A NEW COMPOSITE IMPLICIT ITERATIVE PROCESS FOR A FINITE FAMILY OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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The purpose of this paper is to study the weak and strong convergence of implicit iteration process with errors to a common fixed point for a finite family of nonexpansive mappings in Banach spaces. The results presented in this paper extend and improve the corresponding results of Chang and Cho (2003), Xu and Ori (2001), and Zhou and Chang (2002).

1. Introduction and preliminaries

Throughout this paper we assume that E is a real Banach space and $T : E \to E$ is a mapping. We denote by $F(T)$ and $D(T)$ the set of fixed points and the domain of T, respectively.

Recall that E is said to satisfy *Opial condition* [11], if for each sequence $\{x_n\}$ in E, the condition that the sequence $x_n \to x$ weakly implies that

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y|| \tag{1.1}$$

for all $y \in E$ with $y \neq x$. It is well known that (see, e.g., Dozo [9]) inequality (1.1) is equivalent to

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||. \tag{1.2}$$

Definition 1.1. Let D be a closed subset of E and let $T : D \to D$ be a mapping.

(1) T is said to be *demiclosed* at the origin, if for each sequence $\{x_n\}$ in D, the conditions $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly imply $Tx_0 = 0$.

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(2) T is said to be *semicompact*, if for any bounded sequence $\{x_n\}$ in D such that $\|x_n - Tx_n\| \to 0$ $(n \to \infty)$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in D$.

(3) T is said to be *nonexpansive*, if $\|Tx - Ty\| \le \|x - y\|$, for all $n \ge 1$ for all $x, y \in D$.

Let E be a Hilbet space, let K be a nonempty closed convex subset of E, and let $\{T_1, T_2, \ldots, T_N\} : K \to K$ be N nonexpansive mappings. In 2001, Xu and Ori [19] introduced the following implicit iteration process $\{x_n\}$ defined by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{n(\mathrm{mod}\, N)} x_n, \quad \forall n \ge 1, \tag{1.3}$$

where $x_0 \in K$ is an initial point, $\{\alpha_n\}_{n \ge 1}$ is a real sequence in $(0,1)$ and proved the weakly convergence of the sequence $\{x_n\}$ defined by (1.3) to a common fixed point $p \in F = \bigcap_{i=1}^{N} F(T_i)$.

Recently concerning the convergence problems of an implicit (or nonimplicit) iterative process to a common fixed point for a finite family of asymptotically nonexpansive mappings (or nonexpansive mappings) in Hilbert spaces or uniformly convex Banach spaces have been considered by several authors (see, e.g., Bauschke [1], Chang and Cho [3], Chang et al. [4], Chidume et al. [5], Goebel and Kirk [6], Górnicki [7], Halpern [8], Lions [10], Reich [12], Rhoades [13], Schu [14], Shioji and Takahashi [15], Tan and Xu [16, 17], Wittmann [18], Xu and Ori [19], and Zhou and Chang [20]).

In this paper, we introduce the following new implicit iterative sequence $\{x_n\}$ with errors:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + \beta_1 T_1 (\hat{\alpha}_1 x_0 + \hat{\beta}_1 T_1 x_1 + \hat{\gamma}_1 v_1) + \gamma_1 u_1, \\
x_2 &= \alpha_2 x_1 + \beta_2 T_2 (\hat{\alpha}_2 x_1 + \hat{\beta}_2 T_2 x_2 + \hat{\gamma}_2 v_2) + \gamma_2 u_2, \\
&\;\;\vdots \\
x_N &= \alpha_N x_{N-1} + \beta_N T_N (\hat{\alpha}_N x_{N-1} + \hat{\beta}_N T_N x_N + \hat{\gamma}_N v_N) + \gamma_N u_N, \\
x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} T_1 (\hat{\alpha}_{N+1} x_N + \hat{\beta}_{N+1} T_1 x_{N+1} + \hat{\gamma}_{N+1} v_{N+1}) + \gamma_{N+1} u_{N+1}, \\
&\;\;\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} T_N (\hat{\alpha}_{2N} x_{2N-1} + \hat{\beta}_{2N} T_N x_{2N} + \hat{\gamma}_{2N} v_{2N}) + \gamma_{2N} u_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} T_1 (\hat{\alpha}_{2N+1} x_{2N} + \hat{\beta}_{2N+1} T_1 x_{2N+1} + \hat{\gamma}_{2N+1} v_{2N+1}) + \gamma_{2N+1} u_{2N+1}, \\
&\;\;\vdots
\end{aligned} \tag{1.4}$$

