NONEXPANSIVE MAPPINGS DEFINED ON UNBOUNDED DOMAINS

A. KAEWCHAROEN AND W. A. KIRK

Received 18 January 2006; Accepted 23 January 2006

We obtain fixed point theorems for nonexpansive mappings defined on unbounded sets. Our assumptions are weaker than the asymptotically contractive condition recently introduced by Jean-Paul Penot.

1. Introduction

In the study of nonexpansive mappings and fixed point theory the domain of the mapping is usually assumed to be bounded or, as in certain approximation results (see, e.g., [20]), fixed points are assumed to exist. However in [21] Penot used uniform asymptotic concepts which he had earlier introduced in [22] to extend the Browder-Göhde-Kirk theorem to unbounded sets. The term "asymptotic" is used in this context to describe the behavior of the mapping at infinity rather than the behavior of its iterates. Precisely, we have the following.

Definition 1.1. Let C be a subset of a Banach space X. A mapping $f : C \to X$ is said to be *asymptotically contractive* on C if there exists $x_0 \in C$ such that

$$\limsup_{x\in C,\|x\|\to\infty} \frac{\|f(x)-f(x_0)\|}{\|x-x_0\|} < 1. \tag{1.1}$$

As Penot observed, it is easy to see that this definition is independent of the choice of x_0.

Penot proved that if $f : C \to C$ is a nonexpansive and asymptotically contractive mapping defined on a closed convex subset C of a uniformly convex Banach space, then f has a fixed point. To prove this result he used the well-known fact that $I - f$ is demiclosed on C for nonexpansive f. Since mappings defined on bounded sets are vacuously asymptotically contractive, this result contains the Browder-Göhde-Kirk [3, 12, 14] result as a

Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2006, Article ID 82080, Pages 1–13
DOI 10.1155/FPTA/2006/82080

special case. However, as Penot himself observes, one can also deduce his result from the Browder-Göhde-Kirk result by applying the latter to a sufficiently large ball.

Among other things, we show here that demiclosedness of $I - f$ is not needed for Penot's result; in fact, a more general result holds under a weaker assumption on f. We then turn to the question of commuting families of nonexpansive mappings defined on unbounded domains. Finally, we consider non-self-mappings which satisfy Leray-Schauder-type boundary conditions on unbounded domains.

2. Basic results

We first show that a result more general than Penot's follows from three simple facts, the third of which is implicit in [14] (cf., proof of the corollary).

Let X be a Banach space with $C \subseteq X$. For a mapping $f : C \to X$ and $\delta > 0$, let

$$F_\delta(f) = \{x \in C : \|x - f(x)\| \leq \delta\}. \tag{2.1}$$

Lemma 2.1. *Suppose $f : C \to X$ is asymptotically contractive. Then for each $\delta > 0$, $F_\delta(f)$ is bounded.*

Proof. Suppose for some $\delta > 0$, $F_\delta(f)$ is nonempty and unbounded. Then there exists a sequence (x_n) in C such that $\|x_n - f(x_n)\| \leq \delta$ for every n, while $\|x_n\| \to \infty$ as $n \to \infty$. If x_0 is the point of Definition 1.1, we have

$$\|x_n - x_0\| \leq \|x_n - f(x_n)\| + \|f(x_n) - f(x_0)\| + \|f(x_0) - x_0\|. \tag{2.2}$$

Dividing both sides by $\|x_n - x_0\|$ and letting $n \to \infty$ leads to an obvious contradiction. □

Lemma 2.2. *Suppose C is a nonempty closed convex subset of X, and suppose $f : C \to C$ is nonexpansive. Suppose there exists $\delta > 0$ for which $F_\delta(f)$ is nonempty and bounded. Then there exists $p \in C$ such that $(f^n(p))$ is a bounded subset of C.*

Proof. Since f is nonexpansive, for $x \in F_\delta(f)$ we have

$$\|f(x) - f^2(x)\| \leq \|x - f(x)\| \leq \delta, \tag{2.3}$$

so $f : F_\delta(f) \to F_\delta(f)$. Thus $(f^n(x))$ is bounded for $x \in F_\delta(f)$. □

Lemma 2.3. *With C as above, suppose $f : C \to X$ is nonexpansive, and suppose $(f^n(p))$ is a bounded subset of C for some $p \in C$. Then there is a nonempty bounded closed convex subset K of X for which $f(K \cap C) \subseteq K$. In particular if $f : C \to C$, then there is a bounded closed convex subset of C which is mapped into itself by f.*

Proof. Let $S = (f^n(p))$ and choose $r > 0$ so that $S \subseteq B(p; r)$. Let

$$W = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} B(f^n(p); r). \tag{2.4}$$

Clearly $p \in W$, so $W \neq \emptyset$. If $x \in W \cap C$, then there exists $k \in \mathbb{N}$ such that $\|x - f^n(p)\| \leq r$ for all $n \geq k$. Hence $\|f(x) - f^n(p)\| \leq r$ for all $n \geq k+1$, and so $f : W \cap C \to W$. As the union of an ascending sequence of convex sets, W is convex, so we can take $K = \overline{W}$. □

A Banach space is said to have the FPP if each of its bounded closed convex subsets has the fixed point property for nonexpansive self-mappings.

