NIELSEN NUMBER OF A COVERING MAP

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We consider a finite regular covering $p_H : \widetilde{X}_H \to X$ over a compact polyhedron and a map $f : X \to X$ admitting a lift $\widetilde{f} : \widetilde{X}_H \to \widetilde{X}_H$. We show some formulae expressing the Nielsen number $N(f)$ as a linear combination of the Nielsen numbers of its lifts.

1. Introduction

Let X be a finite polyhedron and let H be a normal subgroup of $\pi_1(X)$. We fix a covering $p_H : \widetilde{X}_H \to X$ corresponding to the subgroup H, that is, $p_\#(\pi_1(\widetilde{X}_H)) = H$.

We assume moreover that the subgroup H has finite rank, that is, the covering p_H is finite. Let $f : X \to X$ be a map satisfying $f(H) \subset H$. Then f admits a lift

$$\begin{array}{ccc} \widetilde{X}_H & \xrightarrow{\widetilde{f}} & \widetilde{X}_H \\ {\scriptstyle p_H}\downarrow & & \downarrow{\scriptstyle p_H} \\ X & \xrightarrow{f} & X \end{array} \tag{1.1}$$

Is it possible to find a formula expressing the Nielsen number $N(f)$ by the numbers $N(\widetilde{f})$ where $\widetilde{f}$ runs the set of all lifts? Such a formula seems very desirable since the difficulty of computing the Nielsen number often depends on the size of the fundamental group. Since $\pi_1\widetilde{X} \subset \pi_1 X$, the computation of $N(\widetilde{f})$ may be simpler. We will translate this problem to algebra. The main result of the paper is Theorem 4.2 expressing $N(f)$ as a linear combination of $\{N(\widetilde{f}_i)\}$, where the lifts are representing all the H-Reidemeister classes of f.

The discussed problem is analogous to the question about "the Nielsen number product formula" raised by Brown in 1967 [1]. A locally trivial fibre bundle $p : E \to B$ and a

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fibre map $f : E \to E$ were given and the question was how to express $N(f)$ by $N(\overline{f})$ and $N(f_b)$, where $\overline{f} : B \to B$ denoted the induced map of the base space and f_b was the restriction to the fibre over a fixed point $b \in \text{Fix}(\overline{f})$. This problem was intensively investigated in 70ties and finally solved in 1980 by You [4]. At first sufficient conditions for the "product formula" were formulated: $N(f) = N(\overline{f})N(f_b)$ assuming that $N(f_b)$ is the same for all fixed points $b \in \text{Fix}(\overline{f})$. Later it turned out that in general it is better to expect the formula

$$N(f) = N(f_{b_1}) + \cdots + N(f_{b_s}), \tag{1.2}$$

where $b_1,\dots,b_s$ represent all the Nielsen classes of $\overline{f}$. One may find an analogy between the last formula and the formulae of the present paper. There are also other analogies: in both cases the obstructions to the above equalities lie in the subgroups $\{\alpha \in \pi_1 X;\ f_\# \alpha = \alpha\} \subset \pi_1 X$.

2. Preliminaries

We recall the basic definitions [2, 3]. Let $f : X \to X$ be a self-map of a compact polyhedron. Let $\text{Fix}(f) = \{x \in X;\ f(x) = x\}$ denote the *fixed point set* of f. We define the *Nielsen relation* on $\text{Fix}(f)$ putting $x \sim y$ if there is a path $\omega : [0,1] \to X$ such that $\omega(0) = x$, $\omega(1) = y$ and the paths ω, $f\omega$ are fixed end point homotopic. This relation splits the set $\text{Fix}(f)$ into the finite number of classes $\text{Fix}(f) = A_1 \cup \cdots \cup A_s$. A class $A \subset \text{Fix}(f)$ is called *essential* if its fixed point index $\text{ind}(f;A) \neq 0$. The number of essential classes is called the *Nielsen number* and is denoted by $N(f)$. This number has two important properties. It is a homotopy invariant and is the lower bound of the number of fixed points: $N(f) \leq \#\text{Fix}(g)$ for every map g homotopic to f.

Similarly we define the *Nielsen relation modulo a normal subgroup* $H \subset \pi_1 X$. We assume that the map f preserves the subgroup H, that is, $f_\# H \subset H$. We say that then $x \sim_H y$ if $\omega = f\omega \text{mod} H$ for a path ω joining the fixed points x and y. This yields H-*Nielsen* classes and H-*Nielsen number* $N_H(f)$. For the details see [4].

Let us notice that each Nielsen class $\text{mod} H$ splits into the finite sum of ordinary Nielsen classes (i.e., classes modulo the trivial subgroup): $A = A_1 \cup \cdots \cup A_s$. On the other hand $N_H(f) \leq N(f)$.

We consider a regular finite covering $p : \widetilde{X}_H \to X$ as described above.

Let

$$\mathbb{O}_{XH} = \{\gamma : \widetilde{X}_H \longrightarrow \widetilde{X}_H;\ p_H \gamma = p_H\} \tag{2.1}$$

denote the group of natural transformations of this covering and let

$$\text{lift}_H(f) = \left\{\widetilde{f} : \widetilde{X}_H \longrightarrow \widetilde{X}_H;\ p_H \widetilde{f} = f p_H\right\} \tag{2.2}$$

denote the set of all lifts.

We start by recalling classical results giving the correspondence between the coverings and the fundamental groups of a space.

