TOWARDS VISCOSITY APPROXIMATIONS OF HIERARCHICAL FIXED-POINT PROBLEMS

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We introduce methods which seem to be a new and promising tool in hierarchical fixed-point problems. The goal of this note is to analyze the convergence properties of these new types of approximating methods for fixed-point problems. The limit attained by these curves is the solution of the general variational inequality, $0 \in (I - Q)x_\infty + N_{\mathrm{Fix}P}(x_\infty)$, where $N_{\mathrm{Fix}P}$ denotes the normal cone to the set of fixed point of the original nonexpansive mapping P and Q a suitable nonexpansive mapping criterion. The link with other approximation schemes in this field is also made.

1. Introduction

In nonlinear analysis, a common approach to solving a problem with multiple solutions is to replace it by a family of perturbed problems admitting a unique solution, and to obtain a particular solution as the limit of these perturbed solutions when the perturbation vanishes. Here, we will introduce a more general approach which consists in finding a particular part of the solution set of a given fixed-point problem, that is, fixed points which solve a variational inequality "criterion." More precisely, the main purpose of this note consists in building methods which hierarchically lead to fixed points of a nonexpansive mapping P with the aid of a nonexpansive mapping Q, in the following sense:

$$\text{find } \tilde{x} \in \mathrm{Fix}(P) \quad \text{such that } \langle \tilde{x} - Q(\tilde{x}), x - \tilde{x} \rangle \geq 0 \quad \forall x \in \mathrm{Fix}(P), \tag{1.1}$$

where $\mathrm{Fix}(P) = \{\bar{x} \in C;\ \bar{x} = P(\bar{x})\}$ is the set of fixed points of P and C is a closed convex subset of a real Hilbert space $\mathscr{H}$.

It is not hard to check that solving (1.1) is equivalent to the fixed-point problem

$$\text{find } \tilde{x} \in C \quad \text{such that } \tilde{x} = \mathrm{proj}_{\mathrm{Fix}(P)} \circ Q(\tilde{x}), \tag{1.2}$$

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where $\operatorname{proj}_{\operatorname{Fix}(P)}$ stands for the metric projection on the convex set $\operatorname{Fix}(P)$, and by using the definition of the normal cone to $\operatorname{Fix}(P)$, that is,

$$N_{\operatorname{Fix}P}: x \longmapsto \begin{cases} \{u \in \mathcal{H};\ (\forall y \in \operatorname{Fix}P)\ \langle y - x, u\rangle \leq 0\}, & \text{if } x \in \operatorname{Fix}P, \\ \varnothing, & \text{otherwise}, \end{cases} \tag{1.3}$$

we easily obtain that (1.1) is equivalent to the variational inequality

$$0 \in (I - Q)\widetilde{x} + N_{\operatorname{Fix}P}(\widetilde{x}). \tag{1.4}$$

It is worth mentioning that when the solution set, S, of (1.1) is a singleton (which is the case, e.g., when Q is a contraction) the problem reduces to the viscosity fixed-point solution introduced in [6] and further developed in [3, 8].

Throughout, $\mathcal{H}$ is a real Hilbert space, $\langle\cdot,\cdot\rangle$ denotes the associated scalar product, and $\|\cdot\|$ stands for the corresponding norm. To begin with, let us recall the following concepts are of common use in the context of convex and nonlinear analysis, see, for example, Rockafellar-Wets [7]. An operator is said to be monotone if

$$\langle u - v, x - y\rangle \geq 0 \quad \text{whenever } u \in A(x),\ v \in A(y). \tag{1.5}$$

It is said to be maximal monotone if, in addition, the graph, $\operatorname{gph}A := \{(x,y) \in \mathcal{H}\times\mathcal{H} : y \in A(x)\}$, is not properly contained in the graph of any other monotone operator. It is well known that the single-valued operator $J_\lambda^A := (I + \lambda A)^{-1}$, called the resolvent of A of parameter λ, is a nonexpansive mapping which is everywhere defined. Recall also that a mapping P is nonexpansive if for all x, y, one has

