : M \times K_1 \to K_0$ with the following properties:*

(1) *$\Phi(x,\cdot) : K_1 \to K_0$ is a homeomorphism for each $x \in M$, and $\Psi(x,y) := (\Phi(x,\cdot)^{-1})(y)$ is also continuous on $M \times K_0$;*

(2) *for each $x \in D$ the equality $\Phi(x, F_1(x)) = F_0(x)$ holds.*

Then p is an (F_0,M,D,K_0)-homotopy-surjection/injection on $A \subseteq M$ if and only if p is an (F_1,M,D,K_1)-homotopy-surjection/injection on A.

Proof. Since the assumptions are symmetric with respect to F_0 and F_1, we prove, without loss of generality, the "only if" part.

Assume first that p is an (F_0,M,D,K_0)-homotopy-surjection on A. Let $q_1 : p^{-1}(M) \to K_1$ be continuous with $\operatorname{COIN}_A(F_1,p,q_1) = \varnothing$. Put

$$q_0(z) := \Phi\left(p(z), q_1(z)\right). \tag{4.6}$$

We have $\operatorname{COIN}_A(F_0,p,q_0) = \varnothing$. Indeed, if $x = p(z) \in A$ would satisfy $F_0(x) = q_0(z)$, then $\Phi(x,q_1(z)) = q_0(z) = F_0(x)$, and so $q_1(z) = \Psi(x,F_0(x)) = F_1(x)$ which would contradict the choice of q_1. Since p is an (F_0,M,D,K_0)-homotopy-surjection, we thus find corresponding continuous maps φ_0 and h_0 with $h_0(0,\cdot) = q$, $h_0(1,\cdot) = \varphi_0 \circ p$, and $\operatorname{COIN}_{A\cap D}(F_0, p, h_0(t,\cdot)) = \varnothing$. Put now

$$\begin{aligned} \varphi_1(x) &:= \Psi\left(x, \varphi_0(x)\right), \\ h_1(t,z) &:= \Psi\left(p(z), h_0(t,z)\right). \end{aligned} \tag{4.7}$$

Then $h_1(0,z) = \Psi(p(z),q_0(z)) = q_0(z)$ and $h_1(1,z) = \Psi(p(z),\varphi_0(p(z))) = \varphi(p(z))$. Moreover, $\operatorname{COIN}_{A\cap D}(F_1,p,h_1(t,\cdot)) = \varnothing$. Indeed, if $x = p(z) \in A \cap D$ would satisfy $F_1(x) =$

$h_1(t,z)$, then $\Psi(x,F_0(x)) = F_1(x) = h_1(t,z) = \Psi(x,h_0(t,z))$. The injectivity of $\Psi(x,\cdot)$ implies that $F_0(x) = h_0(t,z)$ which would contradict the choice of h_0. This proves that p is an (F_1,M,D,K_1)-homotopy-surjection on A.

Assume now that p is an (F_0,M,D,K_0)-homotopy-injection on A. Let φ_1 and $\widetilde{\varphi}_1$ be two maps as in Definition 4.1 (with $F := F_1$), that is, there is a continuous map $h_1 = h : [0,1]\times p^{-1}(D) \to K$ with (4.1) such that $h_1(0,\cdot) = \varphi_1 \circ p$ and $h_1(1,\cdot) = \widetilde{\varphi}_1 \circ p$. Define φ_0 and h_0 by the relation (4.7). Define analogously $\widetilde{\varphi}_0$, that is, put

$$\widetilde{\varphi}_0(x) := \Phi(x,\widetilde{\varphi}_1(x)). \tag{4.8}$$

An analogous argument as above shows that φ_0 and $\widetilde{\varphi}_0$ are as in Definition 4.1 (with $F := F_0$). Hence, we find a corresponding homotopy H_0 with $H_0(0,\cdot) = \varphi_0$, $H_0(1,\cdot) = \widetilde{\varphi}_0$, and $\operatorname{coin}_{A\cap D}(F_0,H_0(t,\cdot)) = \varnothing$. Putting

$$H_1(t,x) := \Psi(x,H_0(t,x)), \tag{4.9}$$

we have $H_1(0,x) = \Psi(x,\varphi_0(x)) = \varphi_1(x)$; analogously $H_1(1,\cdot) = \widetilde{\varphi}_1$. Moreover, we have $\operatorname{COIN}_{A\cap D}(F_1,p,H_1(t,\cdot)) = \varnothing$. To see this, note that if $x = p(z) \in A\cap D$ would satisfy $F_1(x) = H_1(t,x)$, then $F_0(x) = \Phi(x,F_1(x)) = \Phi(x,H_1(t,x)) = H_0(t,x)$, a contradiction to the choice of H_0. Hence, p is an (F_1,M,D,K_1)-homotopy-injection on A. $\square$