for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^{N} : K \to K$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha}_n\}$, $\{\hat{\beta}_n\}$, and $\{\hat{\gamma}_n\}$ are six sequences in [0, 1] satisfying $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1$ for all $n \ge 1$, x_0 is a given point in K, as well as $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K, which can be written in the following compact form:

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + \beta_n T_{n(\bmod N)} y_n + \gamma_n u_n, \\ y_n &= \hat{\alpha}_n x_{n-1} + \hat{\beta}_n T_{n(\bmod N)} x_n + \hat{\gamma}_n v_n, \quad \forall n \ge 1. \end{aligned} \tag{1.5}$$

Especially, if $\{T_i\}_{i=1}^N : K \to K$ are N nonexpansive mappings, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0, 1]$, and x_0 is a given point in K, then the sequence $\{x_n\}$ defined by

$$x_n = \alpha_n x_{n-1} + \beta_n T_{n(\bmod N)} x_{n-1} + \gamma_n u_n, \quad \forall n \ge 1 \tag{1.6}$$

is called the explicit iterative sequence for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$.

The purpose of this paper is to study the weak and strong convergence of iterative sequence $\{x_n\}$ defined by (1.5) and (1.6) to a common fixed point for a finite family of nonexpansive mappings in Banach spaces. The results presented in this paper not only generalized and extend the corresponding results of Chang and Cho [3], Xu and Ori [19], and Zhou and Chang [20], but also in the case of $\gamma_n = \hat{\gamma}_n = 0$ or $\hat{\beta}_n = \hat{\gamma}_n = 0$ are also new.

In order to prove the main results of this paper, we need the following lemmas.

Lemma 1.2 [2]. *Let E be a uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then $I - T$ is semiclosed at zero, that is, for each sequence $\{x_n\}$ in K, if $\{x_n\}$ converges weakly to $q \in K$ and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)q = 0$.*

Lemma 1.3 [17]. *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences satisfying the following condition: $a_{n+1} \le a_n + b_n$ for all $n \ge n_0$, where n_0 is some nonnegative integer. If $\sum_{n=0}^\infty b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 1.4 [14]. *Let E be a uniformly convex Banach space, let b and c be two constants with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a sequence in $[b,c]$ and $\{x_n\}$ and $\{y_n\}$ are two sequence in E such that $\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = d$, $\limsup_{n\to\infty} \|x_n\| \le d$, and $\limsup_{n\to\infty} \|y_n\| \le d$ hold for some $d \ge 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 1.5. *Let E be a real Banach space, let K be a nonempty closed convex subset of E, and let $\{T_1, T_2, \dots, T_N\} : K \to K$ be N nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in K, and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha}_n\}$, $\{\hat{\beta}_n\}$, and $\{\hat{\gamma}_n\}$ be six sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1$, *for all* $n \ge 1$;

(ii) $\tau = \sup\{\beta_n : n \ge 1\} < 1$;

(iii) $\sum_{n=1}^\infty \gamma_n < \infty$, $\sum_{n=1}^\infty \hat{\gamma}_n < \infty$.

If $\{x_n\}$ is the implicit iterative sequence defined by (1.5), then for each $p \in F = \bigcap_{i=1}^N F(T_i)$ the limit $\lim_{n\to\infty} \|x_n - p\|$ exists.

