HYPERBOLIC MONOTONICITY IN THE HILBERT BALL

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We first characterize ρ-monotone mappings on the Hilbert ball by using their resolvents and then study the asymptotic behavior of compositions and convex combinations of these resolvents.

1. Introduction

Monotone operator theory has been intensively developed with many applications to Convex and Nonlinear Analysis, Partial Differential Equations, and Optimization. In this note we intend to apply the concept of (hyperbolic) monotonicity to Complex Analysis. As we will see, this application involves the generation theory of one-parameter continuous semigroups of holomorphic mappings.

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$, and let $\mathbb{B} := \{x \in H : |x| < 1\}$ be its open unit ball. The *hyperbolic metric* ρ on $\mathbb{B} \times \mathbb{B}$ [5, page 98] is defined by

$$\rho(x, y) := \operatorname{argtanh} \left(1 - \sigma(x, y)\right)^{1/2}, \tag{1.1}$$

where

$$\sigma(x, y) := \frac{\left(1 - |x|^2\right)\left(1 - |y|^2\right)}{\left|1 - \langle x, y \rangle\right|^2}, \quad x, y \in \mathbb{B}. \tag{1.2}$$

A mapping $g : \mathbb{B} \to \mathbb{B}$ is said to be *ρ-nonexpansive* if

$$\rho\left(g(x), g(y)\right) \leq \rho(x, y) \tag{1.3}$$

for all $x, y \in \mathbb{B}$. It is known (see, for instance, [5, page 91]) that each holomorphic self-mapping of $\mathbb{B}$ is ρ-nonexpansive.

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Recall that if C is a subset of H, then a (single-valued) mapping $f : C \to H$ is said to be *monotone* if

$$\operatorname{Re}\langle x-y, f(x)-f(y)\rangle \geq 0, \quad x,y \in C. \tag{1.4}$$

Equivalently, f is monotone if

$$\operatorname{Re}\left[\langle x, f(x)\rangle + \langle y, f(y)\rangle\right] \geq \operatorname{Re}\left[\langle y, f(x)\rangle + \langle x, f(y)\rangle\right], \quad x,y \in C. \tag{1.5}$$

It is also not difficult to see that f is monotone if and only if

$$|x-y| \leq \left|x+rf(x)-(y+rf(y))\right|, \quad x,y \in C, \tag{1.6}$$

for all (small enough) $r > 0$.

Let I denote the identity operator. A mapping $f : C \to H$ is said to satisfy the *range condition* if

$$(I+rf)(C) \supset C, \quad r > 0. \tag{1.7}$$

If f is monotone and satisfies the range condition, then the mapping $J_r : C \to C$, well-defined for positive r by $J_r := (I+rf)^{-1}$, is called a (nonlinear) *resolvent* of f. It is clearly *nonexpansive*, that is, 1-Lipschitz:

$$\left|J_rx - J_ry\right| \leq |x-y|, \quad x,y \in C. \tag{1.8}$$

As a matter of fact, this resolvent is even *firmly nonexpansive*:

$$\left|J_rx - J_ry\right| \leq \left|J_rx - J_ry + s(x - J_rx - (y - J_ry))\right| \tag{1.9}$$

for all x and y in C and for all positive s.

This is a direct consequence of (1.6) because $x - J_rx = rf(J_rx)$ and $y - J_ry = rf(J_ry)$ for all x and y in C. We remark in passing that, conversely, each firmly nonexpansive mapping is a resolvent of a (possibly set-valued) monotone operator. To see this, let $T : C \to C$ be firmly nonexpansive. Then the operator

$$M := \bigcup \{[Tx, x - Tx] : x \in C\} \tag{1.10}$$

is monotone because T satisfies (1.9).

We now turn to the concept of *hyperbolic monotonicity* which was introduced in [19, page 244]; there it was called *ρ-monotonicity*. In the present paper we will use both terms interchangeably.

We say that a mapping $f : \mathbb{B} \to H$ is *ρ-monotone* on $\mathbb{B}$ if for each pair of points $(x,y) \in \mathbb{B} \times \mathbb{B}$,

$$\rho(x,y) \leq \rho(x+rf(x), y+rf(y)) \tag{1.11}$$

for all $r > 0$ such that the points $x+rf(x)$ and $y+rf(y)$ belong to $\mathbb{B}$.

We say that $f : \mathbb{B} \to H$ satisfies the *range condition* if

$$(I + rf)(\mathbb{B}) \supset \mathbb{B}, \quad r > 0. \tag{1.12}$$

If a ρ-monotone f satisfies the range condition (1.12), then for each $r > 0$, the resolvent $J_r := (I + rf)^{-1}$ is a single-valued, ρ-nonexpansive self-mapping of $\mathbb{B}$. As a matter of fact, this resolvent is *firmly nonexpansive of the second kind* in the sense of [5, page 129] (see Lemma 4.2 below). We remark in passing that this resolvent is different from the one introduced in [17] which is *firmly nonexpansive of the first kind* [5, page 124].

Our first aim in this note is to establish the following characterization of ρ-monotone mappings. Recall that a subset of $\mathbb{B}$ is said to lie strictly inside $\mathbb{B}$ if its distance from the boundary of $\mathbb{B}$ (the unit sphere of H) is positive.

Theorem 1.1. *Let $\mathbb{B}$ be the open unit ball in a complex Hilbert space H, and let $f : \mathbb{B} \to H$ be a continuous mapping which is bounded on each subset strictly inside $\mathbb{B}$ (equivalently, on each ρ-ball). Then f is ρ-monotone if and only if for each $r > 0$, its resolvent $J_r := (I + rf)^{-1}$ is a single-valued, ρ-nonexpansive self-mapping of $\mathbb{B}$.*

This result shows that in some cases the hyperbolic monotonicity of $f : \mathbb{B} \to H$ already implies the range condition (1.12). This is in analogy with the Euclidean Hilbert space case, where it is known that if $f : H \to H$ is continuous and monotone, then the range $R(I + rf) = H$ for all $r > 0$. To see this, we may first note that a continuous and monotone $f : H \to H$ is maximal monotone and then invoke Minty's classical theorem [11] to conclude that $R(I + rf)$ is indeed all of H for all positive r.

