# FIXED POINTS AND COINCIDENCE POINTS FOR MULTIMAPS WITH NOT NECESSARILY BOUNDED IMAGES

S. V. R. NAIDU

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In metric spaces, single-valued self-maps and multimaps with closed images are considered and fixed point and coincidence point theorems for such maps have been obtained without using the (extended) Hausdorff metric, thereby generalizing many results in the literature including those on the famous conjecture of Reich on multimaps.

## 1. Introduction

Many authors have been using the Hausdorff metric to obtain fixed point and coincidence point theorems for multimaps on a metric space. In most cases, the metric nature of the Hausdorff metric is not used and the existence part of theorems can be proved without using the concept of Hausdorff metric under much less stringent conditions on maps. The aim of this paper is to illustrate this and to obtain fixed point and coincidence point theorems for multimaps with not necessarily bounded images. Incidentally we obtain improvements over the results of Chang [3], Daffer et al. [6], Jachymski [9], Mizoguchi and Takahashi [12], and Węgrzyk [17] on the famous conjecture of Reich on multimaps (Conjecture 3.12).

## 2. Notation

Throughout this paper, unless otherwise stated, $(X,d)$ is a metric space; $C(X)$ is the collection of all nonempty, closed subsets of $X$; $B(X)$ is the collection of all nonempty, bounded subsets of $X$; $\mathrm{CB}(X)$ is the collection of all nonempty, bounded, closed subsets of $X$; $S$, $T$ are self-maps on $X$; $I$ is the identity map on $X$; $F$, $G$ are mappings from $X$ into $C(X)$; for a nonempty subset $A$ of $X$ and $x \in X$, $d(x,A) = \inf\{d(x,y) : y \in A\}$; for nonempty subsets $A$, $B$ of $X$,

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}; \tag{2.1}$$


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$f, g$, and $\rho$ are functions on $X$ defined as $f(x) = d(Sx,Fx)$, $g(x) = d(Tx,Gx)$, and $\rho(x) = d(x,Fx)$ for all $x$ in $X$; for a nonempty subset $A$ of $X$, $\alpha_A = \inf\{f(x) : x \in A\}$, $\beta_A = \inf\{g(x) : x \in A\}$, $\gamma_A = \inf\{\rho(x) : x \in A\}$, and $\delta(A) = \sup\{d(x,y) : x,y \in A\}$; for $x,y$ in $X$ and a nonnegative constant $k$,

$$\begin{aligned}
&A(x,y) = \max\{d(Sx,Ty), d(Sx,Fx), d(Ty,Gy)\},\\
&B_k(x,y) = \max\{A(x,y), k[d(Sx,Gy) + d(Ty,Fx)]\},\\
&A_0(x,y) = \max\{d(Sx,Sy), d(Sx,Fx), d(Sy,Fy)\},\\
&C_0(x,y) = \max\{A_0(x,y), (1/2)[d(Sx,Fy) + d(Sy,Fx)]\},\\
&A_1(x,y) = \max\{d(x,y), d(x,Fx), d(y,Fy)\},\\
&C_1(x,y) = \max\{A_1(x,y), (1/2)[d(x,Fy) + d(y,Fx)]\},\\
&m(x,y) = \max\{d(x,y), d(x,Fx), d(y,Gy), (1/2)[d(x,Gy) + d(y,Fx)]\};
\end{aligned}$$

$\mathbb{N}$ is the set of all positive integers; $\mathbb{R}^+$ is the set of all nonnegative real numbers; $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$; for a real-valued function $\theta$ on a subset $E$ of the real line, $\tilde{\theta}$ and $\hat{\theta}$ are the functions on $E$ defined as $\tilde{\theta}(t) = \limsup_{r\to t+} \theta(r)$ and $\hat{\theta}(t) = \max\{\theta(t), \tilde{\theta}(t)\}$ for all $t$ in $E$; for a self-map $h$ on an arbitrary set $E$, $h^1 = h$, and for a positive integer $n$, $h^{n+1}$ is the composition of $h$ and $h^n$; for $s \in (0,\infty]$, $\Gamma_s = \{\varphi : \varphi$ is increasing on $[0,s)$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty\ \forall t$ in $[0,s)\}$; $\Gamma = \{\varphi : \varphi \in \Gamma_s$ for some $s \in (0,\infty]\}$; $\Gamma^* = \{\varphi \in \Gamma : \varphi(t) < t\ \forall t \in (0,\infty)\}$, $\Gamma' = \{\varphi \in \Gamma^* : \varphi$ is upper semicontinuous from the right on $(0,\infty)\}$; $\mathfrak{I} = \{\varphi : \hat{\varphi}(t) < 1\ \forall t \in (0,\infty)\}$; $\mathfrak{I}_0 = \{\varphi \in \mathfrak{I} : \tilde{\varphi}(0) = 1\}$, and $\mathfrak{I}' = \{\varphi : \varphi(t) < 1\ \forall t \in (0,\infty)\}$. The class $\Gamma_\infty$ was considered by Węgrzyk [17] (with the additional assumption that $\varphi$ is strictly monotonic), whereas the class $\Gamma'$ was introduced independently by Chang [3] and Jachymski [9].

*Remark 2.1.* $H$ restricted to CB($X$) is a metric on CB($X$) and is known as the Hausdorff metric on CB($X$). It is well known that CB($X$) equipped with the Hausdorff metric is a complete metric space. $H$ restricted to $C(X)$ has all the properties of a (complete) metric except that it takes the value $+\infty$ also when $(X,d)$ is unbounded.

## 3. Preliminaries

LEMMA 3.1. *Let $s \in (0,\infty]$ and let $\theta$ be an increasing self-map on $[0,s)$ such that $\theta(t+) < t$ for all $t$ in $(0,s)$ and $\sum_{n=1}^{\infty} \theta^n(t_0) < +\infty$ for some $t_0 \in (0,s)$. Then $\hat{\theta}(0) = 0$ and $\sum_{n=1}^{\infty} \theta^n(t) < +\infty$ for all $t$ in $[0,s)$.*

*Proof.* Since $0 \le \theta(0) \le \theta(t) \le \theta(t+) < t$ for all $t$ in $(0,s)$, we have $\theta(0) = 0$ and $\theta(0+) = 0$. Hence $\hat{\theta}(0) = 0$. Let $r \in [0,t_0)$. Since $\theta$ is increasing on $[0,s)$, it follows that $\theta^n(r) \le \theta^n(t_0)$ for all $n \in \mathbb{N}$. Hence, from the convergence of the series $\sum_{n=1}^{\infty} \theta^n(t_0)$, it follows that the series $\sum_{n=1}^{\infty} \theta^n(r)$ is convergent. We now take $r \in (t_0,s)$. Since $\theta(0) = 0 \le \theta(t) < t$ for all $t$ in $(0,s)$, it follows that $\{\theta^n(r)\}_{n=1}^{\infty}$ decreases to a nonnegative real number $r_0$. We have $r_0 = \lim_{n\to\infty} \theta(\theta^n(r)) \le \theta(r_0+)$. Since $\theta(t+) < t$ for all $t$ in $(0,s)$, we must have $r_0 = 0$. Hence there exists a positive integer $N$ such that $\theta^N(r) < t_0$. Hence, from what we have already proved, it follows that the series $\sum_{n=1}^{\infty} \theta^n(\theta^N(r))$ is convergent. Hence $\sum_{n=1}^{\infty} \theta^n(r)$ is convergent. □

*Remark 3.2.* Let $s \in (0,\infty]$.

(i) If $\theta$ is an increasing self-map on $[0,s)$ and $t_0 \in (0,s)$ is such that $\sum_{n=1}^{\infty} \theta^n(t_0) < +\infty$, then $\theta(t_0) < t_0$.

(ii) If $\theta$ is a self-map on $[0,s)$ such that $\theta(0) = 0$ and $\hat{\theta}(t) < t$ for all $t$ in $(0,s)$, then $\{\theta^n(t)\}_{n=1}^{\infty}$ decreases to zero for all $t$ in $[0,s)$.

(iii) If $\theta$ is a self-map on $[0,s)$ such that $\theta(0) = 0$, $\hat{\theta}(t) < t$ for all $t$ in $(0,s)$, and $\sum_{n=1}^{\infty}\theta^n(t) < +\infty$ for all $t$ in $(0,s_0)$ for some $s_0 \in (0,s)$, then $\sum_{n=1}^{\infty}\theta^n(t) < +\infty$ for all $t$ in $[0,s)$.

(iv) If $\theta \in \Gamma'$, then $\sum_{n=1}^{\infty}\theta^n(t) < +\infty$ for all $t$ in $[0,\infty)$.

(v) If $\theta \in \Gamma$, then $\hat{\theta}(0) = 0$.

(vi) If $\theta$ is a self-map on $[0,s)$ such that $\theta(t) > 0$ and $\hat{\theta}(t) < t$ for all $t$ in $(0,s)$, then $\{\theta^n(t)\}_{n=1}^{\infty}$ strictly decreases to zero for all $t$ in $(0,s)$.