We now have the following generalization of [21, Corollary 3].

Theorem 2.4. *Let X be a Banach space which has the FPP, let C be a closed convex subset of X, and suppose $f : C \to C$ is a nonexpansive mapping for which $F_\delta(f)$ is nonempty and bounded for some $\delta > 0$. Then f has a fixed point.*

Proof. By Lemma 2.2, $(f^n(p))$ is bounded for some $p \in C$, and by Lemma 2.3 some bounded closed convex subset of C is mapped into itself by f. □

In view of Lemma 2.1 we now have the following corollary.

Corollary 2.5. *Let X be a Banach space which has the FPP, let C be a closed convex subset of X, and suppose $f : C \to C$ is a nonexpansive mapping which is asymptotically contractive. Then f has a fixed point.*

Remark 2.6. The assumption that $F_\delta(f)$ is nonempty and bounded is properly weaker than the assumption that f is asymptotically contractive, even for nonexpansive mappings. For example, consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x - 1 & \text{if } x > 1, \\ 0 & \text{if } -1 \leq x \leq 1, \\ 1 - x & \text{if } x < -1. \end{cases} \tag{2.5}$$

Obviously $F_\delta(f)$ is nonempty and bounded for $\delta \in (0,1)$. On the other hand for $x_0 \in \mathbb{R}$,

$$\frac{|f(x) - f(x_0)|}{|x - x_0|} \longrightarrow 1 \quad \text{as } x \longrightarrow \infty. \tag{2.6}$$

As we have seen, if $f : C \to X$ is asymptotically contractive, then for each $\delta > 0$, $F_\delta(f)$ is bounded. It is natural to ask whether there is a similar but weaker asymptotic condition which only implies the existence of some $\delta > 0$ for which $F_\delta(f)$ is nonempty and bounded. For this it seems to be sufficient to assume there exists $x_0 \in C$ such that

$$\inf_{\delta > 0} \left\{ \limsup_{\|x\| \to \infty, x \in C \cap F_\delta(f)} \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} \right\} < 1. \tag{2.7}$$

This condition is also independent of the choice of $x_0 \in C$.

The preceding observations also yield an extension of Luc [18, Theorem 5.1]. For this we need some definitions. A set C is said to be *asymptotically compact* (see, e.g., [19]) if for any sequence (x_n) in C for which $(\|x_n\|) \to \infty$, the sequence $(x_n/\|x_n\|)$ has a convergent subsequence. If C is asymptotically compact it is possible to weaken the asymptotic condition imposed on C. A mapping $f : C \to C$ is said to be *radially asymptotically contractive* [18] if for some $x_0 \in C$ and for any u in the asymptotic cone

$$C_\infty := \limsup_{t\to\infty} t^{-1}C := \{v \in X : \exists (t_n) \longrightarrow \infty,\ (v_n) \longrightarrow v,\ t_n v_n \in C\ \forall n\} \tag{2.8}$$

of C, one has

$$\limsup_{t\to\infty, x_0+tu\in C} \frac{1}{t}||f(x_0+tu) - f(x_0)|| < 1. \tag{2.9}$$

In [21] it is shown that if C is asymptotically compact, then any radially asymptotically contractive $f : C \to C$ which is nonexpansive is asymptotically contractive. Thus the following is a consequence of Corollary 2.5.

THEOREM 2.7. *Let X be a Banach space which has the FPP. Let C be an asymptotically compact closed convex subset of X, and let $f : C \to C$ be a nonexpansive mapping which is radially asymptotically contractive on C. Then f has a fixed point in C.*

By using Corollary 2.5 above instead of [21, Corollary 3] one sees immediately that [21, Theorem 12] also extends to Banach spaces which have the FPP.

3. Families of nonexpansive mappings

We now take up the question of common fixed points for families of nonexpansive mappings defined on unbounded domains, beginning with a generalization of Lemma 2.3.