However, as pointed out on [14, page 393], Minty's theorem is equivalent to the Kirszbraun-Valentine extension theorem which is no longer valid, generally speaking, outside Hilbert space, or for the Hilbert ball of dimension larger than 1 [8, 9]. On the other hand, it is known [10] that if E is any Banach space and $f : E \to E$ is continuous and accretive, then f is m-accretive, that is, $R(I + rf) = E$ for all $r > 0$.

Our proof of Theorem 1.1 uses finite dimensional projections. The separable case is due to Itai Shafrir (see [19, Theorem 2.3]). This proof is presented in Section 3, which also contains a discussion of continuous semigroups of holomorphic mappings and their (infinitesimal) generators (see Corollary 3.2). It is preceded by three preliminary results in Section 2. In Section 4, the last section of our note, we study the asymptotic behavior of compositions and convex combinations of resolvents of ρ-monotone mappings (see Theorems 4.14 and 4.15). Theorem 4.14, in particular, provides two methods for finding a common null point of finitely many (continuous) ρ-monotone mappings.

2. Preliminaries

We precede the proof of Theorem 1.1 with the following three preliminary results.

Given $z \in \mathbb{B}$, let $\{u_\alpha : \alpha \in \mathcal{A}\}$ be a complete orthonormal system in H which contains $z/|z|$ if $z \neq 0$. Let Γ be the set of all finite dimensional subspaces of H which contain z and are spanned by vectors from $\{u_\alpha : \alpha \in \mathcal{A}\}$, ordered by containment. For each $F \in \Gamma$, let $P_F : H \to F$ be the orthogonal projection of H onto F.

Lemma 2.1. *For each $y \in H$, the net $\{P_F y\}_{F \in \Gamma}$ converges to y.*

Proof. Let $y = \sum_{i=1}^{\infty} \langle y, u_{\alpha_i} \rangle u_{\alpha_i}$ and let $\epsilon > 0$. There is $N = N(\epsilon)$ such that if $n \geq N$, then

$$\left| \sum_{i=1}^{n} \langle y, u_{\alpha_i} \rangle u_{\alpha_i} - y \right|^2 = \left| \sum_{i=n+1}^{\infty} \langle y, u_{\alpha_i} \rangle u_{\alpha_i} \right|^2 = \sum_{i=n+1}^{\infty} \left| \langle y, u_{\alpha_i} \rangle \right|^2 < \epsilon^2. \tag{2.1}$$

Let $F_0 := \operatorname{span}\{u_{\alpha_1}, u_{\alpha_2}, \dots, u_{\alpha_N}\}$.

If $F \in \Gamma$, $F \supset F_0$, and $F = \operatorname{span}\{u_{\alpha_1}, \dots, u_{\alpha_N}, v_1, \dots, v_m\}$, then $|P_F y - y|^2 = |\sum_{i=1}^{N} \langle y, u_{\alpha_i} \rangle u_{\alpha_i} + \sum_{j=1}^{m} \langle y, v_j \rangle v_j - y|^2$. If $\langle y, v_j \rangle \neq 0$, then $v_j \in \{u_{\alpha_i} : i \geq N+1\}$ and therefore $|P_F y - y|^2 \leq \sum_{i=N+1}^{\infty} |\langle y, u_{\alpha_i} \rangle|^2 < \epsilon^2$. □

Next, we recall a characterization ([19, Theorem 2.1]) of ρ-monotone mappings in terms of the inner product of H.

Proposition 2.2. *A mapping $f : \mathbb{B} \to H$ is ρ-monotone if and only if for each $x, y \in \mathbb{B}$,*

$$\frac{\operatorname{Re}\langle x, f(x) \rangle}{1 - |x|^2} + \frac{\operatorname{Re}\langle y, f(y) \rangle}{1 - |y|^2} \geq \operatorname{Re}\left\{ \frac{\langle y, f(x) \rangle + \langle x, f(y) \rangle}{1 - \langle x, y \rangle} \right\}. \tag{2.2}$$

Note that (2.2) is the hyperbolic analog of the Euclidean (1.5).

Finally, we recall a fixed point theorem which will be used in the proof of Theorem 1.1.

Let C be a subset of a vector space E and let the point x belong to C. Recall that the inward set $I_C(x)$ of x with respect to C is defined by

$$I_C(x) := \{z \in E : z = x + a(y - x) \text{ for some } y \in C,\ a \geq 0\}. \tag{2.3}$$

If E is a topological vector space, then a mapping $f : C \to E$ is said to be *weakly inward* if $f(x)$ belongs to the closure of $I_C(x)$ for each $x \in C$.

Theorem 2.3. *Let C be a nonempty, compact and convex subset of a locally convex, Hausdorff topological vector space E. If a continuous $f : C \to E$ is weakly inward, then it has a fixed point.*

This theorem is due to Halpern and Bergman [6]. A simple proof can be found in [13].

3. The range condition

We begin this section with the proof of Theorem 1.1.

Proof of Theorem 1.1. One direction is clear: if J_r is ρ-nonexpansive, and the points x, y, $x + rf(x)$, $y + rf(y)$ all belong to $\mathbb{B}$, then

$$\rho(x, y) = \rho(J_r(x + rf(x)), J_r(y + rf(y))) \leq \rho(x + rf(x), y + rf(y)). \tag{3.1}$$

Thus, it is enough to prove that if f is ρ-monotone, then for each $z \in \mathbb{B}$ and $r > 0$, there exists a solution $x \in \mathbb{B}$ to the equation $x + rf(x) = z$. Fix $z \in \mathbb{B}$ and consider the corresponding directed set Γ of finite dimensional subspaces of H.