$$\|P(x) - P(y)\| \leq \|x - y\|, \tag{1.6}$$

and finally that, a sequence A_n is said to be graph convergent to A, if

$$\limsup_{n\to+\infty} \operatorname{gph}A_n \subset \operatorname{gph}A \subset \liminf_{n\to+\infty} \operatorname{gph}A_n, \tag{1.7}$$

where the lower limit of the sequence $\{\operatorname{gph}A_n\}$ is the subset defined by

$$\liminf_{n\to+\infty} \operatorname{gph}A_n = \{(x,y) \in \mathcal{H}\times\mathcal{H}/\exists(x_n,y_n) \longrightarrow (x,y), (x_n,y_n) \in \operatorname{gph}A_n\ n \in \mathbb{N}^*\} \tag{1.8}$$

and the upper limit of the sequence $\{\operatorname{gph}A_n\}$ is the closed subset defined by

$$\limsup_{n\to+\infty} \operatorname{gph}A_n = \{(x,y)/\exists(n_\nu)_{\nu\in\mathbb{N}}, \exists(x_\nu,y_\nu) \longrightarrow (x,y), (x_\nu,y_\nu) \in \operatorname{gph}A_{n_\nu}\ \nu \in \mathbb{N}^*\}. \tag{1.9}$$

2. Convergence of approximating curves

2.1. A hierarchical fixed-point method. Let $P, Q : C \to C$ be two nonexpansive mappings on a closed convex set C and assume that $\operatorname{Fix}(P)$ and the solution set S of (1.1) are nonempty.

Given a real number $t \in (0,1)$, we define a mapping

$$P_t^Q : C \longrightarrow C \quad \text{by } P_t^Q(x) = tQ(x) + (1-t)P(x). \tag{2.1}$$

For simplicity we will write P_t for P_t^Q. It is clear that P_t is nonexpansive on C. Throughout the paper we will also assume that

$$\operatorname{Fix}(P_t) \neq \varnothing \quad \text{and bounded,} \tag{2.2}$$

this is the case for instance if Q is a contraction or under a compactness condition on C.

Now, let us state two preliminary results which will be needed in the sequel.

Lemma 2.1. *Let A be a maximal monotone operator, then $(t^{-1}A)$ graph converges to $N_{A^{-1}(0)}$ as $t \to 0$ provided that $A^{-1}(0) \neq \varnothing$.*

Proof. It is well known, see [4, Proposition 2], that if $A^{-1}(0) \neq \varnothing$, then for any $x \in \mathcal{H}$, $J_{t^{-1}}^A(x)$ pointwise converges to $\operatorname{proj}_{A^{-1}(0)} x$. Since $J_{t^{-1}}^A(x) = J_1^{t^{-1}A}(x)$ and $\operatorname{proj}_{A^{-1}(0)} x = J_1^{N_{A^{-1}(0)}}(x)$, thanks to the fact that the pointwise convergence of the resolvents is equivalent to the graph convergence of the corresponding operators (see, e.g., [7, Theorem 12.32]), we easily deduce that $t^{-1}A$ graph converges to $N_{A^{-1}(0)}$ as $t \to 0$. □

The following lemma contains stability and closure results of the class of maximal monotone operators under graph convergence, see, for example, [1] or [2].

Lemma 2.2. *Let (A_t) be a sequence of maximal monotone operators. If B is a Lipschitz maximal monotone operator, then $A_t + B$ is maximal monotone. Furthermore, if (A_t) graph converges to A, then A is maximal monotone and $(A_t + B)$ graph converges to $A + B$.*

Now, we are in position to study the convergence of an arbitrary curve $\{x_t\}$ in $\operatorname{Fix}(P_t)$ as $t \to 0$.

Proposition 2.3. *Every weak-cluster point x_∞ of $\{x_t\}$ is solution of (1.1), or equivalently a fixed point of (1.2) or equivalently a solution of the variational inequality*

$$\text{find } x_\infty \in C; \quad 0 \in (I-Q)x_\infty + N_S(x_\infty), \tag{2.3}$$

N_S being the normal cone to the closed convex set S.

Proof. $\{x_t\}$ is assumed to be bounded, so are $\{P(x_t)\}$ and $\{Q(x_t)\}$. As a result,

$$\lim_{t\to 0} \|x_t - P(x_t)\| = \lim_{t\to 0} t\|P(x_t) - Q(x_t)\| = 0. \tag{2.4}$$

Let x_∞ be a weak cluster point of $\{x_t\}$, say $\{x_{t_\nu}\}$ weakly converges to x_∞, we will show that x_∞ is a solution of the variational inequality (1.1).