Proof. Since $F=\bigcap_{n=1}^{N}F(T_i)\neq\emptyset$, for any given $p\in F$, it follows from (1.5) that

$$\begin{aligned}||x_n-p|| &= ||(1-\beta_n-\gamma_n)x_{n-1}+\beta_n T_{n(\bmod N)}y_n+\gamma_n u_n-p||\\ &\leq (1-\beta_n-\gamma_n)||x_{n-1}-p||+\beta_n||T_{n(\bmod N)}y_n-p||+\gamma_n||u_n-p||\\ &= (1-\beta_n-\gamma_n)||x_{n-1}-p||+\beta_n||T_{n(\bmod N)}y_n-T_{n(\bmod N)}p||+\gamma_n||u_n-p||\\ &\leq (1-\beta_n)||x_{n-1}-p||+\beta_n||y_n-p||+\gamma_n||u_n-p||.\end{aligned}\tag{1.7}$$

Again it follows from (1.5) that

$$\begin{aligned}||y_n-p|| &= ||(1-\widehat{\beta}_n-\widehat{\gamma}_n)x_{n-1}+\widehat{\beta}_n T_{n(\bmod N)}x_n+\widehat{\gamma}_n v_n-p||\\ &\leq (1-\widehat{\beta}_n-\widehat{\gamma}_n)||x_{n-1}-p||+\widehat{\beta}_n||T_{n(\bmod N)}x_n-p||+\widehat{\gamma}_n||v_n-p||\\ &= (1-\widehat{\beta}_n-\widehat{\gamma}_n)||x_{n-1}-p||+\widehat{\beta}_n||T_{n(\bmod N)}x_n-T_{n(\bmod N)}p||+\widehat{\gamma}_n||v_n-p||\\ &\leq (1-\widehat{\beta}_n)||x_{n-1}-p||+\widehat{\beta}_n||x_n-p||+\widehat{\gamma}_n||v_n-p||.\end{aligned}\tag{1.8}$$

Substituting (1.8) into (1.7), we obtain that

$$\begin{aligned}||x_n-p|| \leq{}& (1-\beta_n\widehat{\beta}_n)||x_{n-1}-p||+\beta_n\widehat{\beta}_n||x_n-p||\\ &+\beta_n\widehat{\gamma}_n||v_n-p||+\gamma_n||u_n-p||.\end{aligned}\tag{1.9}$$

Simplifying we have

$$(1-\beta_n\widehat{\beta}_n)||x_n-p|| \leq (1-\beta_n\widehat{\beta}_n)||x_{n-1}-p||+\sigma_n,\tag{1.10}$$

where $\sigma_n=\beta_n\widehat{\gamma}_n\|v_n-p\|+\gamma_n\|u_n-p\|$. By condition (iii) and the boundedness of the sequences $\{\beta_n\}$, $\{\|u_n-p\|\}$, and $\{\|v_n-p\|\}$, we have $\sum_{n=1}^{\infty}\sigma_n<\infty$. From condition (ii) we know that

$$\beta_n\widehat{\beta}_n\leq\beta_n\leq\tau<1 \quad \text{and so} \quad 1-\beta_n\widehat{\beta}_n\geq 1-\tau>0;\tag{1.11}$$

hence, from (1.10) we have

$$||x_n-p||\leq||x_{n-1}-p||+\frac{\sigma_n}{1-\tau}=||x_{n-1}-p||+b_n,\tag{1.12}$$

where $b_n=\sigma_n/(1-\tau)$ with $\sum_{i=1}^{\infty}b_n<\infty$.

Taking $a_n=\|x_{n-1}-p\|$ in inequality (1.12), we have $a_{n+1}\leq a_n+b_n$, for all $n\geq 1$, and satisfied all conditions in Lemma 1.3. Therefore the limit $\lim_{n\to\infty}\|x_n-p\|$ exists. This completes the proof of Lemma 1.5. □

2. Main results

We are now in a position to prove our main results in this paper.