Lemma 2.1. *There is a bijection* $\mathcal{O}_{XH} = p_H^{-1}(x_0) = \pi_1(X)/H$ *which can be described as follows:*

$$\gamma \sim \gamma(\tilde{x}_0) \sim p_H(\tilde{\gamma}). \tag{2.3}$$

We fix a point $\tilde{x}_0 \in p_H^{-1}(x_0)$. *For a natural transformation* $\gamma \in \mathcal{O}_{XH}$, $\gamma(\tilde{x}_0) \in p_H^{-1}(x_0)$ *is a point and* $\tilde{\gamma}$ *is a path in* $\tilde{X}_H$ *joining the points* $\tilde{x}_0$ *and* $\gamma(\tilde{x}_0)$. *The bijection is not canonical. It depends on the choice of* x_0 *and* $\tilde{x}_0$.

Let us notice that for any two lifts $\tilde{f}, \tilde{f}' \in \operatorname{lift}_H(f)$ there exists a unique $\gamma \in \mathcal{O}_{XH}$ satisfying $\tilde{f}' = \gamma\tilde{f}$. More precisely, for a fixed lift $\tilde{f}$, the correspondence

$$\mathcal{O}_{XH} \ni \alpha \longrightarrow \alpha\tilde{f} \in \operatorname{lift}_H(f) \tag{2.4}$$

is a bijection. This correspondence is not canonical. It depends on the choice of $\tilde{f}$.

The group $\mathcal{O}_{XH}$ is acting on $\operatorname{lift}_H(f)$ by the formula

$$\alpha \circ \tilde{f} = \alpha \cdot \tilde{f} \cdot \alpha^{-1} \tag{2.5}$$

and the orbits of this action are called *Reidemeister classes* mod H and their set is denoted $\mathcal{R}_H(f)$. Then one can easily check [3]

(1) $p_H(\operatorname{Fix}(\tilde{f})) \subset \operatorname{Fix}(f)$ is either exactly one H-Nielsen class of the map f or is empty (for any $\tilde{f} \in \operatorname{lift}_H(f)$)

(2) $\operatorname{Fix}(f) = \bigcup_{\tilde{f}} p_H(\operatorname{Fix}(\tilde{f}))$ where the summation runs the set $\operatorname{lift}_H(f)$

(3) if $p_H(\operatorname{Fix}(\tilde{f})) \cap p_H(\operatorname{Fix}(\tilde{f}')) \neq \emptyset$ then $\tilde{f}$, $\tilde{f}'$ represent the same Reidemeister class in $\mathcal{R}_H(f)$

(4) if $\tilde{f}$, $\tilde{f}'$ represent the same Reidemeister class then $p_H(\operatorname{Fix}(\tilde{f})) = p_H(\operatorname{Fix}(\tilde{f}'))$.

Thus $\operatorname{Fix}(f) = \bigcup_{\tilde{f}} p_H(\operatorname{Fix}(\tilde{f}))$ is the disjoint sum where the summation is over a subset containing exactly one lift $\tilde{f}$ from each H-Reidemeister class. This gives the natural inclusion from the set of Nielsen classes modulo H into the set of H-Reidemeister classes

$$\mathcal{N}_H(f) \longrightarrow \mathcal{R}_H(f). \tag{2.6}$$

The H-Nielsen class A is sent into the H-Reidemeister class represented by a lift $\tilde{f}$ satisfying $p_H(\operatorname{Fix}(\tilde{f})) = A$. By (1) and (2) such lift exists, by (3) the definition is correct and (4) implies that this map is injective.

3. Lemmas

For a lift $\widetilde{f} \in \mathrm{lift}_H(f)$, a fixed point $x_0 \in \mathrm{Fix}(f)$ and an element $\beta \in \pi_1(X;x_0)$ we define the subgroups

$$\begin{aligned} \mathscr{Z}(\widetilde{f}) &= \left\{\gamma \in \mathbb{O}_{XH};\ \widetilde{f}\gamma = \gamma\widetilde{f}\right\} \\ C(f_\#,x_0;\beta) &= \{\alpha \in \pi_1(X;x_0);\ \alpha\beta = \beta f_\#(\alpha)\} \\ C_H(f_\#,x_0;\beta) &= \{[\alpha]_H \in \pi_1(X;x_0)/H(x_0);\ \alpha\beta = \beta f_\#(\alpha) \text{ modulo } H\}. \end{aligned} \tag{3.1}$$

If $\beta = 1$ we will write simply $C(f_\#,x_0)$ or $C_H(f_\#,x_0)$.

We notice that the canonical projection $j : \pi_1(X;x_0) \to \pi_1(X;x_0)/H(x_0)$ induces the homomorphism $j : C(f_\#,x_0;\beta) \to C_H(f_\#,x_0;\beta)$.

Lemma 3.1. *Let $\widetilde{f}$ be a lift of f and let $\widetilde{A}$ be a Nielsen class of $\widetilde{f}$. Then $p_H(\widetilde{A}) \subset \mathrm{Fix}(f)$ is a Nielsen class of f. On the other hand if $A \subset \mathrm{Fix}(f)$ is a Nielsen class of f then $p_H^{-1}(A) \cap \mathrm{Fix}(\widetilde{f})$ splits into the finite sum of Nielsen classes of $\widetilde{f}$.*