THEOREM 3.1. *Let C be a closed convex subset of a Banach space X, let $\mathfrak{F}$ be a finite commuting family of nonexpansive self-mappings of C, and suppose $(f^n(p))$ is bounded for some $p \in C$ and all $f \in \mathfrak{F}$. Then there is a nonempty bounded closed convex subset of C which is mapped into itself by each member of $\mathfrak{F}$.*

Proof. We prove the theorem in the case $\mathfrak{F} = \{f,g\}$. The general case follows by induction.

First observe that $(f^n \circ g^m(p))_{n,m=1}^\infty$ is bounded. Hence there exists $r > 0$ such that

$$f^n \circ g^m(p) \in B(p;r) \tag{3.1}$$

for all $m,n \in \mathbb{N}$. Now let

$$S_{n,m} := \{u \in C : ||u - f^i \circ g^j(p)|| \le r\ \forall i \ge n,\ j \ge m\}, \tag{3.2}$$

and let

$$S = \bigcup_{n,m=1}^{\infty} S_{n,m}. \tag{3.3}$$

Since each of the sets $S_{n,m}$ is convex and since the family $(S_{n,m})_{n,m=1}^{\infty}$ is directed upward by $\subset$, S is convex. Also, if $u \in S_{n,m}$, then $f(u) \in S_{n+1,m}$ and $g(u) \in S_{n,m+1}$. Therefore S is invariant under both f and g. It follows that $\overline{S}$ is bounded, closed, convex, and invariant under both f and g. □

The preceding theorem shows that for mappings with bounded iterates, the question of the existence of common fixed points for a finite commuting family of nonexpansive mappings reduces to the bounded case. In particular, it shows that the assumption of strict convexity is not needed in [7, Theorem 4]. In fact, we show below that if C is locally weakly compact, it suffices to assume that only one of the mappings has a bounded orbit.

Bula [7] has observed that Theorem 3.1 does not hold for infinite families. In this case we need the stronger assumption of Lemma 2.2, namely that an approximate fixed point set is bounded.

Theorem 3.2. *Let C be a closed convex locally weakly compact subset of a Banach space X, and suppose the bounded closed convex subsets of C have the fixed point property for nonexpansive self-mappings. Let $\mathfrak{F} := \{f_\alpha\}_{\alpha\in I}$ be a family of commuting nonexpansive self-mappings of C, and suppose $F_\delta(f_\alpha)$ is nonempty and bounded for some $\alpha \in I$ and $\delta > 0$. Then the common fixed point set of $\mathfrak{F}$ is a nonempty nonexpansive retract of some bounded closed convex subset of C.*

Corollary 3.3. *Under the assumptions of the above theorem, the common fixed point set of $\mathfrak{F}$ is a nonempty nonexpansive retract of some bounded closed convex subset of C whenever one member of $\mathfrak{F}$ is an asymptotic contraction.*

Theorem 3.2 parallels a corresponding result due to R. E. Bruck in the bounded case, and it relies heavily on results of Bruck.

A subset C of a Banach space has the fixed point property for nonexpansive mappings (abbreviated FPP) if every nonexpansive $f : C \to C$ has a fixed point, and C has the conditional fixed point property for nonexpansive self-mappings (abbreviated CFPP) if every nonexpansive $f : C \to C$ satisfies CFP: either f has no fixed points in C, or f has a fixed point in every nonempty bounded closed convex f-invariant subset of C. We use $\mathrm{Fix}(f)$ to denote the fixed point set of a mapping f.

We will need the following results.

Theorem 3.4 [5]. *If C is a closed convex locally weakly compact subset of a Banach space X, and if $f : C \to C$ is nonexpansive and satisfies CFP, then* $\mathrm{Fix}(f)$ *is a nonexpansive retract of C.*

Lemma 3.5 [6]. *Suppose C is a closed convex weakly compact subset of a Banach space X, and suppose C has both the FPP and CFPP. Then if $\mathfrak{R}$ is any family of nonempty nonexpansive retracts of C which is directed downward by $\supset$, $\bigcap\{R : R \in \mathfrak{R}\}$ is a nonempty nonexpansive retract of C.*

Proof of Theorem 3.2. Suppose $F_\delta(f_\alpha)$ is nonempty and bounded for $\delta > 0$. Then $(f_\alpha^n(p))$ is bounded for $p \in F_\delta(f_\alpha)$, so by Lemma 2.3 there is a nonempty bounded closed convex subset H of C such that $f_\alpha(H) \subset H$. Since H has the FPP, $F = \mathrm{Fix}(f_\alpha)$ is a nonempty subset of H. By Theorem 3.4 there exists a nonexpansive retraction r of H onto F. Since