Definition 4.8. Let $M \subseteq X$, $F : M \to Y$, and $p : \Gamma \to Y$. A family $\mathcal{K}$ of subsets $K \subseteq Y$ is *(F,p)-grouping* if the following holds for each $K \in \mathcal{K}$.

(1) K is homeomorphic to a topological (not necessarily commutative) group.
(2) K is an extensor set for M and for $[0,1]\times p^{-1}(M)$.
(3) $F^{-1}(K)$ is closed.
(4) At least one of the following is true: K is contained in a compact subset of Y, or $F^{-1}(K)$ is compact.
(5) The (restricted) map $F : F^{-1}(K) \to K$ is continuous.
(6) For each compact $K_0 \subseteq Y$ there is some $K \in \mathcal{K}$ with $K_0 \subseteq K$.

The main result of this section now can be summarized as follows.

Theorem 4.9. *Let $M \subseteq X$ be closed and $F : M \to Y$. Let $p : \Gamma \to X$ be continuous and proper with $p(\Gamma) \supseteq M$. Let $\mathcal{K}$ be (F,p)-grouping. Assume that for each $K \in \mathcal{K}$ and each closed $D \subseteq F^{-1}(K)$ there is some $y_0 \in K$ such that one of the following properties is satisfied:*

(1) *p is an (y_0,M,D,K)-homotopy-bijection on D, that is, $p|_D$ induces a bijection between the homotopy classes of $[D,K\setminus\{y_0\}]$ and $[p^{-1}(D),K\setminus\{y_0\}]$;*
(2) *all fibres $p^{-1}(x)$ $(x \in D)$ are R_δ-sets (i.e., $p|_{p^{-1}(D)}$ is a cell-like map), K is homeomorphic to an open subset of a metric ANR, D and $p^{-1}(D)$ are metrizable, and one of the following holds:*
   (a) *the inductive dimension of D is finite, or*
   (b) *for all sufficiently large n the homotopy groups $\pi_n(K\setminus\{y_0\})$ are trivial;*
(3) *all fibres $p^{-1}(x)$ $(x \in D)$ are acyclic with respect to Čech cohomology with coefficients in $\mathbb{Z}$ (i.e., $p|_{p^{-1}(D)}$ is a $\mathbb{Z}$-Vietoris map). In addition, K is homeomorphic to an open subset of a metric ANR, $K\setminus\{y_0\}$ is homotopically n-simple for each $n \geq 1$; D and*

$p^{-1}(D)$ are metrizable. Moreover, $\dim X < \infty$, *where* $\dim$ *denotes the covering dimension, and*

$$\sup_{x\in D}\dim p^{-1}(x) < \infty. \tag{4.10}$$

Then p is an (F,M)-compact-homotopy-bijection on each closed set $A \subseteq M$.

Proof. Let $A \subseteq M$ be closed. We apply Proposition 4.6 with $K_0 \subseteq K \in \mathcal{K}$ and $D := A \cap F^{-1}(K)$. Note that either D (and thus also $p^{-1}(D)$) is compact or K is contained in a compact set. It remains to verify that p is an (F,M,D,K)-homotopy-bijection on A. To see this, we apply Proposition 4.7 with $K_0 := K_1 := K$, $F_1 := F$, and $F_0(x) \equiv y_0 \in K$, where y_0 is as in the hypothesis. We may assume that K itself is a topological group which we write additively (although we do not require commutativity). Then the functions

$$\begin{aligned}\Phi(x,y) &:= c + y - F_1(x),\\ \Psi(x,y) &= -c + y + F_1(x)\end{aligned} \tag{4.11}$$

satisfy the assumptions of Proposition 4.7. It remains to prove that p is an (y_0,M,D,K)-homotopy-bijection on A. Since $D \subseteq A$, the latter means that p is an (y_0,M,D,K)-homotopy-bijection on D. This is true in each case of our hypothesis. Note that if K is homeomorphic to an open subset of a metric ANR, then also $K \setminus \{y_0\}$ has this property, and so $K \setminus \{y_0\}$ actually *is* an ANR.