Theorem 2.1. *Let E be a real Banach space, let K be a nonempty closed convex subset of E, and let* $\{T_1,T_2,\ldots,T_N\}: K \to K$ *be N nonexpansive mappings with* $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ *(the set of common fixed points of* $\{T_1,T_2,\ldots,T_N\}$*). Let* $\{u_n\}$ *and* $\{v_n\}$ *be two bounded sequences in K, and let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\hat{\alpha}_n\}$, $\{\hat{\beta}_n\}$, *and* $\{\hat{\gamma}_n\}$ *be six sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n+\beta_n+\gamma_n = \hat{\alpha}_n+\hat{\beta}_n+\hat{\gamma}_n = 1$, *for all* $n \geq 1$;

(ii) $\tau = \sup\{\beta_n : n \geq 1\} < 1$;

(iii) $\sum_{n=1}^\infty \gamma_n < \infty$, $\sum_{n=1}^\infty \hat{\gamma}_n < \infty$.

Then the implicit iterative sequence $\{x_n\}$ *defined by (1.5) converges strongly to a common fixed point* $p \in F = \bigcap_{i=1}^N F(T_i)$ *if and only if*

$$\liminf_{n\to\infty} d(x_n,F) = 0. \tag{2.1}$$

Proof. The necessity of condition (2.1) is obvious.

Next we prove the sufficiency of Theorem 2.1. For any given $p \in F$, it follows from (1.12) in Lemma 1.5 that

$$||x_n - p|| \leq ||x_{n-1} - p|| + b_n \quad \forall n \geq 1, \tag{2.2}$$

where $b_n = \sigma_n/(1-\tau)$ with $\sum_{n=1}^\infty b_n < \infty$. Hence, we have

$$d(x_n,F) \leq d(x_{n-1},F) + b_n \quad \forall n \geq 1. \tag{2.3}$$

It follows from (2.3) and Lemma 1.3 that the limit $\lim_{n\to\infty} d(x_n,F)$ exists. By condition (2.1), we have $\lim_{n\to\infty} d(x_n,F) = 0$.

Next we prove that the sequence $\{x_n\}$ is a Cauchy sequence in K. In fact, for any positive integers m and n, from (2.2), it follows that

$$\begin{aligned} ||x_{n+m} - p|| &\leq ||x_{n+m-1} - p|| + b_{n+m} \leq ||x_{n+m-2} - p|| + b_{n+m-1} + b_{n+m} \\ &\leq \cdots \leq ||x_n - p|| + \sum_{i=n+1}^{n+m} b_i \leq ||x_n - p|| + \sum_{i=n+1}^{\infty} b_i. \end{aligned} \tag{2.4}$$

Since $\lim_{n\to\infty} d(x_n,F) = 0$ and $\sum_{n=1}^\infty b_n < \infty$, for any given $\epsilon > 0$, there exists a positive integer n_0 such that $d(x_n,F) < \epsilon/8$, $\sum_{i=n+1}^\infty b_i < \epsilon/2$, for all $n \geq n_0$. Therefore there exists $p_1 \in F$ such that $\|x_n - p_1\| < \epsilon/4$, for all $n \geq n_0$. Consequently, for any $n \geq n_0$ and for all $m \geq 1$, from (2.4), we have

$$\begin{aligned} ||x_{n+m} - x_n|| &\leq ||x_{n+m} - p_1|| + ||x_n - p_1|| \\ &\leq 2||x_n - p_1|| + \sum_{i=n+1}^{\infty} b_i < 2 \cdot \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{2.5}$$

This implies that $\{x_n\}$ is a Cauchy sequence in K. By the completeness of K, we can assume that $\lim_{n\to\infty} x_n = x^* \in K$. Moreover, since the set of fixed points of a nonexpansive mapping is closed, so is F; thus $x^* \in F$ from $\lim_{n\to\infty} d(x_n,F) = 0$, that is, x^* is a common fixed point of $T_1,T_2,\ldots,T_N$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let E be a real Banach space, let K be a nonempty closed convex subset of E, and let $\{T_1,T_2,\dots,T_N\}: K \to K$ be N nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_1,T_2,\dots,T_N\}$). Let $\{u_n\}$ be a bounded sequence in K, and let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n+\beta_n+\gamma_n = 1$, *for all* $n \geq 1$;

(ii) $\tau = \sup\{\beta_n : n \geq 1\} < 1$;

(iii) $\sum_{n=1}^\infty \gamma_n < \infty$.