$\mathfrak{F}$ is a commutative family, $f_\beta : \mathrm{Fix}(f_\gamma) \to \mathrm{Fix}(f_\gamma)$ for all $\beta, \gamma \in I$. In particular for $\beta \in I$ we have $f_\beta \circ r : H \to F$. Since by assumption H has the FPP, $\mathrm{Fix}(f_\beta \circ r) = \mathrm{Fix}(f_\alpha) \cap \mathrm{Fix}(f_\beta) \neq \emptyset$. Moreover $\mathrm{Fix}(f_\beta \circ r)$ is a nonexpansive retract of H. Now suppose $F \cap (\bigcap_{\beta \in J} \mathrm{Fix}(f_\beta))$ is a nonempty nonexpansive retract of H whenever $|J| = n$, and suppose $J = \{\beta_1, \dots, \beta_{n+1}\}$. By assumption there exists a nonexpansive retraction r of H onto $G := F \cap (\bigcap_{i=1}^n \mathrm{Fix}(f_{\beta_i}))$, and by commutativity, $f_{\beta_{n+1}} : G \to G$. Thus $f_{\beta_{n+1}} \circ r : H \to G$, and we conclude that $\mathrm{Fix}(f_{\beta_{n+1}} \circ r) = \mathrm{Fix}(f_{\beta_{n+1}}) \cap G$ is a nonempty nonexpansive retract of H. However

$$\mathrm{Fix}\left(f_{\beta_{n+1}}\right) \cap G = F \cap \left(\bigcap_{i=1}^{n+1} \mathrm{Fix}\left(f_{\beta_i}\right) \right). \tag{3.4}$$

By induction we conclude that the fixed point set of every finite subfamily of $\mathfrak{F}$ is a nonempty nonexpansive retract of H. Lemma 3.5 now implies that the common fixed point set of $\mathfrak{F}$ is a nonempty nonexpansive retract of H. □

Remark 3.6. The preceding argument shows that the common fixed point set of $\mathfrak{F}$ is a nonempty nonexpansive retract of any bounded closed convex set which is left invariant by some $f \in \mathfrak{F}$. The question remains whether it is a nonexpansive retract of C itself.

Remark 3.7. If the space X in Theorem 3.4 is assumed to be uniformly smooth, then $\mathrm{Fix}(f)$ is a sunny nonexpansive retract of C (see [11, Theorem 13.2]). (A retraction R from C onto $E \subset C$ is said to be *sunny* if

$$R(R(x) + t(x - R(x))) = R(x) \tag{3.5}$$

for all $x \in C$ and $t \geq 0$ for which $R(x) + t(x - R(x)) \in C$.) In their recent paper [2] (for an update, see [1]), Aleyner and Reich show that under certain assumptions there is an explicit algorithmic scheme for constructing the unique sunny nonexpansive retraction onto the common fixed point set of a nonlinear semigroup of nonexpansive mappings.

It seems unlikely that the boundedness assumption on $F_\delta(f_\alpha)$ in Theorem 3.2 could be replaced by the assumption that $(f_\alpha^n(p))$ is bounded for some $p \in C$ and $\alpha \in I$. However the following is true.

Theorem 3.8. *Let C be a closed convex locally weakly compact subset of a Banach space X, and suppose the bounded closed convex subsets of C have the fixed point property for nonexpansive self-mappings. Let $\mathfrak{F} := \{f_\alpha\}_{\alpha \in I}$ be a family of commuting nonexpansive self-mappings of C, and suppose $(f_\alpha^n(p))$ is bounded for some (hence all) $p \in C$ and all $\alpha \in I$. Then the common fixed point set of any finite subfamily of $\mathfrak{F}$ is a nonempty nonexpansive retract of C.*

Proof. Let $\alpha \in I$. By Lemma 2.3 some bounded closed convex subset of C is mapped into itself by f_α, so $F := \mathrm{Fix}(f_\alpha) \neq \emptyset$. By Theorem 3.4 there is a nonexpansive retraction r of C onto F. Now let $\beta \in I$ and consider the mapping $f_\beta \circ r$. Given $p \in C$, $(f_\beta \circ r)^n(p) = f_\beta^n \circ r(p)$ is bounded, so again by Lemma 2.3 some bounded closed convex subset is mapped into itself by $f_\beta \circ r$. It follows that $\mathrm{Fix}(f_\beta \circ r) \neq \emptyset$, and also that $\mathrm{Fix}(f_\beta \circ r)$ is a nonexpansive retract of C. However, since $f_\beta : F \to F$ and $r : C \to F$, if $f_\beta \circ r(x) = x$, then $f_\beta \circ r(x) = f_\beta(x)$. Therefore $\text{Fix}(f_\beta \circ r) = \text{Fix}(f_\alpha) \cap \text{Fix}(f_\beta)$. The conclusion follows by induction. □

4. Boundary conditions

Several fixed point theorems for nonexpansive mappings involve mappings $f : C \to X$ in conjunction with boundary and inwardness conditions. It is customary in these results to assume that the domain C is bounded. In this section we show that this assumption can be replaced with the assumptions of Lemmas 2.2 and 2.3.