For each $F \in \Gamma$, let $\mathbb{B}_F := \mathbb{B} \cap F$ and denote the composition $P_F \circ f$ by f_F. The (restricted) mapping $f_F : \mathbb{B}_F \to F$ is also ρ-monotone because when $x, y \in \mathbb{B}_F$, we have

$\langle x, P_F f(x)\rangle = \langle x, f(x)\rangle$ and $\langle y, P_F f(x)\rangle = \langle y, f(x)\rangle$, and f_F is seen to be ρ-monotone by the characterization (2.2).

Now we want to show that there is a point $w_F \in \mathbb{B}_F$ such that

$$w_F + r f_F(w_F) = z. \tag{3.2}$$

Indeed, consider the mapping $h : \mathbb{B}_F \to F$ defined by

$$h_F(x) := z - r f_F(x), \quad x \in \mathbb{B}_F. \tag{3.3}$$

Using (2.2) with $y = 0$, we get

$$\operatorname{Re}\langle f_F(x), x\rangle \geq (1 - |x|^2)\operatorname{Re}\langle x, f_F(0)\rangle \tag{3.4}$$

for all $x \in \mathbb{B}_F$. Hence

$$\operatorname{Re}\langle h_F(x), x\rangle = \operatorname{Re}\langle z, x\rangle - r\operatorname{Re}\langle f_F(x), x\rangle \leq |z||x| - r(1 - |x|^2)\operatorname{Re}\langle x, f_F(0)\rangle. \tag{3.5}$$

Since $|f_F(0)| = |P_F f(0)| \leq |f(0)|$, it follows that there is $|z| < s < 1$ (independent of F) such that $\operatorname{Re}\langle h_F(x), x\rangle \leq |x|^2$ for all $x \in F$ with $|x| = s$. Thus h_F is weakly inward on $\{x \in F : |x| \leq s\}$ by [12, Proposition 2] (alternatively, it satisfies the Leray-Schauder condition on $\{x \in F : |x| = s\}$) and therefore has a fixed point by Theorem 2.3. This fixed point $w_F \in \overline{\mathbb{B}_F(0,s)} \subset \overline{\mathbb{B}(0,s)}$ is a solution of (3.2).

Let $\{v_E : E \in \Delta\}$ be a subnet of $\{w_F : F \in \Gamma\}$ which converges weakly to $v \in \overline{\mathbb{B}(0,s)}$. We can assume that $\{|v_E|\}_{E\in\Delta}$ converges to t, with $|v| \leq t \leq s < 1$. Since f is bounded on $\mathbb{B}(0,s)$, we can also assume that $\{f(v_E)\}_{E\in\Delta}$ converges weakly to $p \in H$.

Our next claim is that $|v| = t$.

To see this, note first that

$$\langle v_E, y\rangle + \langle r g_E(v_E), y\rangle = \langle z, y\rangle \tag{3.6}$$

for all $E \in \Delta$ and $y \in H$, where $\{g_E\}_{E\in\Delta}$ is a subnet of $\{f_F\}_{F\in\Gamma}$.

Also, if $\varphi : \Delta \to \Gamma$ is the mapping associated with the subnet $\{v_E : E \in \Delta\}$, then $g_E = f_{\varphi(E)}$ and $\langle g_E(v_E), y\rangle = \langle f_{\varphi(E)}(v_E), y\rangle = \langle P_{\varphi(E)} f(v_E), y\rangle = \langle f(v_E), P_{\varphi(E)} y\rangle = \langle f(v_E), y\rangle + \langle f(v_E), P_{\varphi(E)} y - y\rangle \to \langle p, y\rangle$ because $\{f(v_E)\}_{E\in\Delta}$ is bounded and $\{P_E y\}_{E\in\Delta}$ converges to y by Lemma 2.1. Hence $\langle v, y\rangle + r\langle p, y\rangle = \langle z, y\rangle$ for all $y \in H$, and $v + rp = z$.

Writing (2.2) with $x := v$ and $y := v_E$, we see that

$$\begin{aligned} &\operatorname{Re}\Big\{\langle v, f(v)\rangle/(1 - |v|^2) + \langle v_E, f(v_E)\rangle/\Big(1 - |v_E|^2\Big)\Big\} \\ &\quad \geq \operatorname{Re}\{(\langle v, f(v_E)\rangle + \langle f(v), v_E\rangle)/(1 - \langle v, v_E\rangle)\}. \end{aligned} \tag{3.7}$$

Also, $\langle v_E, v_E\rangle + r\langle g_E(v_E), v_E\rangle = \langle z, v_E\rangle$. Hence (letting $Q_F = I - P_F$),

$$\begin{aligned} \langle v_E, f(v_E)\rangle &= \langle w_{\varphi(E)}, P_{\varphi(E)} f(v_E) + Q_{\varphi(E)} f(v_E)\rangle = \langle w_{\varphi(E)}, P_{\varphi(E)} f(v_E)\rangle \\ &= \langle v_E, g_E(v_E)\rangle = \Big(\langle v_E, z\rangle - |v_E|^2\Big)/r, \\ \operatorname{Re} r\langle v_E, f(v_E)\rangle &= \operatorname{Re}\Big\{\langle v_E, z\rangle - |v_E|^2\Big\}. \end{aligned} \tag{3.8}$$