$x_{t_\nu} \in \operatorname{Fix} P_{t_\nu}$ can be rewritten as

$$\left(I - Q + \frac{1-t_\nu}{t_\nu}(I-P)\right)(x_{t_\nu}) = 0. \tag{2.5}$$

Now, in the light of Lemma 2.2 the family $(I-Q+((1-t_\nu)/t_\nu)(I-P))$ graph converges to $(I-Q)+N_{\operatorname{Fix}P}$, because $((1-t_\nu)/t_\nu)(I-P)$ graph converges to the normal cone of $(I-P)^{-1}(0)=\operatorname{Fix}P$ according to Lemma 2.1 and the operator $I-Q$ is a Lipschitz continuous maximal monotone operator.

By passing to the limit in the equality (2.5) as $t_\nu \to 0$, and by taking into account the fact that the graph of $(I-Q)+N_{\operatorname{Fix}P}$ is weakly-strongly closed, we obtain $0\in(I-Q)x_\infty+N_{\operatorname{Fix}P}(x_\infty)$. By using the definition of the normal cone, this amounts to writing $\langle x_\infty - Q(x_\infty), x_\infty - x\rangle \le 0\ \forall x\in\operatorname{Fix}P$, that is, x_∞ solves the variational inequality (1.1). □

Now, we would like to mention some interesting particular cases.

Example 2.4 (monotone inclusions). By setting $Q=I-\gamma\mathcal{F}$, where $\mathcal{F}$ is κ-Lipschitzian and η-strongly monotone with $\gamma\in(0,2\kappa/\eta^2)$, (1.1) reduces to

$$\text{find } \tilde{x}\in\operatorname{Fix}P \quad \text{such that } \langle x-\tilde{x},\mathcal{F}(\tilde{x})\rangle \ge 0 \quad \forall\, x\in\operatorname{Fix}P, \tag{2.6}$$

a variational inequality studied in Yamada [9].

On the other hand, if we set $C=\mathcal{H}$, $P=J_\lambda^A$, and $Q=J_\lambda^B$ with A, B two maximal monotone operators and J_λ^A, J_λ^B the corresponding resolvent mappings, the variational inequality (1.1) reduces to

$$\text{find } \tilde{x}\in\mathcal{H}; \quad 0\in\left(I-J_\lambda^B\right)(\tilde{x})+N_{A^{-1}(0)}(\tilde{x}), \tag{2.7}$$

where $N_{A^{-1}(0)}$ denotes the normal cone to, $A^{-1}(0)=\operatorname{Fix}J_\lambda^A$, the set of zeroes of A. The inclusion (2.7) can be rewritten as find $\tilde{x}$; $0\in B_\lambda(\tilde{x})+N_{A^{-1}(0)}(\tilde{x})$, $B_\lambda := (\lambda I + B^{-1})^{-1}$ being the Yosida approximate of B.

Example 2.5 (convex programming). By setting

$$P=\operatorname{prox}_{\lambda\varphi} := \operatorname{arg\,min}\left\{\varphi(y)+\frac{1}{2\lambda}\|\cdot - y\|^2\right\}, \tag{2.8}$$

φ a lower semicontinuous convex function and $Q=I-\gamma\nabla\psi$, ψ a convex function such that $\nabla\psi$ is κ-strongly monotone and η-Lipschitzian (which is equivalent to the fact that $\nabla\psi$ is η^{-1} cocoercive) with $\gamma\in(0,2/\eta)$, and thanks to the fact that $\operatorname{Fix}(\operatorname{prox}_{\lambda\varphi}) = (\partial\varphi)^{-1}(0)=\operatorname{arg\,min}\varphi$, (1.1) reduces to the hierarchical minimization problem:

$$\min_{x\in\operatorname{arg\,min}\varphi} \psi(x). \tag{2.9}$$

On the other hand, if we set in (2.7), $A=\partial\varphi$ and $B=\partial\psi$, subdifferential operators of lower semicontinuous convex functions φ and ψ, the inclusion (1.1) reduces to the following hierarchical minimization problem: $\min_{x\in\operatorname{arg\,min}\varphi}\psi_\lambda(x)$, where $\psi_\lambda(x)=\inf_y\{\psi(y)+(1/2\lambda)\|x-y\|^2\}$, is the Moreau-Yosida approximate of ψ.