The following theorem was proved in [15].

Theorem 4.1 [15]. *Let C be a bounded closed convex subset of a Banach space X, with $\text{int}(C) \neq \emptyset$, and suppose C has the fixed point property for nonexpansive self-mappings. Suppose $f : C \to X$ is nonexpansive, and suppose*

(i) *there exists $w \in \text{int}(C)$ such that*

$$f(x) - w \neq \lambda(x - w) \quad \forall x \in \partial C,\ \lambda > 1; \tag{4.1}$$

(ii) $\inf\{\|x - f(x)\| : x \in \partial C$ *and* $f(x) \notin C\} > 0$.

Then f has a fixed point.

Theorem 4.2. *Suppose C is a closed convex subset of a Banach space X, with $\text{int}(C) \neq \emptyset$, and suppose the bounded closed convex subsets of X have the fixed point property for nonexpansive self-mappings. Suppose $f : C \to X$ is a nonexpansive mapping for which $F_\delta(f)$ is nonempty and bounded for some $\delta > 0$. Suppose also*

(i) *there exists $w \in F_\delta(f) \bigcap \text{int}(C)$ such that*

$$f(x) - w \neq \lambda(x - w) \quad \forall x \in \partial C,\ \lambda > 1; \tag{4.2}$$

(ii) $\inf\{\|x - f(x)\| : x \in \partial C$ *and* $f(x) \notin C\} > 0$.

Then f has a fixed point.

Proof. Assume f does not have a fixed point. Since $F_\delta(f)$ is bounded, it is possible to choose n so large that $\|x - f(x)\| > \delta$ if $x \in C$ and $\|x - w\| \geq n$. Let $H_n = B(w;n) \bigcap C$. We now have

$$\inf\{\|x - f(x)\| : x \in \partial H_n,\ f(x) \notin H_n\} > 0, \tag{4.3}$$

so by Theorem 4.1 there exists $x \in \partial H_n$ such that

$$f(x) - w = \lambda(x - w) \quad \text{for some } \lambda > 1. \tag{4.4}$$

By (i) it must be the case that $\|x - w\| = n$; hence $\|x - f(x)\| > \delta$. We now have

$$\|x - f(x)\| = \|f(x) - w\| - \|x - w\| = \lambda n - n = (\lambda - 1)n. \tag{4.5}$$

Using the triangle inequality and the fact that $\|w - f(w)\| \leq \delta$ we have

$$\lambda n = ||f(x) - w|| \leq ||f(x) - f(w)|| + ||f(w) - w|| \leq \|x - w\| + \delta = n + \delta. \tag{4.6}$$

Therefore we have the contradiction

$$\delta < (\lambda - 1)n \leq n + \delta - n = \delta. \tag{4.7}$$

It follows that f has a fixed point. □

Corollary 4.3. *Suppose C is a closed convex subset of a Banach space X, with $\operatorname{int}(C) \neq \emptyset$, and suppose the bounded closed convex subsets of C have the fixed point property for nonexpansive self-mappings. Suppose $f : C \to X$ is a nonexpansive mapping which is also asymptotically contractive, and suppose*

(i) *there exists $w \in \operatorname{int}(C)$ such that*

$$f(x) - w \neq \lambda(x - w) \quad \forall x \in \partial C,\ \lambda > 1; \tag{4.8}$$

(ii) $\inf\{\|x - f(x)\| : x \in \partial C \text{ and } f(x) \notin C\} > 0$.

Then f has a fixed point.

Proof. By Lemma 2.1 $F_\delta(f)$ is bounded for each $\delta > 0$, and $w \in F_\delta(f)$ for $\delta = \|w - f(w)\|$. □

If X is uniformly convex, Condition (ii) of Theorem 4.2 may be dropped. This is a consequence of the following special case of a result of Petryshyn [23] (also see [10]).

Theorem 4.4 [23]. *Let C be an open subset of a Banach space and let $f : \overline{C} \to X$ be a contraction mapping. Suppose there exists $w \in C$ such that*

$$f(x) - w \neq \lambda(x - w) \quad \forall x \in \partial C,\ \lambda > 1. \tag{4.9}$$

Then f has a fixed point.