The case of cell-like maps and finite inductive dimension now follows from [21, Theorems 4.3.1 and 10.4.5] if D (and $p^{-1}(D)$) is compact, respectively, from [20] in the general case. The other case for cell-like maps is contained in [66] if D (and $p^{-1}(D)$) is compact, respectively, in [19] in the general case. See also [42, Theorem 2.19].

Finally, the case of finite covering dimension for $\mathbb{Z}$-Vietoris maps follows from [42, Theorem 2.17] (observe [42, Remark 2.24(ii)]) provided that $\dim p^{-1}(D) < \infty$. The latter holds in view of (4.10) by [56] (see also [42, Remark 4.3(ii)]). □

Corollary 4.10. *Let Y be a locally convex metrizable vector space. Let $M \subseteq X$ be compact and closed and let $F : M \to Y$ be continuous. Let $p : \Gamma \to X$ be continuous with $p(\Gamma) \supseteq M$ and such that $p^{-1}(M)$ is compact and Hausdorff. Suppose that one of the following properties holds.*

(1) *All fibres $p^{-1}(x)$ $(x \in M)$ are R_δ-sets and the inductive dimension of M is finite.*
(2) *All fibres $p^{-1}(x)$ $(x \in M)$ are acyclic with respect to Čech cohomology with coefficients in $\mathbb{Z}$, and (4.10) holds with $D := M$.*

Then p is an (F,M)-compact-homotopy-bijection on each closed $A \subseteq M$.

Proof. Put $\mathcal{K} := \{Y\}$ and $c = 0$ in Theorem 4.9. Note that Y is an AR and thus an extensor set for each T_4-space by Proposition 4.4. Observe that $[0,1] \times p^{-1}(M)$ is a T_4-space, because it is compact and Hausdorff. □

Currently, the only effective way that we know to employ the previous observations to find a large class of (F,M)-compact-homotopy-bijections is by assuming that M or

$F(M)$ are compact (as in Corollary 4.10). Unfortunately, this essentially restricts the applications to the case when X or Y are finite-dimensional spaces. Therefore, one way to proceed for the triple-degree in infinite dimensions is to reduce it to a finite-dimensional situation. This will be done in the forthcoming paper [63]. However, this reduction step is rather difficult, and the author strongly feels that also other homotopic methods can be invented in infinite-dimensional spaces which allow to verify that maps are (F,M)-compact-homotopy-bijections. We intend now to prove one such result.

Definition 4.11. A topological space Y has the *open Hilbert cube property* if for each compact $K_0 \subseteq Y$ there is a set $K \subseteq Y$ with $K_0 \subseteq K$ such that K is homeomorphic to the open Hilbert cube $(0,1)^{\mathbb{N}} \cong \mathbb{R}^{\mathbb{N}}$.

All infinite-dimensional Banach or Fréchet spaces have the open Hilbert cube property as the following result shows.

Proposition 4.12. *A topological space Y has the open Hilbert cube property if and only if for each compact $K \subseteq Y$ there is a set $Y_0 \subseteq Y$ such that $K \subseteq Y_0$ and Y_0 is homeomorphic to an infinite-dimensional Banach or Fréchet space.*

Proof. Since $\mathbb{R}^{\mathbb{N}} \cong s$ is homeomorphic to a Fréchet space, necessity of this condition is trivial. To see that the condition is sufficient, let $K_0 \subseteq Y$ be compact. By hypothesis, we find then an infinite-dimensional Fréchet space $Y_0 \subseteq Y$ which contains K_0. Since K_0 is a compact subset of Y_0 and thus separable, the closed linear span of K_0 is a separable Fréchet space. Hence, without loss of generality, we may assume that Y_0 is separable.

Now one might finish the proof by using the fact that each infinite-dimensional separable Fréchet space Y_0 is homeomorphic to $\mathbb{R}^{\mathbb{N}}$; see [2]. However, the proof of this fact requires to show that $\ell_2 \cong \mathbb{R}^{\mathbb{N}}$ which is highly nontrivial [3]. Therefore, it might be of interest to have a simpler proof of our claim which does not use the fact that Y_0 is homeomorphic to $\mathbb{R}^{\mathbb{N}}$. We use only the more elementary fact that all separable infinite-dimensional Banach spaces are homeomorphic to each other (see [38] or [11, 39]) and so (see [10, Remark 1]) that either Y_0 is homeomorphic to $s \cong \mathbb{R}^{\mathbb{N}}$ (in which case we are done) or that Y_0 is homeomorphic to the space c_0 of all null sequences with the sup-norm.