(vii) If $k$ is a constant in $[0,1)$ and $\theta$ is a self-map on $[0,s)$ defined as $\theta(t) = kt$ for all $t$ in $[0,s)$, then $\theta^n(t) = k^n t$ for all $n \in \mathbb{N}$ and for all $t \in [0,s)$ and $\sum_{n=1}^{\infty}\theta^n(t) = (\sum_{n=1}^{\infty}k^n)t = kt/(1-k) < +\infty$ for all $t \in [0,s)$.

The following lemmas throw light on the richness of the class of continuous functions in $\Gamma_\infty$ and its subclass $\{\varphi \in \Gamma_\infty : \varphi$ is continuous on $\mathbb{R}^+$ and $\lim_{t\to 0+}(\varphi(t)/t) = 1\}$.

Lemma 3.3. *Let $s \in (0,\infty]$ and let $\{c_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence in $(0,s)$. Then there exists a strictly increasing continuous function $\theta : [0,s) \to [0,s)$ such that $\theta(t) < t$ for all $t \in (0,s)$ and $\theta(c_n) = c_{n+1}$ for all $n \in \mathbb{N}$.*

*Proof.* Define $\theta$ on $[0,s)$ as $\theta(0) = 0$, $\theta(t) = (c_{n+1}(t - c_{n+1}) + c_{n+2}(c_n - t))/(c_n - c_{n+1})$ if $c_{n+1} < t \le c_n$ for some $n \in \mathbb{N}$, and $\theta(t) = c_2 t/c_1$ if $c_1 < t < s$. Then $\theta$ has the desired properties. □

*Remark 3.4.* Let $s$ and $\{c_n\}_{n=1}^{\infty}$ be as in Lemma 3.3. Let $h$ be a real-valued increasing map on $[0,1]$ such that $h(0) = 0$ and $h(1) > 0$. Define $\theta$ on $[0,s)$ as $\theta(0) = 0$, $\theta(t) = c_{n+2} + ((c_{n+1} - c_{n+2})/h(1))h((t - c_{n+1})/(c_n - c_{n+1}))$ if $c_{n+1} < t \le c_n$ for some $n \in \mathbb{N}$, and $\theta(t) = c_2 t/c_1$ if $c_1 < t < s$. Then $\theta$ is an increasing self-map on $[0,s)$ and $\theta(c_n) = c_{n+1}$ for all $n \in \mathbb{N}$. If $h$ is continuous on $[0,1]$, then $\theta$ is continuous on $[0,s)$. If $h(0+) < h(1)$, then $\theta(t+) < t$ for all $t$ in $(0,s)$.

Lemma 3.5. *Let $s \in (0,\infty]$ and let $\{c_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence in $(0,s)$ such that $\sum_{n=1}^{\infty} c_n < +\infty$. Let $\theta : [0,s) \to [0,s)$ be an increasing map such that $\theta(t+) < t$ for all $t$ in $(0,s)$ and $\theta(c_n) = c_{n+1}$ for all $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty}\theta^n(t) < +\infty$ for all $t$ in $[0,s)$. Further, $\theta(t) > 0$ for all $t$ in $(0,s)$. Moreover, $\theta(t)/t \to 1$ as $t \to 0+$ if $c_{n+1}/c_n \to 1$ as $n \to +\infty$.*

*Proof.* Since $\theta(c_n) = c_{n+1}$ for all $n \in \mathbb{N}$, we have $\theta^n(c_1) = c_{n+1}$ for all $n \in \mathbb{N}$. Hence $\sum_{n=1}^{\infty}\theta^n(c_1) = \sum_{n=2}^{\infty} c_n < +\infty$. Hence, from Lemma 3.1, it follows that $\sum_{n=1}^{\infty}\theta^n(t) < +\infty$ for all $t$ in $[0,s)$. Let $r \in (0,s)$. Since $\{c_n\}$ decreases to zero, there is an $N \in \mathbb{N}$ such that $c_N < r$. Since $\theta$ is increasing on $(0,s)$, we have $\theta(c_N) \le \theta(r)$. Since $\theta(c_N) = c_{N+1} > 0$, $\theta(r) > 0$.

Suppose now that $c_{n+1}/c_n \to 1$ as $n \to +\infty$. Let $t \in [c_{n+1}, c_n]$. Since $\theta$ is increasing on $(0,s)$, we have $\theta(c_{n+1}) \le \theta(t) \le \theta(c_n)$. Hence $c_{n+2} \le \theta(t) \le c_{n+1}$. Hence $c_{n+2}/c_n \le \theta(t)/c_n \le \theta(t)/t \le \theta(t)/c_{n+1} \le 1$. We have $c_{n+2}/c_n = (c_{n+2}/c_{n+1})(c_{n+1}/c_n) \to 1$ as $n \to +\infty$. Hence $\theta(t)/t \to 1$ as $t \to 0+$. □

*Remark 3.6.* In view of Lemma 3.5 and Remark 3.2(vi), we can conclude that if $s \in (0,\infty]$ and $\theta : [0,s) \to [0,s)$ is an increasing map such that $0 < \theta(t+) < t$ for all $t$ in $(0,s)$, then

$\sum_{n=1}^{\infty}\theta^n(t) < +\infty$ for all $t$ in $[0,s)$ if and only if there exists a strictly decreasing sequence $\{c_n\}_{n=1}^{\infty}$ in $(0,s)$ such that $\theta(c_n) = c_{n+1}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} c_n < +\infty$.

Lemma 3.7. *Let $p \in (1,\infty)$ be a constant and let $\theta$ be defined on $\mathbb{R}^+$ as $\theta(t) = t/(1+t^{1/p})^p$ for all $t$ in $\mathbb{R}^+$. Then $\theta$ is a strictly increasing continuous function on $\mathbb{R}^+$, $\theta(t) < t$ for all $t$ in $(0,\infty)$, and $\sum_{n=1}^{\infty}\theta^n(t) < +\infty$ for all $t$ in $\mathbb{R}^+$.*

*Proof.* Let $s \in (0,\infty)$. We have $\theta(1/s^p) = 1/(1+s)^p$, and hence $\theta^n(1/s^p) = 1/(n+s)^p$ for all $n \in \mathbb{N}$. Hence $\sum_{n=1}^{\infty}\theta^n(1/s^p) = \sum_{n=1}^{\infty}(1/(n+s))^p < \sum_{n=1}^{\infty}(1/n^p) < +\infty$ since $p > 1$. Now, for any $t \in (0,\infty)$, take $s = t^{-1/p}$ so that $1/s^p = t$ and hence $\sum_{n=1}^{\infty}\theta^n(t) < +\infty$. Let $0 \le t_1 < t_2 < +\infty$. Let $s_1 = t_1^{1/p}$ and $s_2 = t_2^{1/p}$. Then $0 \le s_1 < s_2 < +\infty$. Hence $s_1/(1+s_1) < s_2/(1+s_2)$. Hence $s_1/(1+s_1)^p < s_2/(1+s_2)^p$. Hence $\theta(t_1) < \theta(t_2)$. Hence $\theta$ is strictly increasing on $\mathbb{R}^+$. The rest of the conclusions in the lemma is evident. □

The following lemma is a slight improvement over Theorem 1 of Sastry et al. [16] and can be deduced from Lemmas 2, 5, 6 and 8 of [16]. For our purposes Theorem 1 of Sastry et al. [16] is enough.

Lemma 3.8. *Suppose that $\varphi \in \Gamma_\infty$ and $\varphi(t+) < t$ for all $t$ in $(0,\infty)$. Then there exists a strictly increasing continuous function $\psi$: $\mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(t) < \psi(t)$ and $\sum_{n=1}^{\infty}\psi^n(t) < +\infty$ for all $t$ in $(0,\infty)$.*

The following lemma is similar to the comparison test for the convergence of a series of nonnegative real numbers and serves as a useful tool in proving the convergence of the sequence of iterates of a self-map on $[0,s)$.

Lemma 3.9. *Let $s \in [0,\infty)$. Let $\theta : [0,s) \to [0,\infty)$ and $\psi : [0,s) \to [0,s)$ be such that $\psi$ is increasing on $[0,s)$, $\theta(t) \le \psi(t)$, and $\sum_{n=0}^{\infty}\psi^n(t) < +\infty$ for all $t$ in $[0,s)$. Then $\theta$ is a self-map on $[0,s)$ and $\sum_{n=0}^{\infty}\theta^n(t) < +\infty$ for all $t$ in $[0,s)$.*

*Proof.* Since $0 \le \theta(t) \le \psi(t)$ for all $t$ in $[0,s)$ and $\psi$ is a self-map on $[0,s)$, $\theta$ is a self-map on $[0,s)$. Let $t \in [0,s)$. Suppose that for a positive integer $m$, we have $\theta^m(t) \le \psi^m(t)$. We have $\theta^{m+1}(t) = \theta(\theta^m(t)) \le \psi(\theta^m(t))$ since $\theta \le \psi$ on $[0,s)$. Since $\psi$ is increasing on $[0,s)$, we have $\psi(\theta^m(t)) \le \psi(\psi^m(t)) = \psi^{m+1}(t)$. Hence $\theta^{m+1}(t) \le \psi^{m+1}(t)$. Hence, from the principle of mathematical induction, we have $\theta^n(t) \le \psi^n(t)$ for all $n \in \mathbb{N}$. Hence, from the convergence of the series $\sum_{n=0}^{\infty}\psi^n(t)$, it follows that the series $\sum_{n=0}^{\infty}\theta^n(t)$ is also convergent. □

The function $t \mapsto t - at^b$ for $a > 0$ and $b \in (1,2)$ was considered by Daffer et al. [6] to show that the class of functions $\{k \in \mathfrak{I}_0 : id_{(0,\infty)}k \in \Gamma'\}$ is nonempty. In view of Lemma 3.7, it is evident that the functions $t \mapsto 1/(1+t^{1/p})^p$ $(p > 1)$ belong to this class. Lemmas 3.3 and 3.5 and Remark 3.4 can also be used to generate a number of functions of this class. The following lemma shows that there are functions of the type considered in Lemma 3.7, which dominate the one considered by Daffer et al. in a right neighborhood of zero.