Example 2.6 (minimization on a fixed-point set). By setting $Q=I-\gamma\nabla\varphi$, φ a convex function; $\nabla\varphi$ is κ-strongly monotone and η-Lipschitzian (thus η^{-1} cocoercive) with $\gamma\in(0,2/\eta]$, (1.1) reduces to $\min_{x\in\operatorname{Fix}P}\varphi(x)$, a problem studied in Yamada [9]. On the other hand, when P is a nonexpansive mapping and $Q=I-\tilde{\gamma}(A-\gamma f)$, A being a linear bounded

$\overline{\gamma}$-strongly monotone operator, f a given α-contraction, and $\gamma > 0$ with $\widetilde{\gamma} \in (0,/\|A\| + \overline{\gamma})$, (1.1) reduces to the problem of minimizing a quadratic function over the set of fixed points of a nonexpansive mapping studied in Marino and Xu [5], namely,

$$\langle (A - \gamma f)\overline{x}, x - \overline{x} \rangle \geq 0, \quad \forall x \in \operatorname{Fix} P, \tag{2.10}$$

which is the optimality condition for the minimization problem

$$\min_{x \in \operatorname{Fix} P} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{2.11}$$

where h is a potential function for γf, that is, $h'(x) = \gamma f(x)$, for $x \in \mathcal{H}$.

For $t \in (0,1)$ let $\{x_t\}$ be a fixed point of P_t. Our interest now is to show that any net $\{x_t\}$ obtained in this way is an *approximate fixed-point net* for P.

Proposition 2.7. *Assume that* $\operatorname{Fix} Q \neq \emptyset$. *Then, for any* $t \in (0,1)$,

$$||Qx_t - Px_t|| \leq 2 \inf_{(p,q) \in \operatorname{Fix}(P) \times \operatorname{Fix}(Q)} \|p - q\|. \tag{2.12}$$

Moreover, the net $\{x_t\}$ *is an approximate fixed-point net for the mapping P, that is,*

$$\lim_{t \to 0} ||x_t - Px_t|| = 0. \tag{2.13}$$

Proof. Consider any $p \in \operatorname{Fix}(P)$ and $q \in \operatorname{Fix}(Q)$ and let $p_t := \operatorname{Proj}_{\Delta_t}(p)$ and $q_t := \operatorname{Proj}_{\Delta_t}(q)$ be the metric projections of p and q onto Δ_t, respectively, where the closed convex set Δ_t is defined by $\Delta_t := \{\lambda(Px_t - Qx_t) + x_t;\ \lambda \in \mathbb{R}\}$.

Now, suppose that condition $Px_t \neq Qx_t$ is satisfied. It is then immediate that $x_t \neq Px_t$ and $x_t \neq Qx_t$ provided that $t \in (0,1)$. Set $a_t := (1/2)(x_t + Px_t)$ and $b_t := (1/2)(x_t + Qx_t)$, it is then easily checked that

$$\begin{aligned} \langle Qx_t - b_t, q - b_t \rangle &= \frac{1}{4}\left(||x_t - q||^2 - ||Qx_t - q||^2\right), \\ \langle Px_t - a_t, p - a_t \rangle &= \frac{1}{4}\left(||x_t - p||^2 - ||Px_t - p||^2\right). \end{aligned} \tag{2.14}$$

Thanks to the nonexpansiveness of Q and P, we deduce that

$$\langle Qx_t - b_t, q - b_t \rangle \geq 0, \qquad \langle Px_t - a_t, p - a_t \rangle \geq 0. \tag{2.15}$$