Hence, without loss of generality, we assume $Y_0 = c_0$. Since $K_0 \subseteq c_0$ is compact, it follows from the well-known compactness criterion in c_0 that

$$\alpha_N := \sup_{n \geq N} \sup_{(\xi_n)_n \in K_0} |\xi_n| \tag{4.12}$$

tends to 0 as $N \to \infty$. Choose some null sequence $\beta_n > \alpha_n$ (put, e.g., $\beta_n := \alpha_n + 1/n$), and let $K := \{(\xi_n)_n \in c_0 : |\xi_n| < \beta_n\}$. Then $K_0 \subseteq K \cong (0,1)^{\mathbb{N}}$. Indeed, the closure of K in c_0 is compact (in the norm topology), and so the restriction of the continuous embedding $c_0 \hookrightarrow s$ to this compact closure is automatically a homeomorphism. In particular, K with the norm topology is homeomorphic to K with the topology of $s \cong \mathbb{R}^{\mathbb{N}}$, that is, K is homeomorphic to $\prod_{n=1}^{\infty}(-\beta_n, \beta_n) \cong (0,1)^{\mathbb{N}}$ (with the product topology). $\square$

Lemma 4.13. *Y has the open Hilbert cube property if and only if for each compact $K_0 \subseteq Y$ there is a subset $K \subseteq Y$ with $K_0 \subseteq K$ such that K is contained in a compact metrizable subset of Y and homeomorphic to $(0,1)^{\mathbb{N}} \cong \mathbb{R}^{\mathbb{N}}$.*

Proof. Sufficiency of the condition is clear. To see the necessity, let Y have the open Hilbert cube property, and let $K_0 \subseteq Y$ be compact. There is a set $K_1 \subseteq Y$ with $K_0 \subseteq K_1$ and a homeomorphism f of K_1 onto $H := (0,1)^{\mathbb{N}}$. Let π_n denote the projection of H onto the nth component. Then $(\pi_n \circ f)(K_0)$ is a compact subset of $(0,1)$ and thus contained in an interval $[a_n, b_n]$ with $0 < a_n < b_n < 1$. Put $H_0 := \prod_{n=1}^{\infty}(a_n, b_n)$, and $K := f^{-1}(H_0)$. Then K contains K_0 and is homeomorphic to $H_0 \cong (0,1)^{\mathbb{N}}$. Moreover, since $H_1 := \prod_{n=1}^{\infty}[a_n, b_n]$ is compact and Hausdorff, it follows that $f^{-1}(H_1) \subseteq Y$ is compact and metrizable. Note now that $K \subseteq f^{-1}(H_1)$. □

Theorem 4.14. *Suppose that Y has the open Hilbert cube property. Let $M \subseteq X$ be closed and metrizable. Let $F : M \to Y$ be such that, for each compact metrizable $K \subseteq Y$, the set $F^{-1}(K)$ is closed and the restriction of F to this set is continuous.*

If $F(M)$ is contained in a compact subset of Y, then each continuous map $p : \Gamma \to X$ with metrizable $p^{-1}(M)$ is an (F,M)-compact-homotopy-bijection on each closed $A \subseteq M$.

We point out that the continuity assumption on F is already satisfied if F (or at least its restriction to $F^{-1}(K)$) has a closed graph.

Proof. Let $\mathcal{K}$ be the family of all sets $K \subseteq Y$ containing $F(M)$ and contained in a compact metrizable set and which are homeomorphic to $(0,1)^{\mathbb{N}}$. Then each $K \in \mathcal{K}$ is homeomorphic to an open subset of the Hilbert cube (which is an ANR, even an AR) and thus an ANR. Moreover, since K is contractible, K is even an AR. In particular, K is an extensor set for each metric space by Proposition 4.5. Identifying K with $\mathbb{R}^{\mathbb{N}}$ via a homeomorphism, we see immediately that K becomes a topological (commutative) group. By Lemma 4.13, each compact $K_0 \subseteq Y$ is contained in an element of $\mathcal{K}$. Hence, $\mathcal{K}$ is (F,p)-grouping. The claim thus follows from Theorem 4.9, because $K \setminus \{0\}$ is contractible, and so each homotopy class in $[D, K \setminus \{0\}]$ (for each metric space D) is trivial. □