Lemma 3.10. *Let $a$ be a positive real number and $b \in (1,2)$. Let $p \in (1, 1/(b-1))$. Then there exists $s \in (0,\infty)$ such that $t - at^b < t/(1+t^{1/p})^p$ for all $t$ in $(0,s]$.*

*Proof.* Let $h_1$, $h_2$ be defined on $\mathbb{R}^+$ as $h_1(t) = 1/(1+t^{1/p})^p + at^{b-1}$ and $h_2(t) = t^\gamma/(1+t^{1/p})^{p+1}$ for all $t$ in $\mathbb{R}^+$, where $\gamma = 1/p - b + 1$. Then $h_1'(t) = t^{b-2}[a(b-1) - h_2(t)]$ for all $t$ in $(0,\infty)$ and $\gamma > 0$. Since $h_2$ is continuous on $\mathbb{R}^+$ and $h_2(0) = 0 < a(b-1)$, there exists $s \in (0,\infty)$ such that $h_2(t) < a(b-1)$ for all $t$ in $(0,s)$. Hence $h_1'(t) > 0$ for all $t$ in $(0,s)$, so that $h_1(t) > h_1(0)(=1)$ for all $t$ in $(0,s]$. Hence $th_1(t) > t$ for all $t$ in $(0,s]$, which yields the thesis. □

In the following lemma we give an easy alternative proof of the essential part of Lemma 4 of Daffer et al. [6].

Lemma 3.11 (see [6, Lemma 4]). *Let $a$ be a positive real number, $b \in (1,2)$, and let $\theta$ be defined on $[0,s)$ as $\theta(t) = t - at^b$, where $s = a^{-1/(b-1)}$. Then $\theta$ is a self-map on $[0,s)$, it is strictly increasing on $[0,(ab)^{-1/(b-1)}]$, and $\sum_{n=1}^{\infty}\theta^n(t) < +\infty$ for all $t$ in $[0,s)$.*

*Proof.* Clearly, $\theta(0) = 0$, $\theta(t) < t$ for all $t$ in $[0,s)$, and for a positive real number $t$, $t - at^b > 0$ if and only if $t < s$. Hence $\theta$ is a self-map on $[0,s)$. From Lemmas 3.7, 3.9, and 3.10 and Remark 3.2(iii) it follows that $\sum_{n=1}^{\infty}\theta^n(t) < +\infty$ for all $t$ in $[0,s)$. The strictly increasing nature of $\theta$ in the specified interval follows from the fact that its derivative is positive in the corresponding right open interval. □

The class of functions $\{\varphi \in \Gamma_\infty : \varphi(t+) < t$ for all $t \in (0,\infty)\}$ was first considered by Sastry et al. [16] to obtain common fixed point theorems for a pair of multimaps on a metric space. Later, the class of functions $\Gamma'$ was conceived by Chang [3] (see also [9, Corollary 4.22 and Remark 4.23]) in an attempt to establish the famous conjecture of Reich on multimaps (Conjecture 3.12) partially by using Theorem 1 of Sastry et al. [16].

Conjecture 3.12 [14, 15]. *If $(X,d)$ is complete, $F : X \to \mathrm{CB}(X)$, $k \in \Im$, and*

$$H(Fx,Fy) \le k\big(d(x,y)\big)d(x,y) \tag{3.1}$$

*for all $x,y$ in $X$, then $F$ has a fixed point in $X$.*

In light of the fact that Mizoguchi and Takahashi [12] established the truth of Reich's conjecture (Conjecture 3.12) for $k \in \Im$ under the additional hypothesis $\tilde{k}(0) < 1$ on the control function $k$ (see Corollary 4.17), the class of functions $\Im_0$ has become significant. Daffer et al. [6] tried to establish the conjecture (see [6, Theorem 5]) for a subclass of $\Im_0$ using [3, Theorem 7] (i.e., Corollary 4.31) (see Remark 4.32). In this paper we observe that the conjecture is true for a $k \in \Im$ if there exist an $s \in (0,\infty)$ and an increasing self-map $\psi$ on $[0,s)$ such that $\psi(t+) < t$ and $tk(t) \le \psi(t)$ for all $t$ in $(0,s)$, and $\sum_{n=1}^{\infty}\psi^n(t_0) < +\infty$ for some $t_0 \in (0,s)$. In fact, in place of the condition $\hat{k}(t) < 1$ for all $t$ in $(0,\infty)$, we use the weaker condition $\hat{k}(t) < 1$ for all $t$ in $(0,d(x_0,Fx_0)]$ for some $x_0 \in X$, and in place of inequality (3.1), we use considerably weaker conditions (see Corollary 4.47).

The following lemma is taken in part from the paper by Altman [1].

Lemma 3.13. *Let $s \in (0,\infty]$. Suppose that $\varphi$ is increasing on $[0,s)$, $\varphi(t) < t$ for all $t$ in $(0,s)$, the function $\chi : (0,s) \to (0,\infty)$ defined as $\chi(t) = t/(t-\varphi(t))$ is decreasing on $(0,s)$, and $\int_0^{s_0}\chi(t)dt < +\infty$ for some $s_0 \in (0,\infty)$. Then $\varphi$ is continuous on $[0,s)$ and $\sum_{n=1}^{\infty}\varphi^n(t) < +\infty$ for all $t \in [0,s)$.*

*Proof.* Since $\varphi$ is nonnegative, increasing on $[0,s)$ and $\varphi(t) < t$ for all $t$ in $(0,s)$, we have $0 \le \varphi(0) \le \varphi(t) < t$ for all $t$ in $(0,s)$. Hence $\varphi(0) = 0$ and $\varphi$ is continuous at zero. Since $\varphi(t) < t$ for all $t$ in $(0,s)$, we have $\chi(t) = 1/(1-\varphi(t)/t) > 0$ for all $t$ in $(0,s)$. Since $\chi$ is positive and decreasing on $(0,s)$, $1/\chi$ is increasing on $(0,s)$. Hence $\varphi(t)/t$ is decreasing on $(0,s)$. Let $t_0 \in (0,s)$. Then we have

$$\frac{\varphi(u)}{u} \le \frac{\varphi(t_0)}{t_0} \le \frac{\varphi(v)}{v} \tag{3.2}$$

for all $u \in (t_0,s)$ and for all $v \in (0,t_0)$. Since $\varphi$ is increasing on $(0,s)$, $\varphi(t_0-)$ and $\varphi(t_0+)$ exist and $\varphi(t_0-) \le \varphi(t_0) \le \varphi(t_0+)$. But, on taking limits in inequality (3.2) as $u \to t_0+$ and $v \to t_0-$, we obtain $\varphi(t_0+) \le \varphi(t_0) \le \varphi(t_0-)$. Hence $\varphi(t_0-) = \varphi(t_0) = \varphi(t_0+)$. Hence $\varphi$ is continuous at $t_0$. Thus $\varphi$ is continuous on $[0,s)$. The convergence of the series $\sum_{n=1}^{\infty} \varphi^n(t)$ was proved by Altman [1]. □

The following definition was introduced by Dugundji [7].

*Definition 3.14.* A function $\theta : X \times X \to [0,\infty)$ is said to be compactly positive if $\inf\{\theta(x,y) : x,y \in X \text{ and } a \le d(x,y) \le b\}$ is positive for any positive real numbers $a$ and $b$ with $a \le b$.