Furthermore, it is obvious that there exist two real numbers λ_t and μ_t such that $q_t = b_t + \lambda_t(Qx_t - b_t)$ and $p_t = a_t + \mu_t(Px_t - a_t)$. In the light of the metric projection properties, we can write

$$0 = \langle q_t - q, Qx_t - b_t \rangle = \langle b_t - q, Qx_t - b_t \rangle + \lambda_t ||Qx_t - b_t||^2, \tag{2.16}$$

hence

$$\lambda_t = \frac{\langle q - b_t, Qx_t - b_t \rangle}{||Qx_t - b_t||^2} \geq 0. \tag{2.17}$$

In a similar way, we get

$$0 = \langle p_t - p, Px_t - a_t\rangle = \langle a_t - p, Px_t - a_t\rangle + \mu_t\|Px_t - a_t\|^2, \tag{2.18}$$

and we obtain

$$\mu_t = \frac{\langle p - a_t, Px_t - a_t\rangle}{\|Px_t - a_t\|^2} \geq 0. \tag{2.19}$$

Note also that $b_t - a_t = (1/2)(Qx_t - Px_t)$ and, according to the fact that $x_t \in \operatorname{Fix} P_t$, that $x_t - Px_t = t(Qx_t - Px_t)$ and $x_t - Qx_t = (1-t)(Px_t - Qx_t)$. Hence, we get $x_t - Px_t = 2t(b_t - a_t)$ and $x_t - Qx_t = -2(1-t)(b_t - a_t)$. Moreover, we immediately have $Qx_t - b_t = (1/2)(Qx_t - x_t)$ and $Px_t - a_t = (1/2)(Px_t - x_t)$, so that

$$\begin{aligned} q_t - b_t &= \lambda_t(Qx_t - b_t) = \lambda_t(1-t)(b_t - a_t), \\ a_t - p_t &= -\mu(Px_t - a_t) = \mu t(b_t - a_t). \end{aligned} \tag{2.20}$$

Consequently, we obtain

$$\begin{aligned} q_t - p_t &= (q_t - b_t) + (b_t - a_t) + (a_t - p_t) \\ &= (\lambda_t(1-t) + 1 + t\mu_t)(b_t - a_t) \\ &= \frac{1}{2}(\lambda_t(1-t) + 1 + t\mu_t)(Qx_t - Px_t). \end{aligned} \tag{2.21}$$

Thus

$$\|q_t - p_t\| = \frac{1}{2}(\lambda_t(1-t) + 1 + t\mu_t)\|Qx_t - Px_t\|. \tag{2.22}$$

Finally, by nonexpansiveness of the projection mapping, we have

$$\|q_t - p_t\| = \|\operatorname{Proj}_{\Delta_t}(p) - \operatorname{Proj}_{\Delta_t}(q)\| \leq \|p - q\|, \tag{2.23}$$

which by (0.1) leads to

$$\|p - q\| \geq \frac{1}{2}(\lambda_t(1-t) + 1 + t\mu_t)\|Qx_t - Px_t\| \geq \frac{1}{2}\|Qx_t - Px_t\|. \tag{2.24}$$

By taking the infimum over p in $\operatorname{Fix} P$ and q in $\operatorname{Fix} Q$, we obtain the desired formula. The latter combined with the fact that $x_t - Px_t = t(Qx_t - Px_t)$ leads to the fact that $\{x_t\}$ is an approximate fixed-point net for P. □

2.2. Coupling the hierarchical fixed-point method with viscosity approximation. To begin with, we will assume that

$$S \subset s - \liminf_{t \to 0} \operatorname{Fix} P_t, \quad s \text{ standing for the strong topology}, \tag{2.25}$$

which is satisfied, for example, when Q is a contraction.

Now, given a real number $s \in (0,1)$ and a contraction $f : C \to C$. Define another mapping

$$P_{t,s}^{f}(x) = sf(x) + (1-s)P_t(x), \tag{2.26}$$

for simplicity we will write $P_{t,s}$ for $P_{t,s}^{f}$.

It is not hard to see that $P_{t,s}$ is a contraction on C. Indeed, for $x, y \in C$, we have

$$\begin{aligned} \|P_{t,s}(x) - P_{t,s}(y)\| &= \|s(f(x) - f(y)) + (1-s)(P_t(x) - P_t(y))\| \\ &\leq \alpha s\|x-y\| + (1-s)\|x-y\| \\ &= (1 - s(1-\alpha))\|x-y\|. \end{aligned} \tag{2.27}$$

Let $x_{t,s}$ be the unique solution of the fixed point of $P_{t,s}$, that is, $x_{t,s}$ is the unique solution of the fixed-point equation

$$x_{t,s} = sf(x_{t,s}) + (1-s)P_t(x_{t,s}). \tag{2.28}$$

The purpose of this section is to study the convergence of $\{x_{t,s}\}$ as $t, s \to 0$.