Then the explicit iterative sequence $\{x_n\}$ defined by (1.6) converges strongly to a common fixed point $p \in F = \bigcap_{i=1}^N F(T_i)$ if and only if $\liminf_{n\to\infty} d(x_n,F) = 0$.

Proof. Taking $\widehat{\beta}_n = \widehat{\gamma}_n = 0$, for all $n \geq 1$ in Theorem 2.1, then the conclusion of Theorem 2.2 can be obtained from Theorem 2.1 immediately. This completes the proof of Theorem 2.2. □

Theorem 2.3. *Let E be a real uniformly convex Banach space satisfying Opial condition, let K be a nonempty closed convex subset of E, and let $\{T_1,T_2,\dots,T_N\}: K \to K$ be N nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in K, and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, and $\{\widehat{\gamma}_n\}$ be six sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n+\beta_n+\gamma_n = \widehat{\alpha}_n+\widehat{\beta}_n+\widehat{\gamma}_n = 1$, *for all* $n \geq 1$;

(ii) $0 < \tau_1 = \inf\{\beta_n : n \geq 1\} \leq \sup\{\beta_n : n \geq 1\} = \tau_2 < 1$;

(iii) $\widehat{\beta}_n \to 0$ $(n \to \infty)$;

(iv) $\sum_{n=1}^\infty \gamma_n < \infty$, $\sum_{n=1}^\infty \widehat{\gamma}_n < \infty$.

Then the implicit iterative sequence $\{x_n\}$ defined by (1.5) converges weakly to a common fixed point of $\{T_1,T_2,\dots,T_N\}$.

Proof. First, we prove that

$$\lim_{n\to\infty} \|x_n - T_{n(\mathrm{mod}\,N)+j}x_n\| = 0, \quad \forall j = 1,2,\dots,N. \tag{2.6}$$

Let $p \in F$. Put $d = \lim_{n\to\infty} \|x_n - p\|$. It follows from (1.5) that

$$\begin{aligned} \|x_n - p\| = \|&(1-\beta_n)[x_{n-1} - p + \gamma_n(u_n - x_{n-1})] \\ &+ \beta_n[T_{n(\mathrm{mod}\,N)}y_n - p + \gamma_n(u_n - x_{n-1})]\| \longrightarrow d, \quad n \longrightarrow \infty. \end{aligned} \tag{2.7}$$

Again since $\lim_{n\to\infty}\|x_n - p\|$ exists, so $\{x_n\}$ is a bounded sequence in K. By virtue of condition (iv) and the boundedness of sequences $\{x_n\}$ and $\{u_n\}$ we have

$$\begin{aligned} &\limsup_{n\to\infty} \|x_{n-1} - p + \gamma_n(u_n - x_{n-1})\| \\ &\quad \leq \limsup_{n\to\infty}\|x_{n-1} - p\| + \limsup_{n\to\infty} \gamma_n\|u_n - x_{n-1}\| = d, \quad p \in F. \end{aligned} \tag{2.8}$$

It follows from (1.8) and condition (iii) that

$$\begin{aligned}
&\limsup_{n\to\infty}||T_{n(\mathrm{mod}N)}y_n-p+\gamma_n(u_n-x_{n-1})||\\
&\quad\le\limsup_{n\to\infty}||y_n-p||+\limsup_{n\to\infty}\gamma_n||u_n-x_{n-1}||\\
&\quad=\limsup_{n\to\infty}||y_n-p||\\
&\quad\le\limsup_{n\to\infty}\{(1-\widehat{\beta}_n)||x_{n-1}-p||+\widehat{\beta}_n||x_n-p||+\widehat{\gamma}_n||v_n-p||\}\\
&\quad\le\limsup_{n\to\infty}(1-\widehat{\beta}_n)||x_{n-1}-p||+\limsup_{n\to\infty}\widehat{\beta}_n||x_n-p||+\limsup_{n\to\infty}\widehat{\gamma}_n||v_n-p||\\
&\quad=d,\quad p\in F.
\end{aligned}\tag{2.9}$$