Lemma 3.15. *Let $\theta$ be a compactly positive function on $X \times X$ such that $\theta(x,y) \le d(x,y)$ for all $x,y$ in $X$, and that there exists a positive real number $\epsilon$ such that*

$$\inf\left\{\frac{\theta(x,y)}{d(x,y)} : x,y \in X \text{ and } 0 < d(x,y) \le \epsilon\right\} > 0. \tag{3.3}$$

*Define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(0) = 0$ and $\varphi(t) = t\psi(t)$ if $t > 0$, where $\psi(t) = \sup\{1-\theta(x,y)/d(x,y) : 0 < d(x,y) \le t\}$. Then $\varphi \in \Gamma_\infty$ and $\varphi(t+) < t$ for all $t$ in $(0,\infty)$.*

*Proof.* Evidently, $\psi$ is increasing on $(0,\infty)$. Let $t \in (0,\infty)$. We show that $\psi(t) < 1$. There exist sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $0 < d(x_n,y_n) \le t$ for all $n$ and $\{1-\theta(x_n,y_n)/d(x_n,y_n)\}$ converges to $\psi(t)$. Since $\{d(x_n,y_n)\}$ is a bounded sequence of real numbers, it contains a convergent subsequence. Without loss of generality, we may assume that $\{d(x_n,y_n)\}$ itself is convergent. Let its limit be denoted as $r$.

*Case (i): $r = 0$.* In this case, from inequality (3.3), it follows that there exists a positive real number $c$ $(\le 1)$ such that $\theta(x_n,y_n)/d(x_n,y_n) > c$ for all sufficiently large $n$. Hence $1-\theta(x_n,y_n)/d(x_n,y_n) < (1-c)$ for all sufficiently large $n$. Hence $\psi(t) \le 1-c < 1$.

*Case (ii): $r > 0$.* In this case there exists a positive integer $N$ such that $d(x_n,y_n) \ge r/2$ for all $n \ge N$. Let $\gamma = \inf\{\theta(x,y) : x,y \in X \text{ and } r/2 \le d(x,y) \le t\}$. Since $\theta$ is compactly positive, $\gamma > 0$. We have $\theta(x_n,y_n) \ge \gamma$ for all $n \ge N$. Hence $1-\theta(x_n,y_n)/d(x_n,y_n) \le 1-\gamma/d(x_n,y_n)$ for all $n \ge N$. Hence $\psi(t) \le 1-\gamma/r < 1$.

Since $\psi$ is increasing on $(0,\infty)$ and $\psi(t) < 1$ for all $t \in (0,\infty)$, it follows that $\psi(t+) < 1$ for all $t \in (0,\infty)$. Since $\varphi(0) = 0$, $\varphi(t) = t\psi(t)$ for all $t \in (0,\infty)$ and $\psi$ is nonnegative, it follows that $\varphi$ is increasing on $\mathbb{R}^+$ and $\varphi(t+) < t$ for all $t \in (0,\infty)$. Let $\{t_n\}$ be a sequence in $(0,\infty)$ converging to zero. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $0 < d(x_n, y_n) \le t_n$ for all $n$ and $\psi(t_n) - 1/n < 1 - (\theta(x_n, y_n)/d(x_n, y_n))(\le \psi(t_n))$ for all $n$. Since $\{t_n\}$ converges to zero, $\{d(x_n, y_n)\}$ converges to zero and $\{1 - \theta(x_n, y_n)/d(x_n, y_n)\}$ converges to $\psi(0+)$. As in case (i) it can be seen here that $1 - \theta(x_n, y_n)/d(x_n, y_n) \le k'$ for some real number $k' \in [0,1)$. Hence $\psi(0+) \le k'$. Let $k \in (k',1)$. Then there exists $s \in (0,\infty)$ such that $\psi(t) < k$ for all $t$ in $(0,s)$. Hence $\varphi(t) \le kt$ for all $t$ in $[0,s)$. Hence, from Remark 3.2(vii) and Lemma 3.9, it follows that $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for all $t$ in $[0,s)$. Since $\varphi$ is increasing on $\mathbb{R}^+$ and $\varphi(t+) < t$ for all $t$ in $(0,\infty)$, from Lemma 3.1, it follows that $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for all $t$ in $(0,\infty)$. Hence $\varphi \in \Gamma_\infty$. □

We now state and prove a number of propositions, some of which are interesting in themselves, while the others are useful in proving fixed point and coincidence point theorems.

Proposition 3.16. *Suppose that $\varphi(t) \le t$ for all $t \in \mathbb{R}^+$ and $A$ is a nonempty subset of $X$ such that $Fx \subseteq TA$ and $Gx \subseteq SA$ for all $x$ in $A$, and for $x, y$ in $A$,*

$$d(Sx, Fx) \le \varphi\big(d(Sx, Ty)\big) \quad \text{if } Sx \in Gy, \tag{3.4}$$

$$d(Ty, Gy) \le \varphi\big(d(Sx, Ty)\big) \quad \text{if } Ty \in Fx. \tag{3.5}$$

*Then $\alpha_A = \beta_A$.*

*Proof.* Let $y \in A$. Since $Gy \subseteq SA$, there exists a sequence $\{x_n\}$ in $A$ such that $Sx_n \in Gy$ for all $n \in \mathbb{N}$ and $\{d(Ty, Sx_n)\}_{n=1}^{\infty}$ converges to $d(Ty, Gy)$. From the definition of $\alpha_A$, inequality (3.4), and the hypothesis that $\varphi(t) \le t$ for all $t \in \mathbb{R}^+$, we have $\alpha_A \le d(Sx_n, Fx_n) \le \varphi(d(Sx_n, Ty)) \le d(Ty, Sx_n)$ for all $n$ in $\mathbb{N}$. Hence $\alpha_A \le d(Ty, Gy)$. Since $y \in A$ is arbitrary, it follows from the definition of $\beta_A$ that $\alpha_A \le \beta_A$. On using the hypothesis that $Fx \subseteq TA$ for all $x$ in $A$ and inequality (3.5), it can be shown that $\beta_A \le \alpha_A$. Hence $\alpha_A = \beta_A$. □

Proposition 3.17. *Suppose that $A$ is a nonempty subset of $X$ such that $Gx \subseteq SA$ for all $x$ in $A$ and for $x, y$ in $A$, inequality (3.4) is true. Then $\alpha_A \le \hat{\varphi}(\beta_A)$.*

*Proof.* There exists a sequence $\{y_n\}_{n=1}^{\infty}$ in $A$ such that $\{d(Ty_n, Gy_n)\}_{n=1}^{\infty}$ converges to $\beta_A$. Since $Gx \subseteq SA$ for all $x \in A$, for each $n \in \mathbb{N}$, there exists $x_n \in A$ such that $Sx_n \in Gy_n$ and $d(Sx_n, Ty_n) < d(Ty_n, Gy_n) + 1/n$. Since $\beta_A \le d(Ty_n, Gy_n) \le d(Sx_n, Ty_n)$ for all $n \in \mathbb{N}$, it follows that $\{d(Sx_n, Ty_n)\}_{n=1}^{\infty}$ converges to $\beta_A$ from the right. From the definition of $\alpha_A$ and inequality (3.4) we have $\alpha_A \le d(Sx_n, Fx_n) \le \varphi(d(Sx_n, Ty_n))$ for all $n \in \mathbb{N}$. Hence $\alpha_A \le \hat{\varphi}(\beta_A)$. □

Proposition 3.18. *Suppose that $A$ is a nonempty subset of $X$ such that $Fx \subseteq TA$ and $Gx \subseteq SA$ for all $x$ in $A$, for $x, y$ in $A$, inequalities (3.4) and (3.5) are true, and that $\hat{\varphi}(0) = 0$ and $\hat{\varphi}(t) < t$ for all $t$ in $(0, s']$ for some real number $s' \ge \max\{\alpha_A, \beta_A\}$. Then $\alpha_A = \beta_A = 0$.*

*Proof.* From Proposition 3.17 we have $\alpha_A \le \hat{\varphi}(\beta_A)$. From the analogue of Proposition 3.17 obtained by interchanging $S$ and $T$ and also $F$ and $G$ we obtain $\beta_A \le \hat{\varphi}(\alpha_A)$. Hence, if one of $\alpha_A$, $\beta_A$ is zero, then from the hypothesis that $\hat{\varphi}(0) = 0$, it follows that the other is also zero, and if both are positive, then from the hypothesis that $\hat{\varphi}(t) < t$ for all $t$ in $(0, s']$ for some real number $s' \ge \max\{\alpha_A, \beta_A\}$, we arrive at the contradictory inequalities $\alpha_A < \beta_A$ and $\beta_A < \alpha_A$. Hence $\alpha_A = \beta_A = 0$. □

Proposition 3.19. *Suppose that $A$ is a nonempty subset of $X$ such that one of $\alpha_A$, $\beta_A$ is zero, $Fx \subseteq TA$ and $Gx \subseteq SA$ for all $x$ in $A$, for $x, y$ in $A$, inequalities (3.4) and (3.5) are true, and that $\varphi \in \Gamma_s$ and $\varphi(t+) < t$ for all $t$ in $(0,s)$ for some $s \in (0,\infty]$. Then $\alpha_A = \beta_A = 0$ and there exist a sequence $\{x_n\}_{n=0}^{\infty}$ in $A$ and a sequence $\{y_n\}_{n=0}^{\infty}$ in $X$ such that $y_{2n+1} = Tx_{2n+1} \in Fx_{2n}$, $y_{2n+2} = Sx_{2n+2} \in Gx_{2n+1}$ $(n = 0,1,2,\ldots)$, and $\{y_n\}_{n=0}^{\infty}$ is Cauchy.*

*Proof.* Let $s_0 \in (0,s)$. Define $\varphi_0 : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi_0(t) = \varphi(t)$ if $0 \le t \le s_0$ and $\varphi_0(t) = \varphi(s_0)$ if $t > s_0$. Then $\varphi_0 \in \Gamma_\infty$ and $\varphi_0(t+) < t$ for all $t$ in $(0,\infty)$. Hence, from Lemma 3.8, it follows that there exists a strictly increasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi_0(t) < \psi(t)$ and $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t$ in $(0,\infty)$.