Let us first recall the following diagonal lemma (see, e.g., [1]).

Lemma 2.8. *Let (X,d) be a metric space and $(a_{n,m})$ a "double" sequence in X satisfying*

$$\forall n \in \mathbb{N} \quad \lim_{m\to+\infty} a_{n,m} = a_n, \quad \lim_{n\to+\infty} a_n = a. \tag{2.29}$$

Then, there exists a nondecreasing mapping $k : \mathbb{N} \to \mathbb{N}$ which to m associates $k(m)$ and such that $\lim_{m\to+\infty} a_{k(m),m} = a$.

Now, we are able to give our main result.

Theorem 2.9. *The net $\{x_{t,s}\}$ strongly converges, as $s \to 0$, to x_t, where x_t satisfies $x_t = \mathrm{proj}_{\mathrm{Fix}\,P_t} \circ f(x_t)$ or equivalently x_t is the unique solution of the quasivariational inequality*

$$0 \in (I-f)x_t + N_{\mathrm{Fix}\,P_t}(x_t). \tag{2.30}$$

Moreover, the net $\{x_t\}$ in turn weakly converges, as $t \to 0$, to the unique solution x_∞ of the fixed-point equation $x_\infty = \mathrm{proj}_S \circ f(x_\infty)$ or equivalently $x_\infty \in S$ is the unique solution of the variational inequality

$$0 \in (I-f)x_\infty + N_S(x_\infty). \tag{2.31}$$

Furthermore, if $\dim \mathcal{H} < \infty$, then there exists a subnet $\{x_{s_\nu,s_n}\}$ of $\{x_{t_n,s_n}\}$ which converges to x_∞.

Proof. We first show that $\{x_{t,s}\}$ is bounded. Indeed take $\tilde{x}_t \in \mathrm{Fix}\,P_t$ to derive

$$\|x_{t,s} - \tilde{x}_t\| \leq s\|f(x_{t,s}) - \tilde{x}_t\| + (1-s)\|P_t(x_{t,s}) - P_t(\tilde{x}_t)\|. \tag{2.32}$$

It follows

$$\begin{aligned}\|x_{t,s}-\tilde{x}_t\| &\le \|f(x_{t,s})-\tilde{x}_t\| \le \|f(x_{t,s})-f(\tilde{x}_t)\|+\|f(\tilde{x}_t)-\tilde{x}_t\| \\ &\le \alpha s\|x_{t,s}-\tilde{x}_t\|+\|f(\tilde{x}_t)-\tilde{x}_t\|.\end{aligned} \tag{2.33}$$

Hence

$$\|x_{t,s}\| \le \|\tilde{x}_t\|+\frac{1}{\alpha}\|f(\tilde{x}_t)-\tilde{x}_t\|. \tag{2.34}$$

This ensures that $\{x_{t,s}\}$ is bounded, since $\{\tilde{x}_t\}$ and $\{f(\tilde{x}_t)\}$ are bounded. Now, we will show that $\{x_{t,s_n}\}$ contains a subnet converging to x_t, where $x_t \in \operatorname{Fix} P_t$ is the unique solution of the quasivariational inequality

$$0 \in (I-f)x_t+N_{\operatorname{Fix} P_t}(x_t). \tag{2.35}$$

Since $\{x_{t,s_n}\}$ is bounded, it admits a weak cluster point x_t, that is, there exists a subnet $\{x_{t,s_\nu}\}$ of $\{x_{t,s_n}\}$ which weakly converges to x_t. On the other hand,

$$(I-f)+\frac{1-s}{s}(I-P_t) \text{ graph converges to } (I-f)+N_{\operatorname{Fix} P_t} \quad \text{as } s \longrightarrow 0. \tag{2.36}$$