Theorem 4.5. *Suppose C is a closed convex subset of a uniformly convex Banach space X, with $\operatorname{int}(C) \neq \emptyset$. Suppose $f : C \to X$ is a nonexpansive mapping for which $F_\delta(f)$ is nonempty and bounded for some $\delta > 0$. Suppose also that*

(i) *there exists $w \in F_\delta(f) \bigcap \operatorname{int}(C)$ such that*

$$f(x) - w \neq \lambda(x - w) \quad \forall x \in \partial C,\ \lambda > 1; \tag{4.10}$$

then f has a fixed point.

Proof. Let (t_n) be a sequence in $(0,1)$ with $\lim_{n\to\infty} t_n = 0$ and define the mappings $f_n : \overline{\operatorname{int}(C)} \to X$ by setting $f_n(x) = (1 - t_n)f(x) + t_n w$. Then each of the mappings f_n is a contraction mapping which satisfies the conditions of Theorem 4.4, so for each n there exists $x_n \in C$ such that $f_n(x_n) = x_n$. Letting $\lambda_n = 1/(1 - t_n)$ we now have

$$f(x_n) - w = \lambda_n(x_n - w) \quad \text{with } \lambda_n > 1. \tag{4.11}$$

Also,

$$\begin{aligned}\|f(x_n)-w\|-\|w-f(w)\| &\le \|f(x_n)-f(w)\| \le \|x_n-w\| \\ &= \|f(x_n)-w\|-\|x_n-f(x_n)\|.\end{aligned} \tag{4.12}$$

Thus

$$\|x_n-f(x_n)\| \le \|w-f(w)\| \le \delta. \tag{4.13}$$

Therefore $(\|x_n\|)$ is bounded, and it follows that $\|x_n-f(x_n)\| \to 0$ as $n\to\infty$. One now concludes that f has a fixed point via the fact that $I-f$ is demiclosed on C. □

Theorem 4.6. *Suppose C is a closed convex subset of a uniformly convex Banach space X, with* $\operatorname{int}(C)\neq\emptyset$. *Suppose $f: C\to X$ is a nonexpansive mapping, and suppose $(f^n(p))$ is a bounded subset of C for some $p\in C$. Suppose also that*

(i) *there exists $w\in\operatorname{int}(C)$ such that*

$$f(x)-w\neq\lambda(x-w)\quad \forall x\in\partial C,\ \lambda>1; \tag{4.14}$$

then f has a fixed point.

Proof. Define K as in Lemma 2.3, but choose $r>0$ large enough to insure that $w\in K$. We show that f satisfies the assumptions of Theorem 4.1 on $K\cap C$. Obviously (i) holds for points $x\in\partial(K\cap C)\cap\partial(C)$. On the other hand, if $x\in\partial(K\cap C)\backslash\partial(C)$, then $f(x)-w=\lambda(x-w)$ for $\lambda>1$ implies $f(x)\notin K$, which is a contradiction. If $\inf\{\|x-f(x)\| : x\in\partial(K\cap C)$ and $f(x)\notin K\cap C\}>0$, the conclusion follows from Theorem 4.1. Otherwise the conclusion follows from demiclosedness of $I-f$. □

Definition 4.7. A mapping $f: C\to X$ is said to be *pseudocontractive* if for all $x,y\in C$ and $r>0$,

$$\|x-y\| \le \big\|(1+r)(x-y)-r\big(f(x)-f(y)\big)\big\|. \tag{4.15}$$

The pseudocontractive mappings are clearly more general than the nonexpansive mappings. They arise in nonlinear analysis via the fact that a mapping $f: C\to X$ is pseudocontractive if and only if the mapping $T=I-f$ is accretive; thus for every $x,y\in C$ there exists $j\in J(x-y)$ such that

$$\langle T(x)-T(y),j\rangle \ge 0, \tag{4.16}$$

where $J: X\to 2^{X^*}$ is the normalized duality mapping [4, 13].

The following theorem is proved in [17].

Theorem 4.8 [17]. *Let C be a bounded closed subset of a Banach space X. Suppose $f: C\to X$ is a continuous pseudocontractive mapping, and suppose there exists $z\in\operatorname{int}(C)$ such that*

$$\|z-f(z)\| < \|x-f(x)\| \quad \forall x\in\partial C. \tag{Δ}$$

Then $\inf\{\|x-f(x)\| : x\in C\}=0$. *If, in addition, C has the fixed point property for nonexpansive mappings, f has a fixed point.*

The condition that $F_\delta(f)$ is nonempty and bounded for some $\delta > 0$ seems to be the natural condition needed for an unbounded analogue of Theorem 4.8.