Proof. It is evident that $p_H(\widetilde{A})$ is contained in a Nielsen class $A \subset \mathrm{Fix}(f)$. Now we show that $A \subset p_H(\widetilde{A})$. Let us fix a point $\widetilde{x}_0 \in \widetilde{A}$ and let $x_0 = p_H(\widetilde{x}_0)$. Let $x_1 \in A$. We have to show that $x_1 \in p_H(\widetilde{A})$. Let $\omega : I \to X$ establish the Nielsen relation between the points $\omega(0) = x_0$ and $\omega(1) = x_1$ and let $h(t,s)$ denote the homotopy between $\omega = h(\cdot,0)$ and $f\omega = h(\cdot,1)$. Then the path ω lifts to a path $\widetilde{\omega} : I \to \widetilde{X}_H$, $\widetilde{\omega}(0) = \widetilde{x}_0$. Let us denote $\widetilde{\omega}(1) = \widetilde{x}_1$. It remains to show that $\widetilde{x}_1 \in \widetilde{A}$. The homotopy h lifts to $\widetilde{h} : I \times I \to \widetilde{X}_H$, $\widetilde{h}(0,s) = \widetilde{x}_0$. Then the paths $\widetilde{h}(\cdot,1)$ and $\widetilde{f}\widetilde{\omega}$ as the lifts of $f\omega$ starting from $\widetilde{x}_0$ are equal. Now $\widetilde{f}(\widetilde{x}_1) = \widetilde{f}(\widetilde{\omega}(1)) = \widetilde{h}(1,1) = \widetilde{h}(1,0) = \widetilde{\omega}(1) = \widetilde{x}_1$. Thus $\widetilde{x}_1 \in \mathrm{Fix}(\widetilde{f})$ and the homotopy $\widetilde{h}$ gives the Nielsen relation between $\widetilde{x}_0$ and $\widetilde{x}_1$ hence $\widetilde{x}_1 \in \widetilde{A}$.

Now the second part of the lemma is obvious. □

Lemma 3.2. *Let $\widetilde{A} \subset \mathrm{Fix}(\widetilde{f})$ be a Nielsen class of $\widetilde{f}$. Let us denote $A = p_H(\widetilde{A})$. Then*

(1) *$p_H : \widetilde{A} \to A$ is a covering where the fibre is in bijection with the image $j_\#(C(f_\#,x)) \subset \pi_1(X;x)/H(x)$ for $x \in A$,*

(2) *the cardinality of the fibre (i.e., $\#(p_H^{-1}(x) \cap \widetilde{A})$) does not depend on $x \in A$ and we will denote it by J_A,*

(3) *if $\widetilde{A}'$ is another Nielsen class of $\widetilde{f}$ satisfying $p_H(\widetilde{A}') = p_H(\widetilde{A})$ then the cardinalities of $p_H^{-1}(x) \cap \widetilde{A}$ and $p_H^{-1}(x) \cap \widetilde{A}'$ are the same for each point $x \in A$.*

Proof. (1) Since p_H is a local homeomorphism, the projection $p_H : \widetilde{A} \to A$ is the covering.

(2) We will show a bijection $\phi : j(C(f_\#;x_0)) \to p_H^{-1}(x_0) \cap \widetilde{A}$ (for a fixed point $x_0 \in A$).

Let $\alpha \in C(f_\#)$. Let us fix a point $\widetilde{x}_0 \in p_H^{-1}(x_0)$. Let $\widetilde{\alpha} : I \to \widetilde{X}$ denote the lift of α starting from $\widetilde{\alpha}(0) = \widetilde{x}_0$. We define $\phi([\alpha]_H) = \widetilde{\alpha}(1)$. We show that

(2a) The definition is correct. Let $[\alpha]_H = [\alpha']_H$. Then $\alpha \equiv \alpha' \bmod H$ hence $\widetilde{\alpha}(1) = \widetilde{\alpha}'(1)$. Now we show that $\widetilde{\alpha}(1) \in \widetilde{A}$. Since $\alpha \in C(f_\#)$, there exists a homotopy h between the loops $h(\cdot,0) = \alpha$ and $h(\cdot,1) = f\alpha$. The homotopy lifts to $\widetilde{h} : I \times I \to \widetilde{X}_H$, $\widetilde{h}(0,s) = \widetilde{x}_0$. Then $\widetilde{x}_1 = \widetilde{h}(1,s)$ is also a fixed point of $\widetilde{f}$ and moreover $\widetilde{h}$ is the homotopy between the paths $\widetilde{\omega}$ and $\widetilde{f}\widetilde{\omega}$. Thus $\widetilde{x}_0, \widetilde{x}_1 \in \mathrm{Fix}(\widetilde{f})$ are Nielsen related hence $\widetilde{x}_1 \in \widetilde{A}$.

(2b) ϕ is onto. Let $\widetilde{x}_1 \in p_H^{-1}(x_0) \cap \widetilde{A}$. Now $\widetilde{x}_0, \widetilde{x}_1 \in \operatorname{Fix}(\widetilde{f})$ are Nielsen related. Let $\widetilde{\omega} : I \to \widetilde{X}_H$ establish this relation ($\widetilde{f}\widetilde{\omega} \sim \widetilde{\omega}$). Now

$$f(p_H\widetilde{\omega}) = p_H\widetilde{f}\widetilde{\omega} \sim p_H\widetilde{\omega} \tag{3.2}$$

hence $p_H\widetilde{\omega} \in C(f_\#; x_0)$. Moreover $\phi[p_H\widetilde{\omega}]_H = \widetilde{\omega}(1) = \widetilde{x}_1$.

(2c) ϕ is injective. Let $[\alpha]_H, [\alpha']_H \in j(C(f_\#))$ and let $\widetilde{\alpha}, \widetilde{\alpha}' : I \to \widetilde{X}_H$ be their lifts starting from $\widetilde{\alpha}(0) = \widetilde{\alpha}'(0) = \widetilde{x}_0$. Suppose that $\phi[\alpha]_H = \phi[\alpha']_H$. This means $\widetilde{\alpha}(1) = \widetilde{\alpha}'(1) \in \widetilde{X}_H$. Thus $p_H(\widetilde{\alpha} * \widetilde{\alpha}'^{-1}) = \alpha * \alpha'^{-1} \in H$ which implies $[\alpha]_H = [\alpha']_H$.