The compactness assumption for F in Theorem 4.14 might also be replaced by other conditions. Unfortunately, none of the results obtained in this way can be applied to functions F for which a nontrivial degree exists. In fact, since $[D, K \setminus \{0\}]$ in the above proof is always trivial, an inspection of the proof of Theorem 4.9 shows that the map φ in Definition 3.1 can actually be chosen independent of q, that is, $\mathrm{DEG}(F,p,q,\Omega)$ is actually independent of q. Nevertheless, perhaps a modification of this approach might apply also in infinite dimensions.

Problem 4.15. Does the open Hilbert cube (or, equivalently, some infinite-dimensional separable Banach space or, equivalently, each infinite-dimensional Fréchet space) Y possess the following property? For each compact $K_0 \subseteq Y$ there is a compact retract $K \supseteq K_0$ of Y which is homeomorphic to some topological group, and, for some $y_0 \in K$, the homotopy groups $\pi_n(K \setminus \{y_0\})$ are trivial for all sufficiently large n.

If the answer is positive, Theorem 4.9 implies the following statement, similarly as in the proof of Theorem 4.14. *Let all assumptions of Theorem 4.14 be satisfied (with the* *exception that $F(M)$ is compact). If p is cell-like (i.e., proper with R_δ-fibres $p^{-1}(x)$), then p is an (F,M)-compact-homotopy-bijection on each closed $A \subseteq M$.*

If one wants to give a positive answer to the problem, one must consider sets such that $K \setminus \{y_0\}$ is not contractible (although $\pi_n(K \setminus \{y_0\})$ is trivial for large n). In particular, the *compact* Hilbert cube is no candidate for such a set K.

In fact, our above considerations imply the following surprising side result which in particular implies that for a compact contractible topological group K and $y_0 \in K$ the set $K \setminus \{y_0\}$ is never contractible if $K \neq \{y_0\}$ is an ANR (and thus an AR), even if K has infinite dimension.

THEOREM 4.16. *Let K be a compact AR such that $K \setminus \{y_0\}$ is nonempty and contractible for some $y_0 \in K$. Then K is not homeomorphic to a topological group.*

More general, there is no continuous function $\Phi : K \times K \to K$ such that $\Phi(x,\cdot)$ is a homeomorphism of K such that $(x,y) \mapsto (\Phi(x,\cdot)^{-1})(y)$ is continuous and $\Phi(x,x) = y_0$ for all $x \in K$.

Proof. Assume by contradiction that such an AR K exists. Recall that by Urysohn's first metrization theorem each regular space with a countable base is homeomorphic to a subset of $[0,1]^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$. In particular, the separable metrizable space K is homeomorphic to a subset K_0 of the open Hilbert cube.

Let $X = Y$ be an infinite-dimensional Banach space, $\Omega \subseteq X$ the open unit ball, and $F : \overline{\Omega} \to Y$ the identity map. Since Y has the open Hilbert cube property and thus in particular contains a homeomorphic copy of the Hilbert cube, it contains a homeomorphic copy of K_0 and thus a homeomorphic copy of K. Identifying K with this copy, we may thus assume that $K \subseteq Y$. Since K contains at least two points, we may similarly assume (shifting and stretching K if necessary which are homeomorphic operations) that $0, e \in K$ where $\|e\| > 1$.

Now we argue as in the proof of Theorem 4.14 with $p := \mathrm{id}$, $q := 0$, and $A := \partial\Omega$. Since $[D, K \setminus \{y_0\}]$ is trivial for each D, we find in view of Theorem 4.9 that q is homotopic on A to the constant map $q_0 := e$ where the homotopy $h : [0,1] \times A \to K$ can be chosen such that $\mathrm{coin}_A(F, h(t,\cdot)) = \emptyset$. For $x \in \overline{\Omega}$, we put $h(0,x) := e$ and $h(1,x) := 0$. Since K is an AR, we can extend h to a continuous map $h : [0,1] \times \overline{\Omega} \to K$. Now the homotopy invariance and normalization of the Leray-Schauder degree imply $\deg_{\mathrm{LS}}(F, q_0, \Omega) = \deg_{\mathrm{LS}}(F, q, \Omega) = 1$, contradicting the solution property. □

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