THEOREM 4.9. *Let C be a closed subset of a Banach space X. Suppose $f : C \to X$ is a continuous pseudocontractive mapping for which $F_\delta(f)$ is nonempty and bounded for some $\delta > 0$, and suppose there exists $z \in \text{int}(C)$ such that*

$$\|z - f(z)\| < \|x - f(x)\| \quad \forall x \in \partial C. \tag{Δ'}$$

Then $\inf\{\|x - f(x)\| : x \in C\} = 0$. *If, in addition, the bounded closed convex subsets of C have the fixed point property for nonexpansive mappings, then f has a fixed point.*

Proof. Clearly we may assume $z \in F_\delta(f)$. We may also assume C is unbounded. Otherwise the result is subsumed by Theorem 4.8. For each $n \in \mathbb{N}$, let $H_n := B(0;n) \bigcap C$. For n large enough we can assume $z \in \text{int}(H_n)$. Suppose that condition (Δ) fails on ∂H_n. Then there exists $x_n \in \partial H_n$ such that

$$\|x_n - f(x_n)\| \le \|z - f(z)\| \le \delta. \tag{4.17}$$

Since $\|z - f(z)\| < \|x - f(x)\|$ for all $x \in \partial C$, it must be the case that $\|x_n\| = n$; thus $\|x_n\| \to \infty$ as $n \to \infty$. However this is a contradiction because $x_n \in F_\delta(f)$. Therefore there exists N such that H_N satisfies the boundary condition (Δ). The conclusion now follows upon applying Theorem 4.8 to H_N. □

Remark 4.10. In all the preceding results the condition that $F_\delta(f)$ is nonempty and bounded for some $\delta > 0$ could be replaced by the stronger assumption that the mapping is asymptotically contractive.

Further remarks. There is another approach to the existence of fixed points for mappings defined on unbounded sets. The inward set $I_C(x)$ of x relative to C is the set

$$I_C(x) = \{x + c(u - x) : u \in C,\ c \ge 1\}. \tag{4.18}$$

A mapping $T : C \to X$ is said to be *weakly inward* if $T(x)$ is in the closure $\overline{I_C(x)}$ of $I_C(x)$ for each $x \in C$. Caristi [8] proved that if a closed convex set C has the fixed point property for nonexpansive self-mappings, then every weakly inward Lipschitzian pseudocontractive mapping $T : C \to X$ has a fixed point. While C is not assumed to be bounded in this result, the assumption that C has the fixed point property for unbounded closed convex sets is very strong (and impossible in a Hilbert space). Thus one should require only that bounded closed convex subsets of C have the fixed point property. It turns out that this problem has already been solved, and it also includes the case when the mapping is asymptotically contractive.

THEOREM 4.11 [9]. *Suppose the bounded closed and convex subsets of X have the fixed point property for nonexpansive self-mappings. Let C be a closed convex subset of X and let $f : C \to X$ be a continuous pseudocontractive mapping which is weakly inward on C. Then the following are equivalent.*

(a) *f has a fixed point in C.*

(b) *There exist* $y_0 \in C$ *and* $R > 0$ *such that* $\|x - y_0\| \le \|(1+s)x - sf(x) - y_0\|$ *for all* $x \in C$ *with* $\|x\| \ge R$ *and for all* $s \in [0,1]$.

(c) *There exist* $y_0 \in C$ *and* $R > 0$ *such that if* $x \in C$ *has* $\|x\| \ge R$, *then there exists* $j \in J(x - y_0)$ *satisfying*

$$\langle x - f(x), j \rangle \ge 0. \tag{4.19}$$

(d) *There exist* $y_0 \in X$ *and* $R > 0$ *such that if* $x \in C$ *with* $\|x\| \ge R$, *there exists* $j \in J(x - y_0)$ *satisfying*

$$\langle x - f(x), j \rangle \ge 0. \tag{4.20}$$

(e) *There exists a bounded sequence* (x_n) *in* C *such that* $\|x_n - f(x_n)\| \to 0$ *as* $n \to \infty$.

Since every mapping $f : C \to C$ is trivially weakly inward, and since nonexpansive mappings are pseudocontractive, (e) $\Rightarrow$ (a) of the above theorem gives another proof of Corollary 2.5. If f is asymptotically contractive, one can follow the proof of [21, Proposition 2] to obtain a bounded sequence (x_n) for which $\|x_n - f(x_n)\| \to 0$. In fact more can be said. The only place that nonexpansiveness of f enters into the proof of [21, Proposition 2] is for the conclusion that the auxiliary mappings f_n defined by

$$f_n(x) := (1 - t_n)f(x) + t_n x_0 \tag{4.21}$$

are contraction mappings having unique fixed points. However if f is continuous and pseudocontractive, then the mappings f_n are continuous and strongly pseudocontractive, and such mappings also have unique fixed points (see, e.g., [16, Corollary 4.5]). Theorem 4.11 therefore implies that Corollary 2.5 actually holds for continuous pseudocontractive mappings.