(3) Let $x_0 \in p_H(\widetilde{A}) = p_H(\widetilde{A}')$. Then by the above $\#(p^{-1}(x_0) \cap \widetilde{A}) = j_\#(C(f_\#)) = \#(p^{-1}(x_0) \cap \widetilde{A}')$. □

Lemma 3.3. *The restriction of the covering map $p_H : \operatorname{Fix}(\widetilde{f}) \to p_H(\operatorname{Fix}(\widetilde{f}))$ is a covering. The fibre over each point is in a bijection with the set*

$$\mathfrak{Z}(\widetilde{f}) = \left\{\gamma \in \mathbb{O}_{XH};\ \widetilde{f}\gamma = \gamma\widetilde{f}\right\}. \tag{3.3}$$

Proof. Since the fibre of the covering p_H is discrete, the restriction $p_H : \operatorname{Fix}(\widetilde{f}) \to p_H(\operatorname{Fix}(\widetilde{f}))$ is a locally trivial bundle. Let us fix points $x_0 \in p_H(\operatorname{Fix}(\widetilde{f}))$, $\widetilde{x}_0 \in p_H^{-1}(x_0) \cap \operatorname{Fix}(\widetilde{f})$. We recall that

$$\alpha : p_H^{-1}(x_0) \longrightarrow \mathbb{O}_{XH}, \tag{3.4}$$

where $\alpha_{\widetilde{x}} \in \mathbb{O}_{XH}$ is characterized by $\alpha_{\widetilde{x}}(\widetilde{x}_0) = \widetilde{x}$, is a bijection. We will show that $\alpha(p_H^{-1}(x_0) \cap \operatorname{Fix}(\widetilde{f})) = \mathfrak{Z}(\widetilde{f})$.

Let $\widetilde{f}(\widetilde{x}) = \widetilde{x}$ for an $\widetilde{x} \in p_H^{-1}(x_0)$. Then

$$\widetilde{f}\alpha_{\widetilde{x}}(\widetilde{x}_0) = \widetilde{f}(\widetilde{x}) = \widetilde{x} = \alpha_{\widetilde{x}}(\widetilde{x}_0) = \alpha_{\widetilde{x}}\widetilde{f}(\widetilde{x}_0) \tag{3.5}$$

which implies $\widetilde{f}\alpha_{\widetilde{x}} = \alpha_{\widetilde{x}}\widetilde{f}$ hence $\alpha_{\widetilde{x}} \in \mathfrak{Z}(\widetilde{f})$.

Now we assume that $\widetilde{f}\alpha_{\widetilde{x}} = \alpha_{\widetilde{x}}\widetilde{f}$. Then in particular $\widetilde{f}\alpha_{\widetilde{x}}(\widetilde{x}_0) = \alpha_{\widetilde{x}}\widetilde{f}(\widetilde{x}_0)$ which gives $\widetilde{f}(\widetilde{x}) = \alpha_{\widetilde{x}}(\widetilde{x}_0)$, $\widetilde{f}(\widetilde{x}) = \widetilde{x}$ hence $\widetilde{x} \in \operatorname{Fix}(\widetilde{f})$. □

We will denote by I_{A_H} the cardinality of the subgroup $\#\mathfrak{Z}(\widetilde{f})$ for the H-Nielsen class $A_H = p_H(\operatorname{Fix}(\widetilde{f}))$. We will also write $I_{A_i} = I_{A_H}$ for any Nielsen class A_i of f contained in A.

Lemma 3.4. *Let $A_0 \subset \operatorname{Fix}(f)$ be a Nielsen class and let $\widetilde{A}_0 \subset \operatorname{Fix}(\widetilde{f})$ be a Nielsen class contained in $p_H^{-1}(A_0)$. Then, by Lemma 3.1 $A_0 = p_H(\widetilde{A}_0)$ and moreover*

$$\begin{gathered} \operatorname{ind}(\widetilde{f}; p_H^{-1}(A_0)) = I_{A_0} \cdot \operatorname{ind}(f; A_0) \\ \operatorname{ind}(\widetilde{f}; \widetilde{A}_0) = J_{A_0} \cdot \operatorname{ind}(f; A_0). \end{gathered} \tag{3.6}$$

Proof. Since the index is the homotopy invariant we may assume that $\mathrm{Fix}(f)$ is finite. Now for any fixed points $x_0 \in \mathrm{Fix}(f)$, $\widetilde{x}_0 \in \mathrm{Fix}(\widetilde{f})$ satisfying $p_H(\widetilde{x}_0) = x_0$ we have $\mathrm{ind}(\widetilde{f}_0;\widetilde{x}_0) = \mathrm{ind}(f_0;x_0)$ since the projection p_H is a local homeomorphism. Thus

$$\begin{aligned}\mathrm{ind}\,(\widetilde{f};p_H^{-1}(A_0)) &= \sum_{x\in A_0} \mathrm{ind}\,(\widetilde{f};p_H^{-1}(x)) = \sum_{x\in A_0} I_{A_0}\cdot \mathrm{ind}(f;x)\\ &= I_{A_0}\sum_{x\in A_0}\mathrm{ind}(f;x) = I_{A_0}\cdot \mathrm{ind}\,(f;A_0).\end{aligned} \tag{3.7}$$