Thus,

$$\begin{aligned}&\operatorname{Re}\left\{r\langle v,f(v)\rangle/(1-|v|^2)+\left(\langle v_E,z\rangle-|v_E|^2\right)/\left(1-|v_E|^2\right)\right\}\\&\quad\geq \operatorname{Re}\left\{\left(r\langle v,f(v_E)\rangle+r\langle f(v),v_E\rangle\right)/\left(1-\langle v,v_E\rangle\right)\right\}.\end{aligned}\tag{3.9}$$

Taking limits, we get

$$\begin{aligned}&\operatorname{Re}\left\{r\langle v,f(v)\rangle/(1-|v|^2)+(\langle v,z\rangle-t^2)/(1-t^2)\right\}\\&\quad\geq \operatorname{Re}\left\{\left(r\langle v,p\rangle+r\langle f(v),v\rangle\right)/(1-|v|^2)\right\}.\end{aligned}\tag{3.10}$$

Now $\langle v,v\rangle+r\langle p,v\rangle=\langle z,v\rangle$. Therefore,

$$\begin{aligned}&\operatorname{Re}\langle v,z\rangle/(1-t^2)-\frac{t^2}{1-t^2}\geq \operatorname{Re}\left\{\frac{\langle z,v\rangle-|v|^2}{1-|v|^2}\right\},\\&\operatorname{Re}\langle v,z\rangle\left\{\frac{1}{1-t^2}-\frac{1}{1-|v|^2}\right\}\geq\frac{t^2}{1-t^2}-\frac{|v|^2}{1-|v|^2}.\end{aligned}\tag{3.11}$$

If $|v|<t$, then this inequality yields $\operatorname{Re}\langle v,z\rangle\geq 1$. But $\operatorname{Re}\langle v,z\rangle\leq|v||z|\leq t\leq s<1$, a contradiction. Hence $|v|=t$, as claimed.

Since $\{v_E\}_{E\in\Delta}$ converges weakly to v and $\{|v_E|\}_{E\in\Delta}$ converges to $t=|v|$, $\{v_E\}_{E\in\Delta}$ converges strongly to v. Since f is continuous, $f(v_E)\to f(v)$ and $p=f(v)$. Hence $v+rf(v)=z$ and the proof is complete. □

Why is it important to know that in certain cases a ρ-monotone mapping already satisfies the range condition? To answer this question, let D be a domain (open, connected subset) in a complex Banach space X, and recall that a holomorphic mapping $f:D\to X$ is said to be a semi-complete vector field on D if the Cauchy problem

$$\begin{gathered}\frac{\partial u(t,z)}{\partial t}+f(u(t,z))=0\\u(0,z)=z\end{gathered}\tag{3.12}$$

has a unique global solution $\{u(t,z):t\geq 0\}\subset D$ for each $z\in D$. It is known (see, e.g., [1, 18]) that if a holomorphic $f:D\to X$ is semi-complete, then the family $S_f=\{F_t\}_{t\geq 0}$ defined by

$$F_t(z):=u(t,z),\quad t\geq 0,\ z\in D,\tag{3.13}$$

is a one-parameter (nonlinear) semigroup (semiflow) of holomorphic self-mappings of D, that is,

$$\begin{gathered}F_{t+s}=F_t\circ F_s,\quad t,s\geq 0,\\F_0=I,\end{gathered}\tag{3.14}$$

where I denotes the restriction of the identity operator on X to D. In addition,

$$\lim_{t\to 0^+}F_t(z)=z,\quad z\in D,\tag{3.15}$$

uniformly on each ball which is strictly inside D.

A semigroup $\{F_t\}_{t\geq 0}$ is said to be generated if, for each $z \in D$, there exists the strong limit

$$f(z) := \lim_{t\to 0^+} [z - F_t(z)]/t. \tag{3.16}$$

This mapping f is called the (infinitesimal) generator of the semigroup. It is, of course, a semi-complete vector field. Analogous definitions apply to (continuous) semigroups of ρ-nonexpansive mappings, where ρ is a pseudometric assigned to D by a Schwarz-Pick system [5, page 91].

When is a mapping $f : D \to X$ a generator? An answer to this question is provided by the following result [19, page 239]. Recall that if D is a convex domain, then all the pseudometrics assigned to D by Schwarz-Pick systems coincide. If D is also bounded, then this common pseudometric is, in fact, a metric, which we call the hyperbolic metric of D.

Theorem 3.1. *Let D be a bounded convex domain in a complex Banach space X, and let ρ denote its hyperbolic metric. Suppose that $f : D \to X$ is bounded and uniformly continuous on each ρ-ball in D. Then f is a generator of a ρ-nonexpansive semigroup on D if and only if, for each $r > 0$, the mapping $J_r := (I + rf)^{-1}$ is a well-defined ρ-nonexpansive self-mapping of D.*

If, in the setting of this theorem, $f : D \to X$ is a generator of a ρ-nonexpansive semigroup $\{F_t\}_{t\geq 0}$, then the following exponential formula holds:

$$F_t(z) = \lim_{n\to\infty} \left(I + \frac{t}{n} f\right)^{-n} z, \quad z \in D. \tag{3.17}$$

Combining Theorems 1.1 and 3.1, we obtain the following corollary.

Corollary 3.2. *Let $f : \mathbb{B} \to H$ be bounded and uniformly continuous on each ρ-ball in $\mathbb{B}$. Then f is the generator of a ρ-nonexpansive semigroup on $\mathbb{B}$ if and only if f is ρ-monotone.*

If follows from the Cauchy inequalities that this corollary applies, in particular, to holomorphic mappings which are bounded on each ρ-ball.

Note that all the mappings of the form $f = I - T$, where I is the identity operator and $T : \mathbb{B} \to \mathbb{B}$ is ρ-nonexpansive (in particular, holomorphic), are generators of semigroups of ρ-nonexpansive (resp., holomorphic) mappings. More applications of hyperbolic monotonicity and, in particular, of the characterizations provided by Proposition 2.2 and Corollary 3.2, can be found in [2].