Corollary 4.12. *Let* X *be a Banach space which has the FPP, let* C *be a closed convex subset of* X, *and suppose* $f : C \to C$ *is a continuous pseudocontractive mapping which is asymptotically contractive. Then* f *has a fixed point.*

We conclude with a question.

Question 4.13. Is it possible to add the following condition to the list in Theorem 4.11?

(f) $F_\delta(f)$ is bounded for some $\delta > 0$.

Acknowledgments

This work was conducted while the first author was visiting the University of Iowa. She wishes to express her gratitude to the Department of Mathematics and particularly to Professor W. A. Kirk. This work was supported by the Thailand Research Fund under Grant BRG4780013. The first author was also supported by the Royal Golden Jubilee program under Grant PHD/0250/2545.

References

[1] A. Aleyner and S. Reich, *A note on explicit iterative constructions of sunny nonexpansive retractions in Banach spaces*, Journal of Nonlinear and Convex Analysis **6** (2005), no. 3, 525–533.

[2] ______, *An explicit construction of sunny nonexpansive retractions in Banach spaces*, Fixed Point Theory and Applications **2005** (2005), no. 3, 295–305.

[3] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proceedings of the National Academy of Sciences of the United States of America **54** (1965), 1041–1044.

[4] ______, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bulletin of the American Mathematical Society **74** (1968), 660–665.

[5] R. E. Bruck Jr., *Properties of fixed-point sets of nonexpansive mappings in Banach spaces*, Transactions of the American Mathematical Society **179** (1973), 251–262.

[6] ______, *A common fixed point theorem for a commuting family of nonexpansive mappings*, Pacific Journal of Mathematics **53** (1974), 59–71.

[7] I. Bula, *Some generalizations of W. A. Kirk's fixed point theorems*, Mathematics, Latv. Univ. Zināt. Raksti, vol. 595, Latv. Univ., Riga, 1994, pp. 159–166.

[8] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Transactions of the American Mathematical Society **215** (1976), 241–251.

[9] J. García-Falset, *Fixed points for mappings with the range type condition*, Houston Journal of Mathematics **28** (2002), no. 1, 143–158.

[10] J. A. Gatica and W. A. Kirk, *Fixed point theorems for contraction mappings with applications to nonexpansive and pseudo-contractive mappings*, The Rocky Mountain Journal of Mathematics **4** (1974), 69–79.

[11] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 83, Marcel Dekker, New York, 1984.

[12] D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, Mathematische Nachrichten **30** (1965), 251–258.

[13] T. Kato, *Nonlinear semigroups and evolution equations*, Journal of the Mathematical Society of Japan **19** (1967), 508–520.

[14] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, The American Mathematical Monthly **72** (1965), 1004–1006.

[15] ______, *Fixed point theorems for nonexpansive mappings satisfying certain boundary conditions*, Proceedings of the American Mathematical Society **50** (1975), 143–149.

[16] W. A. Kirk and C. H. Morales, *Nonexpansive mappings: boundary/inwardness conditions and local theory*, Handbook of Metric Fixed Point Theory, Kluwer Academic, Dordrecht, 2001, pp. 299–321.

[17] W. A. Kirk and R. Schöneberg, *Some results on pseudo-contractive mappings*, Pacific Journal of Mathematics **71** (1977), no. 1, 89–100.

[18] D. T. Luc, *Recessively compact sets: properties and uses*, Set-Valued Analysis **10** (2002), no. 1, 15–35.

[19] D. T. Luc and J.-P. Penot, *Convergence of asymptotic directions*, Transactions of the American Mathematical Society **353** (2001), no. 10, 4095–4121.

[20] S.-Y. Matsushita and W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory and Applications **2004** (2004), no. 1, 37–47.

[21] J.-P. Penot, *A fixed-point theorem for asymptotically contractive mappings*, Proceedings of the American Mathematical Society **131** (2003), no. 8, 2371–2377.

[22] ______, *A metric approach to asymptotic analysis*, Bulletin des Sciences Mathématiques **127** (2003), no. 9, 815–833.

[23] W. V. Petryshyn, *Structure of the fixed points sets of k-set-contractions*, Archive for Rational Mechanics and Analysis **40** (1970/1971), 312–328.

A. Kaewcharoen: Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand
E-mail address: akaewcharoen@yahoo.com

W. A. Kirk: Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419, USA
E-mail address: kirk@math.uiowa.edu