Similarly we prove the second equality:

$$\begin{aligned}\mathrm{ind}\,(\widetilde{f};\widetilde{A}_0) &= \sum_{x\in A_0}\mathrm{ind}\left(\widetilde{f};p_H^{-1}(x)\cap \widetilde{A}_0\right) = \sum_{x\in A_0}\sum_{\widetilde{x}\in p_H^{-1}(x)\cap\widetilde{A}_0}\mathrm{ind}\,(\widetilde{f};\widetilde{x})\\ &= \sum_{x\in A_0} J_{A_0}\cdot \mathrm{ind}(f;x) = J_{A_0}\cdot\left(\sum_{x\in A_0}\mathrm{ind}(f;x)\right) = J_{A_0}\cdot \mathrm{ind}\,(f;A_0).\end{aligned} \tag{3.8}$$

□

To get a formula expressing $N(f)$ by the numbers $N(\widetilde{f})$ we will need the assumption that the numbers $J_A = J_{A'}$ for any two H-Nielsen related classes $A,A' \subset \mathrm{Fix}(f)$. The next lemma gives a sufficient condition for such equality.

Lemma 3.5. *Let $x_0 \in p(\mathrm{Fix}(\widetilde{f}))$. If the subgroups $H(x_0), C(f,x_0) \subset \pi_1(X,x_0)$ commute, that is, $h\cdot\alpha = \alpha\cdot h$, for any $h \in H(x_0)$, $\alpha \in C(f,x_0)$, then $J_A = J_{A'}$ for all Nielsen classes $A,A' \subset p(\mathrm{Fix}(\widetilde{f}))$.*

Proof. Let $x_1 \in p(\mathrm{Fix}(\widetilde{f}))$ be another point. The points $x_0,x_1 \in p(\mathrm{Fix}(\widetilde{f}))$ are H-Nielsen related, that is, there is a path $\omega : [0,1] \to X$ satisfying $\omega(0) = x_0$, $\omega(1) = x_1$ such that $\omega * f(\omega^{-1}) \in H(x_0)$. We will show that the conjugation

$$\pi_1(X,x_0) \ni \alpha \longrightarrow \omega^{-1} * \alpha * \omega \in \pi_1(X,x_1) \tag{3.9}$$

sends $C(f,x_0)$ onto $C(f,x_1)$. Let $\alpha \in C(f,x_0)$. We will show that $\omega^{-1} * \alpha * \omega \in C(f,x_1)$. In fact $f(\omega^{-1} * \alpha * \omega) = \omega^{-1} * \alpha * \omega \Leftrightarrow (\omega * f\omega^{-1}) * \alpha = \alpha * (\omega * f\omega^{-1})$ but the last equality holds since $\omega * f\omega^{-1} \in H(x_0)$ and $\alpha \in C(f,x_0)$. □

Remark 3.6. The assumption of the above lemma is satisfied if at least one of the groups $H(x_0)$, $C(f,x_0)$ belongs to the center of $\pi_1(X;x_0)$.

Remark 3.7. Let us notice that if the subgroups $H(x_0), C(f,x_0) \subset \pi_1(X,x_0)$ commute then so do the corresponding subgroups at any other point $x_1 \in p_H(\mathrm{Fix}(\widetilde{f}))$.

Proof. Let us fix a path $\omega : [0,1] \to X$. We will show that the conjugation

$$\pi_1(X,x_0) \ni \alpha \longrightarrow \omega^{-1} * \alpha * \omega \in \pi_1(X,x_1) \tag{3.10}$$

sends $C(f,x_0)$ onto $C(f,x_1)$. Let $\alpha \in C(f,x_0)$. We will show that $\omega^{-1} * \alpha * \omega \in C(f,x_1)$. But the last means $f(\omega^{-1} * \alpha * \omega) = \omega^{-1} * \alpha * \omega$ hence $f(\omega^{-1}) * f\alpha * f\omega = \omega^{-1} * \alpha * \omega \Leftrightarrow f(\omega^{-1}) * \alpha * f\omega = \omega^{-1} * \alpha * \omega \Leftrightarrow (\omega * f\omega^{-1}) * \alpha = \alpha * (\omega * f\omega^{-1})$ and the last holds since $(\omega * f\omega^{-1}) \in H(x_0)$ and $\alpha \in C(f,x_0)$. Now it remains to notice that the elements of $H(x_1)$, $C(f;x_1)$ are of the form $\omega^{-1} * \gamma * \omega$ and $\omega^{-1} * \alpha * \omega$ respectively for some $\gamma \in H(x_0)$ and $\alpha \in C(f,x_0)$. □

Now we will express the numbers I_A, J_A in terms of the homotopy group homomorphism $f_\# : \pi_1(X,x_0) \to \pi_1(X,x_0)$ for a fixed point $x_0 \in \text{Fix}(f)$. Let $\widetilde{f} : \widetilde{X}_H \to \widetilde{X}_H$ be a lift satisfying $\widetilde{x}_0 \in p_H^{-1}(x_0) \cap \text{Fix}(\widetilde{f})$. We also fix the isomorphism

$$\pi_1(X;x_0)/H(x_0) \ni \alpha \longrightarrow \gamma_\alpha \in \mathbb{O}_{XH}, \tag{3.11}$$

where $\gamma_\alpha(\widetilde{x}_0) = \widetilde{\alpha}(1)$ and $\widetilde{\alpha}$ denotes the lift of α starting from $\widetilde{\alpha}(0) = \widetilde{x}_0$.