4. Asymptotic behavior

In this section we study the asymptotic behavior of compositions and convex combinations of resolvents of ρ-monotone mappings.

Consider the function $\psi : [0, \delta] \to [0, \infty)$ defined by

$$\psi(t) := \sigma(x + tu, y + tv), \tag{4.1}$$

where x, y, u and v are any four points in $\mathbb{B}$ and $\delta > 0$ is sufficiently small. We begin by recalling [19, Lemma 2.2]. Note that ψ is differentiable at the origin by Lemma 2.1 there. See also [20, Proposition 4.3].

LEMMA 4.1. *Let the function ψ be defined by (4.1). Then the following are equivalent:*
(a) $\psi(t) \le \psi(0)$, $0 \le t \le \delta$;
(b) ψ *decreases on* $[0,\delta]$;
(c) $\psi'(0) \le 0$.

Let D be a subset of the Hilbert ball $\mathbb{B}$. Recall that a mapping $T : D \to \mathbb{B}$ is said to be *firmly nonexpansive of the second kind* [5, page 129] if the function $\varphi : [0,1] \to [0,\infty)$ defined by

$$\varphi(s) := \rho\big((1-s)x + sTx, (1-s)y + sTy\big), \quad 0 \le s \le 1, \tag{4.2}$$

is decreasing for all points x and y in D.

We denote the family of firmly nonexpansive mappings of the second kind by FN_2.

LEMMA 4.2. *Any resolvent of a ρ-monotone mapping is firmly nonexpansive of the second kind.*

Proof. Fix a positive r and let J_r be a resolvent of a ρ-monotone mapping $f : \mathbb{B} \to H$. Let x and y be any two points in the domain of J_r. To show that the function $\rho(tx + (1-t)J_r x, ty + (1-t)J_r y)$ increases on $[0,1]$, we have to show that the function $\psi : [0,1] \to [0,\infty)$ defined by

$$\psi(t) := \sigma\big(J_r x + t(x - J_r x), J_r y + t(y - J_r y)\big), \quad 0 \le t \le 1, \tag{4.3}$$

decreases on $[0,1]$. To this end, it suffices, according to Lemma 4.1, to check that $\psi(t) \le \psi(0)$ for all $0 \le t \le 1$.

Indeed, since f is ρ-monotone, $x - J_r x = rf(J_r x)$, and $y - J_r y = rf(J_r y)$, we know that, by (1.11),

$$\begin{aligned} \rho(J_r x, J_r y) &\le \rho\big(J_r x + sf(J_r x), J_r y + sf(J_r x)\big) \\ &= \rho\big(J_r x + (s/r)(x - J_r x), J_r y + (s/r)(y - J_r y)\big) \end{aligned} \tag{4.4}$$

for all $0 \le s \le r$. In other words,

$$\psi(0) = \sigma(J_r x, J_r y) \ge \sigma\big(J_r x + t(x - J_r x), J_r y + t(y - J_r y)\big) = \psi(t) \tag{4.5}$$

for all $0 \le t \le 1$, as required. □

We now turn to the class of strongly nonexpansive mappings.

Let $T : D \to \mathbb{B}$ be a ρ-nonexpansive mapping with a nonempty fixed point set $F(T)$. Recall that such a mapping is called *strongly nonexpansive* ([4, 16]) if for any ρ-bounded sequence $\{x_n : n = 1,2,3,\dots\} \subset D$ and every $y \in F(T)$, the condition $\rho(x_n, y) - \rho(Tx_n, y) \to 0$ implies that $\rho(x_n, Tx_n) \to 0$.

To define this concept for fixed point free mappings, we first recall two notations.

If the point b belongs to the boundary of $\mathbb{B}$, let the function $\varphi_b : \mathbb{B} \to (0,\infty)$ be defined by

$$\varphi_b(x) := \left|1 - \langle x, b\rangle\right|^2 / (1 - |x|^2), \tag{4.6}$$

and for positive r consider the ellipsoids $E(b,r) := \{x \in \mathbb{B} : \varphi_b(x) < r\}$.

Now we recall [5, page 126] that if a ρ-nonexpansive mapping $T : \mathbb{B} \to \mathbb{B}$ is fixed point free, then there exists a unique point $e = e(T)$ of norm one (the *sink point* of T) such that all the ellipsoids $E(e,r)$, $r > 0$, are invariant under T. We say that such a mapping is *strongly nonexpansive* if for any sequence $\{x_n : n = 1,2,\dots\} \subset \mathbb{B}$ such that $\{\varphi_e(x_n)\}$ is bounded, the condition $\varphi_e(x_n) - \varphi_e(Tx_n) \to 0$ implies that $x_n - Tx_n \to 0$.

Proofs of the following two lemmas can be found in [15].

Lemma 4.3. *Let $\{x_n\}$ and $\{z_n\}$ be two sequences in $\mathbb{B}$. Suppose that for some y in $\mathbb{B}$, $\limsup_{n\to\infty} \rho(x_n, y) \le M$, $\limsup_{n\to\infty} \rho(z_n, y) \le M$, and $\liminf_{n\to\infty} \rho((x_n + z_n)/2, y) \ge M$. Then $\lim_{n\to\infty} |x_n - z_n| = 0$.*

Lemma 4.4. *Let the point b belong to the boundary of $\mathbb{B}$, and let $\{x_n\}$ and $\{z_n\}$ be two sequences in $\mathbb{B}$. Suppose that $\limsup_{n\to\infty} \varphi_b(x_n) \le M$, $\limsup_{n\to\infty} \varphi_b(z_n) \le M$, and $\liminf_{n\to\infty} \varphi_b((x_n + z_n)/2) \ge M$. Then $\lim_{n\to\infty} |x_n - z_n| = 0$.*

Our interest in strongly nonexpansive mappings stems from the following two facts.