Suppose that $\alpha_A = 0$. Then there exists $x_0 \in A$ such that $d(Sx_0, Fx_0) < s_0$. Let $y_0 = Sx_0$. Choose $y_1 \in Fx_0$ such that $d(Sx_0, y_1) < s_0$ subject to the condition that $y_1 = y_0$ if $Sx_0 \in Fx_0$. Since $y_1 \in Fx_0 \subseteq T(A)$, there exists $x_1 \in A \ni y_1 = Tx_1$.

If $Sx_0 \in Fx_0$, we take $y_2 = y_1$. When $Sx_0 \in Fx_0$, from the selection of $y_1$, we have $y_1 = y_0$, that is, $Sx_0 = Tx_1$ so that from inequality (3.5) and the closedness of $Gx_1$ we have $Tx_1 \in Gx_1$ and hence $y_2 \in Gx_1$. Suppose that $Sx_0 \notin Fx_0$. Then $d(Sx_0, Tx_1) > 0$. We note that $d(Sx_0, Tx_1) < s_0$. Hence $\varphi(d(Sx_0, Tx_1)) < \psi(d(Sx_0, Tx_1))$. Hence, from inequality (3.5), we have $d(Tx_1, Gx_1) < \psi(d(Sx_0, Tx_1))$. Hence we can choose $y_2 \in Gx_1$ such that $d(Tx_1, y_2) < \psi(d(Sx_0, Tx_1))$ subject to the condition that $y_2 = Tx_1$ if $Tx_1 \in Gx_1$. Thus, irrespective of whether $Sx_0$ belongs to $Fx_0$ or not, we can always choose an element $y_2$ of $Gx_1$ such that

$$d(y_1, y_2) \le \psi(d(y_0, y_1)) \tag{3.6}$$

subject to the condition that $y_2 = Tx_1$ if $Tx_1 \in Gx_1$. Since $y_2 \in Gx_1 \subseteq S(A)$, there exists an element $x_2$ of $A$ such that $y_2 = Sx_2$.

If $Tx_1 \in Gx_1$, we take $y_3 = y_2$. When $Tx_1 \in Gx_1$, from the selection of $y_2$, we have $y_2 = y_1$, that is, $Sx_2 = Tx_1$ so that from inequality (3.4) and the closedness of $Fx_2$ we have $Sx_2 \in Fx_2$ and hence $y_3 \in Fx_2$. Suppose that $Tx_1 \notin Gx_1$. Then $d(Tx_1, Sx_2) > 0$. From inequality (3.6) we have $d(y_1, y_2) \le d(y_0, y_1) < s_0$. Hence $\varphi(d(Tx_1, Sx_2)) < \psi(d(Tx_1, Sx_2))$. Hence, from inequality (3.4), we have $d(Sx_2, Fx_2) < \psi(d(Tx_1, Sx_2))$. Hence we can choose $y_3 \in Fx_2$ such that $d(Sx_2, y_3) < \psi(d(Tx_1, Sx_2))$ subject to the condition that $y_3 = Sx_2$ if $Sx_2 \in Fx_2$. Thus, irrespective of whether $Tx_1$ belongs to $Gx_1$ or not, we can always choose an element $y_3$ of $Fx_2$ such that

$$d(y_2, y_3) \le \psi(d(y_1, y_2)) \tag{3.7}$$

subject to the condition that $y_3 = Sx_2$ if $Sx_2 \in Fx_2$. Since $y_3 \in Fx_2 \subseteq T(A)$, there exists an element $x_3$ of $A$ such that $y_3 = Tx_3$.

On proceeding like this, we obtain sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ in $A$ such that $y_{2n+1} = Tx_{2n+1} \in Fx_{2n}$, $y_{2n+2} = Sx_{2n+2} \in Gx_{2n+1}$ $(n = 0,1,2,\ldots)$,

$$d(y_n, y_{n+1}) \le \psi(d(y_{n-1}, y_n)) \quad (n \in \mathbb{N}) \tag{3.8}$$

and subject to the condition that for any nonnegative integer $n$, $y_{2n+1} = y_{2n}$ if $Sx_{2n} \in Fx_{2n}$ and $y_{2n+2} = y_{2n+1}$ if $Tx_{2n+1} \in Gx_{2n+1}$.

On repeatedly using inequality (3.8), we obtain $d(y_n, y_{n+1}) \le \psi^n(d(y_0, y_1))$ for all $n \in \mathbb{N}$. Hence, for $n, m \in \mathbb{N}$ with $m > n$, we have $d(y_n, y_m) \le \sum_{k=n}^{m-1} d(y_k, y_{k+1}) \le \sum_{k=n}^{m-1} \psi^k(t_0)$, where $t_0 = d(y_0, y_1)$. Since $\sum_{k=1}^{\infty} \psi^k(t_0) < +\infty$, it follows that $d(y_n, y_m) \to 0$ as both $m$ and $n$ tend to $+\infty$. Hence $\{y_n\}_{n=o}^{\infty}$ is Cauchy. Since $d(Tx_{2n+1}, Gx_{2n+1}) \le d(y_{2n+1}, y_{2n+2}) \to 0$ as $n \to +\infty$, it follows that $\beta_A = 0$. In a similar manner, it can be shown that $\alpha_A = 0$ if we assume that $\beta_A = 0$. □

Proposition 3.20. *Suppose that* $\varphi(0) = 0$ *and* $A$ *is a nonempty subset of* $X$ *such that* $Fx \subseteq TA$ *and* $Gx \subseteq SA$ *for all* $x$ *in* $A$, *and for* $x, y$ *in* $A$, *inequalities (3.4) and (3.5) are true. Then* $\{Sx : x \in A \text{ and } Sx \in Fx\} = \{Ty : y \in A \text{ and } Ty \in Gy\}$.

*Proof.* Let $x \in A$ be such that $Sx \in Fx$. Since $Fx \subseteq T(A)$, there exists a $y \in A$ such that $Sx = Ty$. Now, from inequality (3.5), we have $d(Ty, Gy) = 0$. Since $Gy$ is closed, $Ty \in Gy$. Conversely, suppose that $y \in A$ is such that $Ty \in Gy$. Since $Gy \subseteq S(A)$, there exists an $x \in A$ such that $Ty = Sx$. Now, from inequality (3.4), we have $d(Sx, Fx) = 0$. Since $Fx$ is closed, $Sx \in Fx$. Hence $\{Sx : x \in A \text{ and } Sx \in Fx\} = \{Ty : y \in A \text{ and } Ty \in Gy\}$. □

Proposition 3.21. *Suppose that* $\varphi(t) < t$ *for all* $t$ *in* $(0, \infty)$ *and* $A$ *is a nonempty subset of* $X$ *such that*

$$H(Fx, Gy) \le \max\{\varphi(d(Sx, Ty)), \varphi(A(x, y)), \varphi(B_{1/2}(x, y))\} \tag{3.9}$$

*for all* $x, y$ *in* $A$. *Then inequalities (3.4) and (3.5) are true for* $x, y$ *in* $A$.

*Proof.* Let $x, y \in A$ be such that $Sx \in Gy$. Then $d(Sx, Fx) \le H(Fx, Gy)$, $d(Ty, Gy) \le d(Sx, Ty)$, $(1/2)[d(Sx, Gy) + d(Ty, Fx)] = (1/2)d(Ty, Fx) \le (1/2)[d(Ty, Sx) + d(Sx, Fx)] \le \max\{d(Sx, Ty), d(Sx, Fx)\} = A(x, y) = B_{1/2}(x, y)$, and the right-hand side of inequality (3.9) is less than or equal to $\max\{\varphi(d(Sx, Ty)), \varphi(d(Sx, Fx))\}$. Hence, from inequality (3.9), we have $d(Sx, Fx) \le \max\{\varphi(d(Sx, Ty)), \varphi(d(Sx, Fx))\}$. Since $\varphi(t) < t$ for all $t$ in $(0, \infty)$, it follows that $d(Sx, Fx) \le \varphi(d(Sx, Ty))$. Similarly, it can be shown that inequality (3.5) is also true for $x, y \in A$. □

*Remark 3.22.* Unless $\varphi$ is increasing on $\mathbb{R}^+$, the right-hand side of inequality (3.9) may not be equal to $\varphi(B_{1/2}(x, y))$.