We will describe the subgroup corresponding to $C(\widetilde{f})$ by this isomorphism and then we will do the same for the other lifts $\widetilde{f}' \in \text{lift}_H(f)$.

Lemma 3.8.

$$\widetilde{f}\gamma_\alpha = \gamma_{f\alpha}\widetilde{f}. \tag{3.12}$$

Proof.

$$\widetilde{f}\gamma_\alpha(\widetilde{x}_0) = \widetilde{f}\widetilde{\alpha}(1) = \gamma_{f\alpha}(\widetilde{x}_0) = \gamma_{f\alpha}\widetilde{f}(\widetilde{x}_0), \tag{3.13}$$

where the middle equality holds since $\widetilde{f}\widetilde{\alpha}$ is a lift of the path $f\alpha$ from the point $\widetilde{x}_0$. □

Corollary 3.9. *There is a bijection between*

$$\mathfrak{X}(\widetilde{f}) = \left\{\gamma \in \mathbb{O}_{XH};\ \widetilde{f}\gamma = \gamma\widetilde{f}\right\},$$
$$C_H(f) = \{\alpha \in \pi_1(X;x_0)/H(x_0);\ f_{H\#}(\alpha) = \alpha\}. \tag{3.14}$$

Thus

$$I_A/J_A = \#\mathfrak{X}(\widetilde{f})/\#j(C(f)) = \#(C_H(f)/j(C(f))). \tag{3.15}$$

Let us emphasize that $C(f)$, $C_H(f)$ are the subgroups of $\pi_1(X;x_0)$ or $\pi_1(X;x_0)/H(x_0)$ respectively where the base point is the chosen fixed point. Now will take another fixed point $x_1 \in \text{Fix}(f)$ and we will denote $C'(f) = \{\alpha' \in \pi_1(X;x_1);\ f_\#\alpha = \alpha\}$ and similarly we define $C'_H(f)$. We will express the cardinality of these subgroups in terms of the group $\pi_1(X;x_0)$.

Lemma 3.10. *Let $\eta : [0,1] \to X$ be a path from x_0 to x_1. This path gives rise to the isomorphism $P_\eta : \pi_1(X;x_1) \to \pi_1(X;x_0)$ by the formula $P_\eta(\alpha) = \eta\alpha\eta^{-1}$. Let $\delta = \eta \cdot (f\eta)^{-1}$. Then*

$$P_\eta(C'(f)) = \{\alpha \in \pi_1(X;x_0);\ \alpha\delta = \delta f_\#(\alpha)\}$$
$$P_\eta(C'_H(f)) = \{[\alpha] \in \pi_1(X;x_0)/H(x_0);\ \alpha\delta = \delta f_\#(\alpha) \text{ modulo } H\}. \tag{3.16}$$

Proof. We notice that δ is a loop based at x_0 representing the Reidemeister class of the point x_1 in $\mathcal{R}(f) = \pi_1(X;x_0)/\mathcal{R}$.

We will denote the right-hand side of the above equalities by $C(f;\delta)$ and $C_H(f;\delta)$ respectively. Let $\alpha' \in \pi_1(X;x_1)$. We denote $\alpha = P_\eta(\alpha') = \eta\alpha'\eta^{-1}$. We will show that $\alpha \in C(f;\delta) \Leftrightarrow \alpha' \in C'(f)$.

In fact $\alpha \in C(f;\delta) \Leftrightarrow \alpha\delta = \delta \cdot f\alpha \Leftrightarrow (\eta\alpha'\eta^{-1})(\eta \cdot f\eta^{-1}) = (\eta \cdot f\eta^{-1})(f\eta \cdot f\alpha' \cdot (f\eta)^{-1}) \Leftrightarrow \eta\alpha' \cdot (f\eta)^{-1} = \eta \cdot f\alpha' \cdot (f\eta)^{-1} \Leftrightarrow \alpha' = f\alpha'$.

Similarly we prove the second equality. □

Thus we get the following formulae for the numbers I_A, J_A.

Corollary 3.11. *Let $\delta \in \pi_1(X;x_0)$ represent the Reidemeister class $A \in \mathcal{R}(f)$. Then $I_A = \#C_H(f;j(\delta))$, $J_A = \#j(C(f;\delta))$.*

4. Main theorem

Lemma 4.1. *Let $A \subset p_H(\operatorname{Fix}(\tilde{f}))$ be a Nielsen class of f. Then $p_H^{-1}A$ contains exactly I_A/J_A fixed point classes of $\tilde{f}$.*

Proof. Since the projection of each Nielsen class $\tilde{A} \subset p_H^{-1}(A) \cap \operatorname{Fix}(\tilde{f})$ is onto A (Lemma 3.1), it is enough to check how many Nielsen classes of $\tilde{f}$ cut $p_H^{-1}(a)$ for a fixed point $a \in A$. But by Lemma 3.3 $p_H^{-1}(a) \cap \operatorname{Fix}(\tilde{f})$ contains I_A points and by Lemma 3.2 each class in this set has exactly J_A common points with $p_H^{-1}(a)$. Thus exactly I_A/J_A Nielsen classes of $\tilde{f}$ are cutting $p_H^{-1}(a) \cap \operatorname{Fix}(\tilde{f})$. □

Let $f : X \to X$ be a self-map of a compact polyhedron admitting a lift $\tilde{f} : \tilde{X}_H \to \tilde{X}_H$. We will need the following auxiliary assumption:

for any Nielsen classes $A, A' \in \operatorname{Fix}(f)$ representing the same class modulo the subgroup H the numbers $J_A = J_{A'}$.