Lemma 4.5. *If a mapping $T \in FN_2$ has a fixed point, then it is strongly nonexpansive.*

Proof. Suppose that the sequence $\{x_n\}$ is ρ-bounded, $y \in F(T)$, and $\rho(x_n, y) - \rho(Tx_n, y) \to 0$. In order to prove that $\rho(x_n, Tx_n) \to 0$, we may assume without loss of generality that $\lim_{n\to\infty} \rho(x_n, y) = \lim_{n\to\infty} \rho(Tx_n, y) = d > 0$. Since $T \in FN_2$, we also have

$$\rho(Tx_n, y) \le \rho((x_n + Tx_n)/2, y) \le \rho(x_n, y). \tag{4.7}$$

Hence $\lim_{n\to\infty} \rho((x_n + Tx_n)/2, y) = d$, too. Now we can invoke Lemma 4.3 to conclude that $x_n - Tx_n \to 0$. Since $\{x_n\}$ is ρ-bounded, it follows that $\rho(x_n, Tx_n) \to 0$ as well. □

Lemma 4.6. *If a mapping $T : \mathbb{B} \to \mathbb{B}$ belongs to FN_2 and is fixed point free, then it is strongly nonexpansive.*

Proof. Let e be the sink point of T and let $\{x_n : n = 1,2,\dots\} \subset \mathbb{B}$ be a sequence such that $\{\varphi_e(x_n)\}$ is bounded and $\varphi_e(x_n) - \varphi_e(Tx_n) \to 0$. In order to prove that $x_n - Tx_n \to 0$, we may assume that $\varphi_e(x_n) \to M$. Hence $\varphi_e(Tx_n) \to M$, too. Since $T \in FN_2$, we know by [5, Lemma 30.7 on page 142] that the function $g : [0,1] \to (0,\infty)$ defined by

$$g(s) := \varphi_e((1-s)x + sTx), \quad 0 \le s \le 1, \tag{4.8}$$

is decreasing for each $x \in \mathbb{B}$. Hence

$$\varphi_e(Tx_n) \le \varphi_e\left(\frac{x_n + Tx_n}{2}\right) \le \varphi_e(x_n) \tag{4.9}$$

for each $n = 1,2,\dots$.

Thus $\lim_{n\to\infty}\varphi_e((x_n+Tx_n)/2)=M$, too, and hence $\lim_{n\to\infty}(x_n-Tx_n)=0$ by Lemma 4.4. □

Next, we recall [16] the following weak convergence result.

Proposition 4.7. *If $T:\mathbb{B}\to\mathbb{B}$ has a fixed point and is strongly nonexpansive, then for each point x in $\mathbb{B}$, the sequence of iterates $\{T^n x\}$ converges weakly to a fixed point of T.*

In view of Lemma 4.5, this result applies, in particular, to all those mappings $T:\mathbb{B}\to\mathbb{B}$ in FN_2 which have a fixed point.

It follows from [8, 9] that in the setting of Proposition 4.7, strong convergence does not hold in general. However, our next result shows that when a strongly nonexpansive mapping is fixed point free, its iterates do converge strongly.

Proposition 4.8. *If $T:\mathbb{B}\to\mathbb{B}$ is strongly nonexpansive and fixed point free, then for each point x in $\mathbb{B}$, the sequence of iterates $\{T^n x\}$ converges strongly to the sink point of T.*

Proof. Let e be the sink point of T and denote $T^n x$ by x_n, $n=1,2,\ldots$. Since $\varphi_e(Tx)\le\varphi_e(x)$ for all $x\in\mathbb{B}$, the sequences $\{\varphi_e(x_n)\}$ and $\{\varphi_e(Tx_n)\}$ decrease to the same limit M. Since T is strongly nonexpansive, it follows that $x_n-Tx_n\to 0$. Since T is fixed point free, this implies that $\{x_n\}$ cannot have a ρ-bounded subsequence. Thus $\lim_{n\to\infty}|x_n|=1$, $\langle x_n,e\rangle\to 1$, and $x_n\to e$, as asserted. □

Now we consider compositions and convex combinations of strongly nonexpansive mappings.

The following result is proved in [16].

Lemma 4.9. *Let the mappings $T_j:\mathbb{B}\to\mathbb{B}$, $1\le j\le m$, be strongly nonexpansive, and let $T=T_mT_{m-1}\cdots T_1$. If $F=\cap\{F(T_j):1\le j\le m\}$ is not empty, then $F=F(T)$ and T is also strongly nonexpansive.*

Here is an analog of this result for the fixed point free case.

Lemma 4.10. *If the fixed point free mappings $T_j:\mathbb{B}\to\mathbb{B}$, $1\le j\le m$, have a common sink point and are strongly nonexpansive, then $T=T_mT_{m-1}\cdots T_1$ is also strongly nonexpansive.*

Proof. Let T_1 and T_2 be two fixed point free and strongly nonexpansive mappings with a common sink point $e=e(T_1)=e(T_2)$. We first note that the composition $T=T_2T_1$ is also fixed point free. Indeed, let $x\in\mathbb{B}$ and consider the iterates $x_n=T^n x$, $n=1,2,\ldots$. Since the decreasing sequence $\{\varphi_e(x_n)\}$ converges, we see that

$$0\le\varphi_e(x_n)-\varphi_e(T_1x_n)\le\varphi_e(x_n)-\varphi_e(Tx_n)\longrightarrow 0, \tag{4.10}$$

and therefore $x_n-T_1x_n\to 0$.