Therefore, from condition (ii), (2.7)–(2.9), and Lemma 1.4 we know that

$$\lim_{n\to\infty}||T_{n(\mathrm{mod}N)}y_n-x_{n-1}||=0.\tag{2.10}$$

From (1.5) and (2.10) we have

$$\begin{aligned}
||x_n-x_{n-1}||&=||\beta_n[T_{n(\mathrm{mod}N)}y_n-x_{n-1}]+\gamma_n(u_n-x_{n-1})||\\
&\le\beta_n||T_{n(\mathrm{mod}N)}y_n-x_{n-1}||+\gamma_n||u_n-x_{n-1}||\longrightarrow 0,\quad n\longrightarrow\infty,
\end{aligned}\tag{2.11}$$

which implies that

$$\lim_{n\longrightarrow\infty}||x_n-x_{n-1}||=0\tag{2.12}$$

and so

$$\lim_{n\to\infty}||x_n-x_{n+j}||=0\quad\forall j=1,2,\dots,N.\tag{2.13}$$

On the other hand, we have

$$\begin{aligned}
||x_n-T_{n(\mathrm{mod}N)}x_n||\le{}&||x_n-x_{n-1}||+||x_{n-1}-T_{n(\mathrm{mod}N)}y_n||\\
&+||T_{n(\mathrm{mod}N)}y_n-T_{n(\mathrm{mod}N)}x_n||.
\end{aligned}\tag{2.14}$$

Now, we consider the third term on the right-hand side of (2.14). From (1.5) we have

$$\begin{aligned}
&||T_{n(\mathrm{mod}N)}y_n-T_{n(\mathrm{mod}N)}x_n||\\
&\quad\le||y_n-x_n||=||\widehat{\alpha}_n(x_{n-1}-x_n)+\widehat{\beta}_n(T_{n(\mathrm{mod}N)}x_n-x_n)+\widehat{\gamma}_n(v_n-x_n)||\\
&\quad\le\widehat{\alpha}_n||x_{n-1}-x_n||+\widehat{\beta}_n||T_{n(\mathrm{mod}N)}x_n-x_n||+\widehat{\gamma}_n||v_n-x_n||.
\end{aligned}\tag{2.15}$$

Substituting (2.15) into (2.14), we obtain that

$$\begin{aligned}
||x_n-T_{n(\mathrm{mod}N)}x_n||\le{}&(1+\widehat{\alpha}_n)||x_n-x_{n-1}||+||x_{n-1}-T_{n(\mathrm{mod}N)}y_n||\\
&+\widehat{\beta}_n||T_{n(\mathrm{mod}N)}x_n-x_n||+\widehat{\gamma}_n||v_n-x_n||.
\end{aligned}\tag{2.16}$$

Hence, by virtue of conditions (iii), (iv), (2.10), (2.12) and the boundedness of sequences $\{\|T_{n(\bmod N)}x_n - x_n\|\}$ and $\{\|v_n - x_n\|\}$ we have

$$\lim_{n\to\infty} ||x_n - T_{n(\bmod N)}x_n|| = 0. \tag{2.17}$$

Therefore, from (2.13) and (2.17), for any $j = 1,2,\ldots,N$, we have

$$\begin{aligned} ||x_n - T_{n(\bmod N)+j}x_n|| &\le ||x_n - x_{n+j}|| + ||x_{n+j} - T_{n(\bmod N)+j}x_{n+j}|| \\ &\quad + ||T_{n(\bmod N)+j}x_{n+j} - T_{n(\bmod N)+j}x_n|| \\ &\le 2||x_n - x_{n+j}|| + ||x_{n+j} - T_{n(\bmod N)+j}x_{n+j}|| \longrightarrow 0, \quad n \longrightarrow \infty. \end{aligned} \tag{2.18}$$

That is, (2.6) holds.