We fix lifts $\tilde{f}_1, \dots, \tilde{f}_s$ representing all H-Nielsen classes of f, that is,

$$\operatorname{Fix}(f) = p_H\Big(\operatorname{Fix}(\tilde{f}_1)\Big) \cup \cdots \cup p_H\Big(\operatorname{Fix}(\tilde{f}_s)\Big) \tag{4.1}$$

is the mutually disjoint sum. Let I_i, J_i denote the numbers corresponding to a (Nielsen class of f) $A \subset p_H(\operatorname{Fix}(\tilde{f}_i))$. By the remark after Lemma 3.3 and by the above assumption these numbers do not depend on the choice of the class $A \subset p_H(\operatorname{Fix}(\tilde{f}_i))$. We also notice that Lemmas 3.3, 3.2 imply

$$\begin{aligned} I_i &= \#\mathcal{L}(\tilde{f}_i) = \#\Big\{\gamma \in \mathcal{O}_{XH};\ \gamma\tilde{f}_i = \tilde{f}_i\gamma\Big\} \\ J_i &= \#j\big(C(f_\#;x)\big) = \#j\big(\{\gamma \in \pi_1(X,x_i);\ f_\#\gamma = \gamma\}\big) \end{aligned} \tag{4.2}$$

for an $x_i \in A_i$.

THEOREM 4.2. *Let X be a compact polyhedron, $P_H : \widetilde{X}_H \to \widetilde{X}$ a finite regular covering and let $f : X \to X$ be a self-map admitting a lift $\widetilde{f} : \widetilde{X}_H \to \widetilde{X}_H$. We assume that for each two Nielsen classes $A, A' \subset \mathrm{Fix}(f)$, which represent the same Nielsen class modulo the subgroup H, the numbers $J_A = J_{A'}$. Then*

$$N(f) = \sum_{i=1}^{s} (J_i/I_i) \cdot N(\widetilde{f}_i), \tag{4.3}$$

where I_i, J_i denote the numbers defined above and the lifts $\widetilde{f}_i$ represent all H-Reidemeister classes of f, corresponding to nonempty H-Nielsen classes.

Proof. Let us denote $A_i = p_H(\mathrm{Fix}(\widetilde{f}_i))$. Then A_i is the disjoint sum of Nielsen classes of f. Let us fix one of them $A \subset A_i$. By Lemma 3.1 $p_H^{-1}A \cap \mathrm{Fix}(\widetilde{f}_i)$ splits into I_A/J_A Nielsen classes in $\mathrm{Fix}(\widetilde{f}_i)$. By Lemma 3.4 A is essential iff one (hence all) Nielsen classes in $p_H^{-1}A \subset \mathrm{Fix}\,\widetilde{f}_i$ is essential. Summing over all essential classes of $\widetilde{f}$ in $A_i = p_A(\mathrm{Fix}(\widetilde{f}_i))$ we get

$$\begin{aligned} &\text{the number of essential Nielsen classes of } f \text{ in } A_i \\ &\quad = \sum_A (J_A/I_A) \cdot (\text{number of essential Nielsen classes of } \widetilde{f}_i \text{ in } p_H^{-1}A), \end{aligned} \tag{4.4}$$

where the summation runs the set of all essential Nielsen classes contained in A_i.

But $J_A = J_i$, $I_A = I_i$ for all $A \subset A_i$ hence

$$(\text{the number of essential Nielsen classes of } f \text{ in } A_i) = J_i/I_i \cdot N(\widetilde{f}_i). \tag{4.5}$$

Summing over all lifts $\{\widetilde{f}_i\}$ representing non-empty H-Nielsen classes of f we get

$$N(f) = \sum_i (J_i/I_i) \cdot N(\widetilde{f}_i) \tag{4.6}$$

since $N(f)$ equals the number of essential Nielsen classes in $\mathrm{Fix}(f) = \bigcup_{i=1}^{s} p_H \mathrm{Fix}(\widetilde{f}_i)$. □

COROLLARY 4.3. *If moreover, under the assumptions of Theorem 4.2, $C = J_i/I_i$ does not depend on i then*

$$N(f) = C \cdot \sum_{i=1}^{s} N(\widetilde{f}_i). \tag{4.7}$$

5. Examples

In all examples given below the auxiliary assumption $J_A = J_{A'}$ holds, since the assumptions of Lemma 3.5 are satisfied (in 1, 2, 3 and 5 the fundamental groups are commutative and in 4 the subgroup $C(f, x_0)$ is trivial).

(1) If $\pi_1 X$ is finite and $p : \widetilde{X} \to X$ is the universal covering (i.e., $H = 0$) then $\widetilde{X}$ is simply connected hence for any lift $\widetilde{f} : \widetilde{X} \to \widetilde{X}$

$$N(\widetilde{f}) = \begin{cases} 1 & \text{for } L(\widetilde{f}) \neq 0 \\ 0 & \text{for } L(\widetilde{f}) = 0. \end{cases} \tag{5.1}$$

But $L(\widetilde{f}) \neq 0$ if and only if the Nielsen class $p(\mathrm{Fix}(\widetilde{f})) \subset \mathrm{Fix}(f)$ is essential (Lemma 3.4). Thus

$$N(f) = \text{number of essential classes} = N(\widetilde{f}_1) + \cdots + N(\widetilde{f}_s), \tag{5.2}$$

where the lifts $\widetilde{f}_1, \ldots, \widetilde{f}_s$ represent all Reidemeister classes of f.