If $\{x_n\}$ were ρ-bounded, then its asymptotic center [5, page 116] would be a fixed point of T_1. Hence $\{x_n\}$ is ρ-unbounded and T is fixed point free, as claimed. Thus $e=e(T)$ is also the sink point of T. To show that T is strongly nonexpansive, let $\{x_n\}\subset\mathbb{B}$ be a sequence such that $\{\varphi_e(x_n)\}$ is bounded and $\varphi_e(x_n) - \varphi_e(Tx_n) \to 0$. Then

$$\begin{aligned} 0 &\le \varphi_e(x_n) - \varphi_e(T_1 x_n) \le \varphi_e(x_n) - \varphi_e(T_2 T_1 x_n), \\ 0 &\le \varphi_e(T_1 x_n) - \varphi_e(T_2 T_1 x_n) \le \varphi_e(x_n) - \varphi_e(T_2 T_1 x_n). \end{aligned} \tag{4.11}$$

Hence

$$\lim_{n\to\infty} (x_n - T_1 x_n) = \lim_{n\to\infty} (T_1 x_n - T_2 T_1 x_n) = 0, \tag{4.12}$$

and so $\lim_{n\to\infty}(x_n - T_2 T_1 x_n) = 0$, too.

The proof can now be completed by using induction on m. □

Turning to convex combinations, we first note the following fact. It is a consequence of [4, Theorem 9.5 (ii)].

Lemma 4.11. *Let the mappings $T_j : \mathbb{B} \to \mathbb{B}$, $1 \le j \le m$, be strongly nonexpansive, and let $T = \sum_{j=1}^m \lambda_j T_j$, where $0 < \lambda_j < 1$ and $\sum_{j=1}^m \lambda_j = 1$. If*

$$F = \cap\{F(T_j) : 1 \le j \le m\} \tag{4.13}$$

is not empty, then $F = F(T)$ and T is also strongly nonexpansive.

We now formulate an analog of this fact for the fixed point free case.

Lemma 4.12. *If the fixed point free mappings $T_j : \mathbb{B} \to \mathbb{B}$, $1 \le j \le m$, have a common sink point and are strongly nonexpansive, then $T = \sum_{j=1}^m \lambda_j T_j$, where $0 < \lambda_j < 1$ and $\sum_{j=1}^m \lambda_j = 1$, is also strongly nonexpansive.*

Proof. Once again, let T_1 and T_2 be two fixed point free and strongly nonexpansive mappings with a common sink point $e = e(T_1) = e(T_2)$. We claim that the convex combination $T = \lambda_1 T_1 + \lambda_2 T_2$, where $0 < \lambda_1$, $\lambda_2 < 1$ and $\lambda_1 + \lambda_2 = 1$, is also fixed point free. To see this, let $x \in \mathbb{B}$ and consider the iterates $x_n = T^n x$, $n = 1, 2, \ldots$. Note that $\varphi_e(x_n) - \varphi_e(Tx_n) \to 0$ because the decreasing sequence $\{\varphi_e(x_n)\}$ is convergent. Assume that $\{x_n\}$ has a ρ-bounded subsequence. Passing to a further subsequence and relabeling, if necessary, we may assume without loss of generality that

$$\varphi_e(T_1 x_n) = \max\{\varphi_e(T_1 x_n), \varphi_e(T_2 x_n)\}. \tag{4.14}$$

Since all the ellipsoids $E(e,r)$ are convex, it follows that $\varphi_e(Tx_n) \le \varphi_e(T_1 x_n)$ and therefore

$$0 \le \varphi(x_n) - \varphi_e(T_1 x_n) \le \varphi_e(x_n) - \varphi_e(Tx_n) \longrightarrow 0. \tag{4.15}$$

Thus $x_n - T_1 x_n \to 0$ and the asymptotic center of $\{x_n\}$ is a fixed point of T_1, a contradiction. Hence $\{x_n\}$ does not have a ρ-bounded subsequence, T is fixed point free, as asserted, and $e = e(T)$ is also the sink point of T.

To show that T is strongly nonexpansive, let $\{x_n\} \subset \mathbb{B}$ be a sequence such that $\{\varphi_e(x_n)\}$ is bounded and $\varphi_e(x_n) - \varphi_e(Tx_n) \to 0$. We have to show that $x_n - Tx_n \to 0$. If this were false, we would obtain by passing to subsequences and relabeling (if necessary), numbers

$\varepsilon > 0$ and $M \geq 0$ such that

$$\begin{gathered} |x_n - Tx_n| \geq \varepsilon, \quad n = 1,2,\dots \\ \varphi_e(T_1x_n) = \max\{\varphi_e(T_1x_n), \varphi_e(T_2x_n)\}, \quad n = 1,2,\dots, \\ \varphi_e(x_n) \longrightarrow M \quad \text{as } n \longrightarrow \infty. \end{gathered} \tag{4.16}$$

Since T_1 is strongly nonexpansive and

$$0 \leq \varphi_e(x_n) - \varphi_e(T_1x_n) \leq \varphi(x_n) - \varphi_e(Tx_n), \tag{4.17}$$

we also see that

$$\lim_{n\to\infty} \varphi_e(T_1x_n) = \lim_{n\to\infty} \varphi_e(Tx_n) = M \tag{4.18}$$

and that

$$\lim_{n\to\infty} (x_n - T_1x_n) = 0. \tag{4.19}$$

Consider now the two sequences $\{u_n\}$ and $\{v_n\}$ determined by the following properties:

$$\begin{gathered} u_n \in \operatorname{co}\{T_1x_n, Tx_n\}, \qquad v_n \in \operatorname{co}\{Tx_n, T_2x_n\}, \\ |u_n - Tx_n| = |v_n - Tx_n| = \min\{|T_1x_n - Tx_n|, |T_2x_n - Tx_n|\}. \end{gathered} \tag{4.20}$$

Then $(u_n + v_n)/2 = Tx_n$ and

$$|T_1x_n - T_2x_n| = |u_n - v_n|/(2\min\{\lambda_1,\lambda_2\}). \tag{4.21}$$

We have

$$\begin{gathered} \varphi_e(u_n) \leq \max\{\varphi_e(T_1x_n), \varphi_e(Tx_n)\} = \varphi_e(T_1x_n), \\ \varphi_e(v_n) \leq \max\{\varphi_e(T_2x_n), \varphi_e(Tx_n)\} \leq \varphi_e(T_1x_n) \end{gathered} \tag{4.22}$$

for all n.