*Definition 3.23.* We say that the pair $(F, S)$ has property $P$ with respect to the pair $(G, T)$ if $d(Sw, Fw) = 0$ whenever $w \in X$ is such that there are sequences $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ in $X$ such that $v_{2n+1} = Tu_{2n+1} \in Fu_{2n}$, $v_{2n+2} = Su_{2n+2} \in Gu_{2n+1}$ for all $n = 0, 1, 2, \ldots$, and $\{v_n\}_{n=0}^{\infty}$ converges to $Sw$.

Proposition 3.24. *If* $\varphi(0) = 0$, $\varphi(t) < t$ *for all* $t \in (0, \infty)$, $k$ *is a constant in* $[0, 1)$, *and*

$$H(Fx, Gy) \le \max\{\varphi(d(Sx, Ty)), \varphi(A(x, y)), \varphi(B_k(x, y))\} \tag{3.10}$$

*for all* $x, y$ *in* $X$, *then* $(F, S)$ *has property* $P$ *with respect to* $(G, T)$ *and vice versa.*

*Proof.* Let $w$, $\{u_n\}$, and $\{v_n\}$ be as in Definition 3.23. If possible, suppose that $d(Sw,Fw) > 0$. We have $B_k(w,u_{2n+1}) = \max\{d(Sw,v_{2n+1}), d(Sw,Fw), d(v_{2n+1},Gu_{2n+1}), k[d(Sw,Gu_{2n+1}) + d(v_{2n+1},Fw)]\}$ for all $n \in \mathbb{N}$. Since $\{d(Sw,v_{2n+1})\}$ converges to zero, $\{d(v_{2n+1},Fw)\}$ converges to $d(Sw,Fw)$, $d(v_{2n+1},Gu_{2n+1}) \le d(v_{2n+1},v_{2n+2}) \to 0$ as $n \to +\infty$, $d(Sw,Gu_{2n+1}) \le d(Sw,v_{2n+2}) \to 0$ as $n \to +\infty$, we have $B_k(w,u_{2n+1}) = d(Sw,Fw)$ for all sufficiently large $n$. Similarly, it can be seen that $A(w,u_{2n+1}) = d(Sw,Fw)$ for all sufficiently large $n$. Since $\varphi(0) = 0$, $\varphi(t) < t$ for all $t \in (0,\infty)$, we have $\hat{\varphi}(0) = 0$. Since $d(Sw,Tu_{2n+1}) = d(Sw,v_{2n+1}) \to 0$ as $n \to +\infty$, it follows that $\varphi(d(Sw,Tu_{2n+1})) \to 0$ as $n \to +\infty$. Hence $\max\{\varphi(d(Sw,Tu_{2n+1})), \varphi(A(w,u_{2n+1})), \varphi(B_k(w,u_{2n+1}))\} \to \varphi(d(Sw,Fw))$ as $n \to +\infty$. Since $d(v_{2n+2},Fw) \le H(Fw, Gu_{2n+1})$, from inequality (3.10), we have

$$d(v_{2n+2},Fw) \le \max\{\varphi(d(Sw,Tu_{2n+1})), \varphi(A(w,u_{2n+1})), \varphi(B_k(w,u_{2n+1}))\} \tag{3.11}$$

for all $n \in \mathbb{N}$. On taking limits on both sides of the above inequality as $n \to +\infty$, we obtain $d(Sw,Fw) \le \varphi(d(Sw,Fw))$. This is a contradiction since $\varphi(t) < t$ for all $t \in (0,\infty)$. Hence we must have $d(Sw,Fw) = 0$. Hence $(F,S)$ has property $P$ with respect to $(G,T)$. Similarly, it can be shown that $(G,T)$ has property $P$ with respect to $(F,S)$. □

*Remark 3.25.* Unless $\varphi$ is increasing on $\mathbb{R}^+$, the right-hand side of inequality (3.10) may not be equal to $\varphi(B_k(x,y))$.

Proposition 3.26. *If $\hat{\varphi}(0) = 0$ and*

$$H(Fx,Gy) \le \varphi(d(Sx,Ty)) \tag{3.12}$$

*for all $x, y$ in $X$, then $(F,S)$ has property $P$ with respect to $(G,T)$ and vice versa.*

*Definition 3.27.* We say that $F$ and $S$ are $w$-compatible (or that the pair $(F,S)$ is $w$-compatible) if $d(Sv_n, FSu_n) \to 0$ as $n \to \infty$ whenever $\{u_n\}$ and $\{v_n\}$ are sequences in $X$ such that $\{Su_n\}$ is convergent in $X$, $v_n \in Fu_n$ for all $n$, and $\{d(Su_n,v_n)\}$ converges to zero.

*Remark 3.28.* For single-valued maps, the notion of $w$-compatibility coincides with the notion of compatibility introduced by Jungck [10]. If $S$ is the identity map on $X$, then $(F,S)$ is $w$-compatible.

*Definition 3.29.* We say that $F$ and $S$ are $w^*$-compatible (or that the pair $(F,S)$ is $w^*$-compatible) if $S^2x \in FSx$ for any $x \in X$ such that $Sx \in Fx$.

*Remark 3.30.* If $(F,S)$ is $w$-compatible, then $(F,S)$ is $w^*$-compatible. If $S = I$, then evidently $(F,S)$ is $w^*$-compatible.

*Definition 3.31* [11]. Let $F : X \to \mathrm{CB}(X)$. We say that $F$ and $S$ are compatible (or that the pair $(F,S)$ is compatible) if $SFx \in \mathrm{CB}(X)$ for all $x \in X$ and if $\lim_{n\to\infty} H(FSu_n, SFu_n) = 0$ whenever $\{u_n\}$ is a sequence in $X$ such that there exists an $A \in \mathrm{CB}(X)$ such that $\{H(Fu_n, A)\}$ converges to zero and $\{Su_n\}$ converges to an element of $A$.

*Remark 3.32.* If $F : X \to \mathrm{CB}(X)$ and $(F,S)$ is compatible, then $(F,S)$ is $w^*$-compatible.

The concept of weakly contractive self-maps on a metric space was introduced by Dugundji and Granas [8]. It was extended for set-valued maps by Daffer and Kaneko [4] in the following form.

*Definition 3.33.* A mapping $F : X \to \mathrm{CB}(X)$ is said to be weakly contractive if there exists a compactly positive function $\theta$ on $X \times X$ such that

$$H(Fx,Fy) \leq d(x,y) - \theta(x,y) \tag{3.13}$$

for all $x, y$ in $X$.

## 4. Fixed point and coincidence point theorems

Theorem 4.1. *Suppose that $(X,d)$ is complete, $A$ is a nonempty subset of $X$ such that $Fx \subseteq TA$ and $Gx \subseteq SA$ for all $x$ in $A$, and for $x, y$ in $A$, inequalities (3.4) and (3.5) are true, and that $\varphi \in \Gamma$ and $\hat{\varphi}(t) < t$ for all $t$ in $(0,s']$ for some positive real number $s' \geq \max\{\alpha_A, \beta_A\}$. Then the following statements are true.*

(1) *If $S$ is continuous on $X$, $f$ is lower semicontinuous on $X$, and $(F,S)$ is $w$-compatible, then $\{x \in X : Sx \in Fx\} \neq \phi$.*
(2) *If $T$ is continuous on $X$, $g$ is lower semicontinuous on $X$, and $(G,T)$ is $w$-compatible, then $\{x \in X : Tx \in Gx\} \neq \phi$.*
(3) *If either (i) $S(A)$ is closed and $(F,S)$ has property $P$ with respect to $(G,T)$ or (ii) $T(A)$ is closed and $(G,T)$ has property $P$ with respect to $(F,S)$, then $\{Sx : x \in A$ and $Sx \in Fx\} = \{Tx : x \in A$ and $Tx \in Gx\} \neq \phi$.*

*Proof.* Since $\varphi \in \Gamma$, $\hat{\varphi}(0) = 0$. Now, from Propositions 3.18 and 3.19, it follows that $\alpha_A = \beta_A = 0$ and that there exist a sequence $\{x_n\}_{n=0}^{\infty}$ in $A$ and a sequence $\{y_n\}_{n=0}^{\infty}$ in $X$ such that $y_{2n+1} = Tx_{2n+1} \in Fx_{2n}$, $y_{2n+2} = Sx_{2n+2} \in Gx_{2n+1}$ $(n = 0,1,2,\ldots)$, and $\{y_n\}_{n=0}^{\infty}$ is Cauchy. Since $(X,d)$ is complete, $\{y_n\}_{n=0}^{\infty}$ converges to an element $z$ of $X$.