By passing to the limit in the following equality:

$$\left((I-f)+\frac{(1-s_\nu)}{s_\nu}P_t\right)(x_{t,s_\nu})=0, \tag{2.37}$$

we obtain that x_t is the unique solution of the quasivariational inequality

$$0 \in \left((I-f)+N_{\operatorname{Fix} P_t}\right)(x_t), \tag{2.38}$$

or equivalently x_t satisfies $x_t=\operatorname{proj}_{\operatorname{Fix} P_t}\circ f(x_t)$. It should be noticed that in contrast with the first section $\{x_t\}$ is unique (a select approximating curve in $\operatorname{Fix} P_t$). Hence the whole net $\{x_{t,s_n}\}$ weakly converges to x_t. In fact the convergence is strong. Indeed, since

$$x_{t,s}-x_t=s\left(f(x_{t,s})-x_t\right)+(1-s)\left(P_t(x_{t,s})-x_t\right), \tag{2.39}$$

we successively have

$$\begin{aligned}\|x_{t,s}-x_t\|^2 &= (1-s)\langle P_t(x_{t,s})-x_t,x_{t,s}-x_t\rangle+s\langle f(x_{t,s})-x_t,x_{t,s}-x_t\rangle \\ &\le (1-s)\|x_{t,s}-x_t\|^2+s\langle f(x_{t,s})-x_t,x_{t,s}-x_t\rangle.\end{aligned} \tag{2.40}$$

Hence

$$\begin{aligned}\|x_{t,s}-x_t\|^2 &\le \langle f(x_{t,s})-x_t,x_{t,s}-x_t\rangle \\ &= \langle f(x_{t,s})-f(x_t),x_{t,s}-x_t\rangle+\langle f(x_t)-x_t,x_{t,s}-x_t\rangle \\ &\le \alpha\|x_{t,s}-\tilde{x}_t\|^2+\langle f(x_t)-x_t,x_{t,s}-x_t\rangle.\end{aligned} \tag{2.41}$$

This implies that

$$\|x_{t,s_n} - x_t\|^2 \leq \frac{1}{1-\alpha}\langle f(x_t) - x_t, x_{t,s_n} - x_t\rangle. \tag{2.42}$$

But $\{x_{t,s_n}\}$ weakly converges to x_t, by passing to the limit in (2.31), it follows that $\{x_{t,s_n}\}$ strongly converges to x_t.

According to the first section, $\{x_t\}$ is bounded and $w - \limsup_{t\to 0} \operatorname{Fix} P_t \subset S$ which together with (2.25) is nothing but $(\operatorname{Fix} P_t)$ converges to S in the sense of Mosco, which in turn amounts to saying, thanks to [7, Proposition 7.4(f)], that the indicator function $(\delta_{\operatorname{Fix} P_t})$ Mosco converges to δ_S. In the light of Attouch's theorem (see [7, Theorem 12.35]), this implies the graph convergence of $(N_{\operatorname{Fix} P_t})$ to N_S. Now, by taking a subnet $\{x_{t_\nu}\}$ which weakly converges to some x_∞ and by passing to the limit in

$$0 \in \left((I - f) + N_{\operatorname{Fix} P_{t_\nu}}\right)(x_{t_\nu}), \tag{2.43}$$

we obtain

$$0 \in \left((I - f) + N_S\right)(x_\infty), \tag{2.44}$$

because $I - f$ is a Lipschitz continuous maximal monotone operator which ensures, by virtue of Lemma 2.2, the fact that the graph convergence of $(N_{\operatorname{Fix} P_t})$ to N_S implies that of $((I - f) + N_{\operatorname{Fix} P_t})$ to $(I - f) + N_S$ and also that the graph of the operator $(I - f) + N_S$ is weakly-strongly closed. The weak cluster point x_∞ being unique, we infer that the whole net $\{x_t\}$ weakly converges to x_∞ which solves (2.28). We conclude by applying the diagonal Lemma 2.8. □

Conclusion. The convergence properties of new types of approximating curves for fixed point problems are investigated relying on the graph convergence. The limits attained by these curves are solutions of variational or quasivariational inequalities involving fixed-point sets. Approximating curves are also relevant to numerical methods since understanding their properties is central in the analysis of parent continuous and discrete dynamical systems, so we envisage to study the related iterative schemes in a forthcoming paper.

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