Since E is uniformly convex, every bounded subset of E is weakly compact. Again since $\{x_n\}$ is a bounded sequence in K, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q \in K$.

Without loss of generality, we can assume that $n_k = i(\bmod N)$, where i is some positive integer in $\{1,2,\ldots,N\}$. Otherwise, we can take a subsequence $\{x_{n_{k_j}}\} \subset \{x_{n_k}\}$ such that $n_{k_j} = i(\bmod N)$. For any $l \in \{1,2,\ldots,N\}$, there exists an integer $j \in \{1,2,\ldots,N\}$ such that $n_k + j = l(\bmod N)$. Hence, from (2.18) we have

$$\lim_{k\to\infty} ||x_{n_k} - T_l x_{n_k}|| = 0, \quad l = 1,2,\ldots,N. \tag{2.19}$$

By Lemma 1.2, we know that $q \in F(T_l)$. By the arbitrariness of $l \in \{1,2,\ldots,N\}$, we know that $q \in F = \bigcap_{j=1}^N F(T_j)$.

Finally, we prove that $\{x_n\}$ converges weakly to q. In fact, suppose the contrary, then there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $q_1 \in K$ and $q_1 \neq q$. Then by the same method as given above, we can also prove that $q_1 \in F = \bigcap_{j=1}^N F(T_j)$.

Taking $p = q$ and $p = q_1$ and by using the same method given in the proof of Lemma 1.5, we can prove that the following two limits exist and $\lim_{n\to\infty} \|x_n - q\| = d_1$ and $\lim_{n\to\infty} \|x_n - q_1\| = d_2$, where d_1 and d_2 are two nonnegative numbers. By virtue of the Opial condition of E, we have

$$\begin{aligned} d_1 &= \limsup_{n_k\longrightarrow\infty} ||x_{n_k} - q|| < \limsup_{n_k\to\infty} ||x_{n_k} - q_1|| = d_2 \\ &= \limsup_{n_j\to\infty} ||x_{n_j} - q_1|| < \limsup_{n_j\to\infty} ||x_{n_j} - q|| = d_1. \end{aligned} \tag{2.20}$$

This is a contradiction. Hence $q_1 = q$. This implies that $\{x_n\}$ converges weakly to q. This completes the proof of Theorem 2.3. □

Theorem 2.4. *Let E be a real uniformly convex Banach space satisfying Opial condition, let K be a nonempty closed convex subset of E, and let $\{T_1, T_2, \ldots, T_N\} : K \to K$ be N nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in K, and let*

$\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $[0,1]$ satisfying the following conditions:

(i) $\alpha_n+\beta_n+\gamma_n=1$, $\forall n\geq 1$;

(ii) $0<\tau_1=\inf\{\beta_n : n\geq 1\}\leq \sup\{\beta_n : n\geq 1\}=\tau_2<1$;

(iii) $\sum_{n=1}^{\infty}\gamma_n<\infty$.

Then the explicit iterative sequence $\{x_n\}$ defined by (1.6) converges weakly to a common fixed point of $\{T_1,T_2,\dots,T_N\}$.

Proof. Taking $\widehat{\beta}_n=\widehat{\gamma}_n=0$, for all $n\geq 1$ in Theorem 2.3, then the conclusion of Theorem 2.4 can be obtained from Theorem 2.3 immediately. This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $\{T_1,T_2,\dots,T_N\}:K\to K$ be N nonexpansive mappings with $F=\bigcap_{i=1}^N F(T_i)\neq\emptyset$ and there exists an T_j, $1\leq j\leq N$, which is semicompact (without loss of generality, assume that T_1 is semicompact). Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in K, and let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, and $\{\widehat{\gamma}_n\}$ be six sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n+\beta_n+\gamma_n=\widehat{\alpha}_n+\widehat{\beta}_n+\widehat{\gamma}_n=1$, *for all* $n\geq 1$;

(ii) $0<\tau_1=\inf\{\beta_n : n\geq 1\}\leq \sup\{\beta_n : n\geq 1\}=\tau_2<1$;

(iii) $\widehat{\beta}_n\to 0$ $(n\to\infty)$;

(iv) $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\widehat{\gamma}_n<\infty$.