Suppose that the hypothesis of statement (1) is true. Since $\{y_{2n}\}_{n=0}^{\infty}$ converges to $z$, from the lower semicontinuity of $f$, we have $d(Sz,Fz) \leq \liminf_{n\to\infty} d(Sy_{2n},Fy_{2n})$. We have $d(Sy_{2n},Fy_{2n}) \leq d(Sy_{2n},Sy_{2n+1}) + d(Sy_{2n+1},Fy_{2n})$ for all $n \in \mathbb{N}$. Since $y_{2n+1} \in Fx_{2n}$, $y_{2n} = Sx_{2n} \to z$ as $n \to +\infty$, $d(y_{2n},y_{2n+1}) \to 0$ as $n \to +\infty$, and $(F,S)$ is $w$-compatible, it follows that $d(Sy_{2n+1},Fy_{2n}) \to 0$ as $n \to +\infty$. Since $S$ is continuous on $X$ and $\{y_n\}_{n=0}^{\infty}$ converges to $z$, $d(Sy_{2n},Sy_{2n+1}) \to 0$ as $n \to +\infty$. Hence $d(Sy_{2n},Fy_{2n}) \to 0$ as $n \to +\infty$. Hence $d(Sz,Fz) = 0$. Since $Fz$ is closed, $Sz \in Fz$. Hence $\{x \in X : Sx \in Fx\}$ is nonempty. In a similar manner, statement (2) can be proved.

Suppose that (i) of statement (3) is true. Since $\{y_{2n}\}$ is a sequence in $S(A)$ converging to $z$ and $S(A)$ is closed, $z \in S(A)$. Hence there exists $w \in A \ni Sw = z$. Since $y_{2n+1} = Tx_{2n+1} \in Fx_{2n}$, $y_{2n+2} = Sx_{2n+2} \in Gx_{2n+1}$ $(n = 0,1,2,\ldots)$, $\{y_n\}_{n=0}^{\infty}$ converges to $z = Sw$, and $(F,S)$ has property $P$ with respect to $(G,T)$, it follows that $d(Sw,Fw) = 0$. Since $Fw$ is closed, $Sw \in Fw$. Hence $\{x \in A : Sx \in Fx\}$ is nonempty. In a similar manner, it can be shown that $\{x \in A : Tx \in Gx\} \neq \phi$ if (ii) of statement (3) is true. Statement (3) now follows from Proposition 3.20. □

Corollary 4.2. *Suppose that $(X,d)$ is complete, $\varphi \in \Gamma$, $\hat{\varphi}(t) < t$ for all $t$ in $(0,s']$ for some positive real number $s' \geq \max\{\alpha_X, \beta_X\}$, $Fx \subseteq TX$ and $Gx \subseteq SX$ for all $x$ in $X$, and that* *for $x,y$ in $X$, inequalities (3.4) and (3.5) are true. Then $\{Sx : x \in X$ and $Sx \in Fx\}$ and $\{Tx : x \in X$ and $Tx \in Gx\}$ are nonempty and equal, provided that one of the following statements is true.*

(1) *$S$ is continuous on $X$, $f$ is lower semicontinuous on $X$, and $(F,S)$ is $w$-compatible.*
(2) *$T$ is continuous on $X$, $g$ is lower semicontinuous on $X$, and $(G,T)$ is $w$-compatible.*
(3) *$S(X)$ is closed and $(F,S)$ has property $P$ with respect to $(G,T)$.*
(4) *$T(X)$ is closed and $(G,T)$ has property $P$ with respect to $(F,S)$.*

*Proof.* The proof follows from Theorem 4.1 and Proposition 3.20 on taking $A = X$. □

Corollary 4.3. *Suppose that $(X,d)$ is complete, either $S(X)$ or $T(X)$ is closed, $Fx \subseteq TX$ and $Gx \subseteq SX$ for all $x$ in $X$, $\varphi \in \Gamma^*$, $\hat{\varphi}(t) < t$ for all $t$ in $(0,s']$ for some positive real number $s' \geq \max\{\alpha_X,\beta_X\}$, and inequality (3.9) is true for all $x,y$ in $X$. Then $\{Sx : x \in X$ and $Sx \in Fx\}$ and $\{Tx : x \in X$ and $Tx \in Gx\}$ are nonempty, closed sets and are equal.*

*Proof.* Let $A = \{Sx : x \in X \text{ and } Sx \in Fx\}$ and $B = \{Tx : x \in X \text{ and } Tx \in Gx\}$. Since $\varphi(t) < t$ for all $t$ in $(0,\infty)$, from Proposition 3.21, it follows that for $x,y \in X$, inequalities (3.4) and (3.5) are true. Since $\varphi \in \Gamma$, $\hat{\varphi}(0) = 0$. From Proposition 3.24 it follows that $(F,S)$ has property $P$ with respect to $(G,T)$ and vice versa. Hence, from Corollary 4.2, it follows that $A$ and $B$ are nonempty and equal. Without loss of generality, we may assume that $T(X)$ is closed. Let $v$ be a limit point of $A$. Then there exists a sequence $\{v_n\}_{n=1}^{\infty}$ in $A$ converging to $v$. For each $n \in \mathbb{N}$, there exists $u_n \in X \ni v_n = Su_n \in Fu_n$. Since $Fu_n \subseteq T(X)$, $v_n \in T(X)$ for all $n \in \mathbb{N}$. Since $T(X)$ is closed, $v \in T(X)$. Hence there exists $w \in X \ni v = Tw$. If possible, suppose that $d(v,Gw) > 0$. Then $B_{1/2}(u_n,w) = \max\{d(v_n,v), d(v,Gw), (1/2)[d(v_n,Gw) + d(v,Fu_n)]\} = d(v,Gw) = A(u_n,w)$ for all sufficiently large $n \in \mathbb{N}$. Since $\hat{\varphi}(0) = 0$, $\varphi(d(Su_n,Tw)) = \varphi(d(v_n,v)) \to 0$ as $n \to +\infty$. Hence $\max\{\varphi(d(Su_n,Tw)), \varphi(A(u_n,w)), \varphi(B_{1/2}(u_n,w))\} \to \varphi(d(v,Gw))$ as $n \to +\infty$. Since $d(v_n,Gw) \leq H(Fu_n,Gw)$, on taking $x = u_n$ and $y = w$ in inequality (3.9), we obtain

$$d(v_n,Gw) \leq \max\{\varphi(d(Su_n,Tw)), \varphi(A(u_n,w)), \varphi(B_{1/2}(u_n,w))\} \tag{4.1}$$

for all $n \in \mathbb{N}$. On taking limits on both sides of the above inequality as $n \to +\infty$, we obtain $d(v,Gw) \leq \varphi(d(v,Gw))$, which is a contradiction since $\varphi(t) < t$ for all $t$ in $(0,\infty)$. Hence $d(v,Gw) = 0$. Since $Gw$ is closed, $v(= Tw) \in Gw$. Hence $v \in B = A$. Hence $A$ is closed. □

Corollary 4.4. *Suppose that $(X,d)$ is complete, either $S(X)$ or $T(X)$ is closed, $Fx \subseteq TX$ and $Gx \subseteq SX$ for all $x$ in $X$, $\varphi \in \Gamma$, $\hat{\varphi}(t) < t$ for all $t$ in $(0,s']$ for some positive real number $s' \geq \max\{\alpha_X,\beta_X\}$, and that inequality (3.12) is true for all $x,y$ in $X$. Then $\{Sx : x \in X$ and $Sx \in Fx\}$ and $\{Tx : x \in X$ and $Tx \in Gx\}$ are nonempty, closed sets and are equal.*

*Proof.* From inequality (3.12) it is evident that for $x,y \in X$, inequalities (3.4) and (3.5) are true. Since $\varphi \in \Gamma$, $\hat{\varphi}(0) = 0$. Hence, from Proposition 3.26, it follows that $(F,S)$ has property $P$ with respect to $(G,T)$ and vice versa. Hence, from Corollary 4.2, it follows that the sets $A$ and $B$ defined as in the proof of Corollary 4.3 are nonempty and equal. Without loss of generality, we may assume that $T(X)$ is closed. Let $v$, $\{v_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, and $w$ be as in the proof of Corollary 4.3. Since $d(v_n,Gw) \leq H(Fu_n,Gw)$, on taking $x = u_n$

and $y = w$ in inequality (3.12), we obtain

$$d(v_n, Gw) \le \varphi(d(Su_n, Tw)) \tag{4.2}$$

for all $n \in \mathbb{N}$. On taking limits on both sides of the above inequality as $n \to +\infty$, we obtain $d(v, Gw) \le \hat{\varphi}(0)$. Since $\hat{\varphi}(0) = 0$, we have $d(v, Gw) = 0$. Since $Gw$ is closed, $v(= Tw) \in Gw$. Hence $v \in B = A$. Hence $A$ is closed. □

*Remark 4.5.* Corollaries 4.3 and 4.4 differ only in the inequalities governing the maps $F$, $G$, $S$, and $T$ and the conditions on the control function $\varphi$. In Corollary 4.4, while the governing inequality is more stringent than that in Corollary 4.3, the control function $\varphi$ is not required to satisfy the condition $\varphi(t) < t$ for all $t \in (0, \infty)$ unlike in Corollary 4.3.