Thus

$$\limsup_{n\to\infty} \varphi_e(u_n) \leq M, \qquad \limsup_{n\to\infty} \varphi_e(v_n) \leq M, \qquad \lim_{n\to\infty} \varphi_e((u_n+v_n)/2) = M. \tag{4.23}$$

Lemma 4.4 now implies that $\lim_{n\to\infty}(u_n - v_n) = 0$. Hence (see (4.19))

$$\lim_{n\to\infty} (T_1x_n - T_2x_n) = 0, \qquad \lim_{n\to\infty} (x_n - T_2x_n) = 0, \qquad \lim_{n\to\infty} (x_n - Tx_n) = 0, \tag{4.24}$$

a contradiction. Thus T is indeed strongly nonexpansive.

The proof can now be finished by using induction on m. □

We continue with a known fact [7].

Lemma 4.13. *Let $\{x_n\}$ and $\{y_n\}$ be two ρ-bounded sequences in $\mathbb{B}$. If $\{x_n\}$ converges weakly to x and $\{y_n\}$ converges weakly to y, then*

$$\rho(x,y) \le \liminf_{n\to\infty} \rho(x_n, y_n). \tag{4.25}$$

We are now ready to formulate and prove the main result of this section.

Theorem 4.14. *For each $1 \le j \le m$, let $f_j : \mathbb{B} \to H$ be a continuous ρ-monotone mapping which is bounded on each ρ-ball. Let r_j be positive and denote the resolvent $(I + r_j f_j)^{-1}$ of f_j by R_j. Furthermore, let $0 < \lambda_j < 1$ satisfy $\sum_{j=1}^m \lambda_j = 1$. If the common null point set*

$$Z := \cap\{f_j^{-1}(0) : 1 \le j \le m\} \tag{4.26}$$

of $\{f_j : 1 \le j \le m\}$ is not empty, then the weak $\lim_{n\to\infty}(R_m R_{m-1} \cdots R_1)^n x = P_1 x$ and the weak $\lim_{n\to\infty}(\sum_{j=1}^m \lambda_j R_j)^n x = P_2 x$ exist and define ρ-nonexpansive retractions of $\mathbb{B}$ onto Z.

Proof. For each $1 \le j \le m$, the resolvent R_j is well-defined on all of $\mathbb{B}$ by Theorem 1.1, and its fixed point set $F(R_j)$ coincides with the null point set $f_j^{-1}(0)$ of f_j. Furthermore, each R_j is firmly nonexpansive of the second kind by Lemma 4.2 and strongly nonexpansive by Lemma 4.5. The composition $R_m R_{m-1} \cdots R_1$ and the convex combination $\sum_{j=1}^m \lambda_j R_j$ are also strongly nonexpansive by Lemmas 4.9 and 4.11, respectively, and their fixed point sets coincide with Z. The existence of the limits $P_1 : \mathbb{B} \to Z$ and $P_2 : \mathbb{B} \to Z$ is now seen to follow from Proposition 4.7. Both P_1 and P_2 are ρ-nonexpansive retractions by Lemma 4.13. □

When a continuous ρ-monotone mapping $f : \mathbb{B} \to H$ is bounded on each ρ-ball and has no null point, then its resolvents $(I + rf)^{-1}$, $r > 0$, which are well-defined on all of $\mathbb{B}$ by Theorem 1.1, are fixed point free and all of them share the same sink point on the boundary $\partial\mathbb{B}$ of $\mathbb{B}$. (This follows from the resolvent identity.) We will refer to this point as the *sink point* of f.

Theorem 4.15. *For each $1 \le j \le m$, let $f_j : \mathbb{B} \to H$ be a continuous ρ-monotone mapping which is bounded on each ρ-ball and has no null point. Let r_j be positive and let $0 < \lambda_j < 1$ satisfy $\sum_{j=1}^m \lambda_j = 1$. Consider the resolvents $R_j = (I + r_j f)^{-1}$. If the mappings $\{f_j\}$ have a common sink point $e \in \partial\mathbb{B}$, then the strong $\lim_{n\to\infty}(R_m R_{m-1} \cdots R_1)^n x =$ the strong $\lim_{n\to\infty}(\sum_{j=1}^m \lambda_j R_j)^n x = e$.*

Proof. Each one of the resolvents $R_j : \mathbb{B} \to \mathbb{B}$, $1 \le j \le m$, is firmly nonexpansive of the second kind by Lemma 4.2 and strongly nonexpansive by Lemma 4.6.

The composition $R_m R_{m-1} \cdots R_1$ and the convex combination $\sum_{j=1}^m \lambda_j R_j$ are also strongly nonexpansive by Lemmas 4.10 and 4.12, respectively. The existence of the strong $\lim_{n\to\infty}(R_m R_{m-1} \cdots R_1)^n x$ and the strong $\lim_{n\to\infty}(\sum_{j=1}^m \lambda_j R_j)^n x$ is now seen to follow from Proposition 4.8. □

Theorems 4.14 and 4.15 provide certain Hilbert ball analogs of [3, Theorems 3.3 and 3.5]. These latter theorems are concerned with the asymptotic behavior of the composition of two resolvents of maximal monotone operators in Hilbert space.

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