(2) Consider the commutative diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{p_l} & S^1 \\ {\scriptstyle p_k}\downarrow & & \downarrow{\scriptstyle p_k} \\ S^1 & \xrightarrow{p_l} & S^1 \end{array} \tag{5.3}$$

Where $p_k(z) = z^k$, $p_l(z) = z^l$, $k, l \geq 2$. The map p_k is regarded as k-fold regular covering map. Then each natural transformation map of this covering is of the form $\alpha(z) = \exp(2\pi p/k) \cdot z$ for $p = 0, \ldots, k-1$ hence is homotopic to the identity map. Now all the lifts of the map p_l are maps of degree l hence their Nielsen numbers equal $l - 1$. On the other hand the Reidemeister relation of the map $p_l : S^1 \to S^1$ modulo the subgroup $H = \mathrm{im}\, p_{k\#}$ is given by

$$\begin{aligned} \alpha \sim \beta &\iff \beta = \alpha + p(l-1) \in k \cdot \mathbb{Z} \quad \text{for a } p \in \mathbb{Z} \\ &\iff \beta = \alpha + p(l-1) + qk \quad \text{for some } p, q \in \mathbb{Z} \\ &\iff \alpha = \beta \text{ modulo g.c.d. } (l-1, k). \end{aligned} \tag{5.4}$$

Thus $\#\mathcal{R}_H(p_l) = \text{g.c.d.}(l-1, k)$. Now the sum

$$\sum_{p_l'} N(p_l') = (\text{g.c.d.}(l-1, k)) \cdot (l-1), \tag{5.5}$$

(where the summation runs the set having exactly one common element with each H-Reidemeister class) equals $N(p_l) = l - 1$ iff the numbers $k, l-1$ are relatively prime.

Notice that in our notation $I = \text{g.c.d.}(l-1, k)$ while $J = 1$.

(3) Let us consider the action of the cyclic group $\mathbb{Z}_8$ on $S^3 = \{(z, z') \in \mathbb{C} \times \mathbb{C};\ |z|^2 + |z'|^2 = 1\}$ given by the cyclic homeomorphism

$$S^3 \ni (z, z') \longrightarrow \left(\exp(2\pi i/8) \cdot z, \exp(2\pi i/8) \cdot z'\right) \in S^3. \tag{5.6}$$

The quotient space is the lens space which we will denote L_8. We will also consider the quotient space of S^3 by the action of the subgroup $2\mathbb{Z}_4 \subset \mathbb{Z}_8$. Now the quotient group is also a lens space which we will denote by L_4. Let us notice that there is a natural 2-fold covering $p_H : L_4 \to L_8$

$$L_4 = S^3/\mathbb{Z}_4 \ni [z,z'] \longrightarrow [z,z'] \in S^3/\mathbb{Z}_8 = L_8. \tag{5.7}$$

The group of natural transformations $\mathbb{O}_L$ of this covering contains two elements: the identity and the map $A[z,z'] = [\exp(2\pi i/8)\cdot z, \exp(2\pi i/8)\cdot z']$. Now we define the map $f : L_8 \to L_8$ putting $f[z,z'] = [z^7/|z|^6, z'^7/|z|'^6]$. This map admits the lift $\widetilde{f} : L_4 \to L_4$ given by the same formula and the lift $A\widetilde{f}$. We notice that each of the maps f, $\widetilde{f}$, $A\widetilde{f}$ is a map of a closed oriented manifold of degree 49. Since $H_1(L;\mathbb{Q}) = H_2(L;\mathbb{Q}) = 0$ for all lens spaces, the Lefschetz number of each of these three maps equals; $L(f) = 1 - 49 = -48 \neq 0$. On the other hand since the lens spaces are Jiang [3], all involved Reidemeister classes are essential hence the Nielsen number equals the Reidemeister number in each case.

Now

$$\mathcal{R}(f) = \operatorname{coker}(\mathrm{id} - 7\cdot \mathrm{id}) = \operatorname{coker}(-6\cdot \mathrm{id}) = \operatorname{coker}(2\cdot \mathrm{id}) = \mathbb{Z}_2. \tag{5.8}$$

Similarly $\mathcal{R}(\widetilde{f}) = \mathbb{Z}_2$ and $\mathcal{R}(A\cdot\widetilde{f}) = \mathcal{R}(\widetilde{f}) = \mathbb{Z}_2$ since A is homotopic to the identity. Thus

$$R(f) = 2 \neq 2 + 2 = R(\widetilde{f}) + R(A\cdot\widetilde{f}). \tag{5.9}$$

Since all the classes are essential, the same inequality holds for the Nielsen numbers.

(4) If the group $\{\alpha \in \pi_1(X;x)/H(x);\ f_\#\alpha = \alpha\}$ is trivial for each $x \in \mathrm{Fix}(f)$ lying in an essential Nielsen class of f then all the numbers $I_i = J_i = 1$ and the sum formula holds.

(5) If $\pi_1 X/H$ is abelian then the rank of the groups

$$C(f_{H\#}) = \{\alpha \in \pi_1(X,x)/H(x);\ f_\#\alpha = \alpha\} = \ker\left(\mathrm{id} - f_\#\right) : \pi_1(X,x)/H(x) \longrightarrow \pi_1(X,x)/H(x) \tag{5.10}$$

does not depend on $x \in \mathrm{Fix}(f)$ hence I is constant. If moreover $\pi_1 X$ is abelian then also the group $C(f_\#) = \ker(\mathrm{id} - f_\#)$ does not depend on $x \in \mathrm{Fix}(f)$. Then we get

$$N(f) = J/I \cdot \left(N(\widetilde{f}_1) + \cdots + N(\widetilde{f}_s)\right). \tag{5.11}$$

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