Then the implicit iterative sequence $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of $\{T_1,T_2,\dots,T_N\}$ in K.

Proof. For any given $p\in F=\bigcap_{i=1}^N F(T_i)$, by the same method as given in proving Lemma 1.5 and (2.19), we can prove that

$$\lim_{n\to\infty}||x_n-p||=d, \tag{2.21}$$

where $d\geq 0$ is some nonnegative number, and

$$\lim_{k\to\infty}||x_{n_k}-T_l x_{n_k}||=0,\quad l=1,2,\dots,N. \tag{2.22}$$

Especially, we have

$$\lim_{k\to\infty}||x_{n_k}-T_1 x_{n_k}||=0. \tag{2.23}$$

By the assumption, T_1 is semicompact; therefore it follows from (2.23) that there exists a subsequence $\{x_{n_{k_i}}\}\subset\{x_{n_k}\}$ such that $x_{n_{k_i}}\to x^*\in K$. Hence from (2.22) we have that

$$||x^*-T_l x^*||=\lim_{k_i\to\infty}||x_{n_{k_i}}-T_l x_{n_{k_i}}||=0\quad \forall l=1,2,\dots,N, \tag{2.24}$$

which implies that $x^*\in F=\bigcap_{i=1}^N F(T_i)$. Take $p=x^*$ in (2.21), similarly we can prove that $\lim_{n\to\infty}\|x_n-x^*\|=d_1$, where $d_1\geq 0$ is some nonnegative number. From $x_{n_{k_i}}\to x^*$ we know that $d_1=0$, that is, $x_n\to x^*$. This completes the proof of Theorem 2.5. □

Theorem 2.6. *Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $\{T_1, T_2, \dots, T_N\} : K \to K$ be N nonexpansive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and there exists an T_j, $1 \leq j \leq N$, which is semicompact (without loss of generality, assume that T_1 is semicompact). Let $\{u_n\}$ be a bounded sequence in K, and let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 1$;

(ii) $0 < \tau_1 = \inf\{\beta_n : n \geq 1\} \leq \sup\{\beta_n : n \geq 1\} = \tau_2 < 1$;

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then the explicit iterative sequence $\{x_n\}$ defined by (1.6) converges strongly to a common fixed point of $\{T_1, T_2, \dots, T_N\}$ in K.

Proof. Taking $\widehat{\beta}_n = \widehat{\gamma}_n = 0$, for all $n \geq 1$ in Theorem 2.5, then the conclusion of Theorem 2.6 can be obtained from Theorem 2.5 immediately. This completes the proof of Theorem 2.6. □

Remark 2.7. Theorems 2.3–2.6 improve and extend the corresponding results in Chang and Cho [3, Theorem 3.1] and Zhou and Chang [20, Theorem 3], and the implicit iterative process $\{x_n\}$ defined by (1.3) is replaced by the more general implicit or explicit iterative process $\{x_n\}$ defined by (1.5) or (1.6).

Remark 2.8. Theorems 2.3–2.6 generalize and improve the main results of Xu and Ori [19] in the following aspects.

(1) The class of Hilbert spaces is extended to that of Banach spaces satisfying Opial's or semicompactness condition.

(2) The implicit iterative process $\{x_n\}$ defined by (1.3) is replaced by the more general implicit or explicit iterative process $\{x_n\}$ defined by (1.5) or (1.6).

Remark 2.9. The iterative algorithm used in this paper is different from those in [1, 8, 10, 14, 18].

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