Corollary 4.6 [13, Theorem 1]. *Suppose that $(X, d)$ is complete, $\varphi \in \Gamma_\infty$, $\varphi(t+) < t$ for all $t$ in $(0, \infty)$, $F$ and $G$ are mappings from $X$ into* CB$(X)$, *there is a nonempty subset $A$ of $X$ such that $SA$ and $TA$ are closed subsets of $X$, $Fx \subseteq TA$ and $Gx \subseteq SA$ for all $x$ in $A$, and*

$$H(Fx, Gy) \le \varphi(B_{1/2}(x, y)) \tag{4.3}$$

*for all $x, y$ in $X$. Then $\{Sx : x \in X \text{ and } Sx \in Fx\}$ and $\{Tx : x \in X \text{ and } Tx \in Gx\}$ are nonempty. Furthermore, both sets are closed and equal if one can take $A = X$.*

*Proof.* The proof follows from Theorem 4.1 and Propositions 3.20, 3.21, and 3.24 except for closedness which can be established as in the proof of Corollary 4.3. □

Corollary 4.7 [16, Theorem 9]. *Suppose that $(X, d)$ is complete, $\varphi \in \Gamma_\infty$, $\varphi(t+) < t$ for all $t$ in $(0, \infty)$, $F$ and $G$ are mappings from $X$ into* CB$(X)$, *and*

$$H(Fx, Gy) \le \varphi(m(x, y)) \tag{4.4}$$

*for all $x, y$ in $X$. Then $\{x \in X : x \in Fx\}$ and $\{x \in X : x \in Gx\}$ are nonempty, closed sets and are equal.*

*Proof.* The proof follows from Corollary 4.3 on taking $S = T = I$. □

Corollary 4.8. *Suppose that $(X, d)$ is complete, $\varphi \in \Gamma$, $\hat{\varphi}(t) < t$ for all $t$ in $(0, s']$ for some positive real number $s' \ge \gamma_X$, and that*

$$H(Fx, Fy) \le \varphi(d(x, y)) \tag{4.5}$$

*for all $x, y$ in $X$. Then $\{x \in X : x \in Fx\}$ is nonempty and closed.*

*Proof.* The proof follows from Corollary 4.4 on taking $S = T = I$ and $G = F$. □

Corollary 4.9 [5, Theorem 4.4]. *Suppose that $(X, d)$ is complete, $\varphi \in \Gamma'$, and for every $x, y \in X$, $u \in Fx$, there exists $v \in Fy$ such that*

$$d(u, v) \le \varphi(d(x, y)). \tag{4.6}$$

*Then $F$ has a fixed point in $X$.*

*Proof.* Let $x, y \in X$. For any $v \in Fy$, we have $d(u,Fy) \le d(u,v)$. Hence, from inequality (4.6), we have $d(u,Fy) \le \varphi(d(x,y))$ for all $u \in Fx$. Hence

$$\sup_{u\in Fx} d(u,Fy) \le \varphi(d(x,y)). \tag{4.7}$$

On reversing the roles of $x$ and $y$ in the above inequality, we obtain

$$\sup_{w\in Fy} d(w,Fx) \le \varphi(d(x,y)). \tag{4.8}$$

From the above two inequalities it follows that inequality (4.5) is true. Hence Corollary 4.9 follows from Corollary 4.8. □

*Remark 4.10.* All the results in this section in which the control function $\varphi$ is assumed to be a member of $\Gamma$ remain valid if this part of the hypothesis is replaced by the hypothesis that there exist an $s \in (0,\infty)$ and an increasing map $\psi : [0,s) \to [0,s)$ such that $\varphi(t) \le \psi(t)$ and $\psi(t+) < t$ for all $t$ in $(0,s)$ and $\sum_{n=1}^{\infty} \psi^n(t_0) < +\infty$ for some $t_0 \in (0,s)$.

For example, from Corollary 4.8, we have the following one.

Corollary 4.11. *Suppose that $(X,d)$ is complete, $\hat{\varphi}(t) < t$ for all $t$ in $(0,s']$ for some positive real number $s' \ge \gamma_X$, there exist an $s \in (0,\infty)$ and an increasing map $\psi : [0,s) \to [0,s)$ such that $\varphi(t) \le \psi(t)$ and $\psi(t+) < t$ for all $t$ in $(0,s)$ and $\sum_{n=1}^{\infty} \psi^n(t_0) < +\infty$ for some $t_0 \in (0,s)$, and inequality (4.5) is true for all $x, y$ in $X$. Then $\{x \in X : x \in Fx\}$ is nonempty and closed.*

*Proof.* Define $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ as $\eta(t) = \psi(t)$ if $t \in [0,s)$ and $\eta(t) = \varphi(t)$ if $t \in [s,\infty)$. From Lemma 3.1 it follows that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t \in [0,s)$. Hence $\sum_{n=1}^{\infty} \eta^n(t) < +\infty$ for all $t \in [0,s)$. Since $\psi$ is increasing on $[0,s)$, $\eta$ is increasing on $[0,s)$. Hence $\eta \in \Gamma_s$. Since $\psi(t+) < t$ for all $t$ in $(0,s)$ and $\hat{\varphi}(t) < t$ for all $t$ in $(0,s']$, we have $\hat{\eta}(t) < t$ for all $t$ in $(0,s']$. Since $\varphi(t) \le \psi(t) = \eta(t)$ for all $t$ in $(0,s)$, we have $\varphi(t) \le \eta(t)$ for all $t$ in $[0,s)$. Hence the truth of inequality (4.5) for all $x, y$ in $X$ implies that of the inequality $H(Fx,Fy) \le \eta(d(x,y))$. Hence Corollary 4.11 follows from Corollary 4.8. □

*Remark 4.12.* For any $s \in (0,\infty)$, Lemmas 3.3, 3.5, and 3.7 and Remark 3.4 show that there are plenty of functions $\psi : [0,s) \to [0,s)$ such that $\psi(t+) < t$ for all $t$ in $(0,s)$, $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t$ in $(0,s)$, and $\lim_{t\to 0+} \psi(t)/t = 1$. Daffer et al. [6] proved that the function $t \mapsto t - at^b$, where $a > 0$ and $b \in (1,2)$, also has this property for $s = (ab)^{-1/(b-1)}$. However, it is dominated in a right neighbourhood of zero by functions of the type considered in Lemma 3.7 (see Lemma 3.10).

Corollary 4.13 [6, Theorem 5]. *Suppose that $(X,d)$ is complete, $F : X \to \mathrm{CB}(X)$, $\varphi$ is upper right semicontinuous, $\varphi(t) < t$ for all $t \in (0,\infty)$, $\varphi(t) \le t - at^b$ for all $t$ in $(0,s_0)$ for some $a > 0$, $b \in (1,2)$ and $s_0 \in (0,\infty)$, and that inequality (4.5) is true for all $x, y$ in $X$. Then $F$ has a fixed point in $X$.*

*Proof.* The proof follows from Corollary 4.11 and Lemma 3.11 on choosing $\psi(t) = t - at^b$ and $0 < s < \min\{(ab)^{-1/(b-1)}, s_0\}$. □

Corollary 4.14 [5, Theorem 4.6]. *Suppose that $(X,d)$ is complete, $\varphi$ is upper right semicontinuous, $\varphi(t) < t$ for all $t \in (0,\infty)$, $\varphi(t) \leq t - at^b$ for all $t$ in $(0,s_0)$ for some $a > 0$, $b \in (1,2)$ and $s_0 \in (0,\infty)$, and that for every $x,y \in X$, $u \in Fx$, there exists $v \in Fy$ such that inequality (4.6) is true. Then $F$ has a fixed point in $X$.*

*Proof.* From the proof of Corollary 4.9 we see that inequality (4.5) is true here for all $x,y$ in $X$. Hence Corollary 4.14 follows from Corollary 4.11 and Lemma 3.11 on choosing $\psi$ and $s$ as in the proof of Corollary 4.13. □

Corollary 4.15. *Suppose that $(X,d)$ is complete, $k : \mathbb{R}^+ \to \mathbb{R}^+$, $\hat{k}(t) < 1$ for all $t$ in $(0,s']$ for some positive real number $s' \geq \gamma_X$, there exist an $s \in (0,\infty)$ and an increasing map $\psi : [0,s) \to [0,s)$ such that $tk(t) \leq \psi(t)$ and $\psi(t+) < t$ for all $t$ in $(0,s)$ and $\sum_{n=1}^{\infty} \psi^n(t_0) < +\infty$ for some $t_0 \in (0,s)$, and that inequality (3.1) is true for all $x,y$ in $X$. Then $\{x \in X : x \in Fx\}$ is nonempty and closed.*

*Proof.* The proof follows from Corollary 4.11 on taking $\varphi(t) = tk(t)$ for all $t$ in $\mathbb{R}^+$. □

Corollary 4.16. *Suppose that $(X,d)$ is complete, $k : \mathbb{R}^+ \to \mathbb{R}^+$, $\hat{k}(t) < 1$ for all $t$ in $(0,s']$ for some positive real number $s' \geq \gamma_X$, $\tilde{k}(0) < 1$, and that inequality (3.1) is true for all $x,y$ in $X$. Then $\{x \in X : x \in Fx\}$ is nonempty and closed.*

*Proof.* Let $\gamma \in (\tilde{k}(0),1)$. Then there exists $s \in (0,\infty)$ such that $k(t) < \gamma$ for all $t$ in $(0,s)$. Define $\psi$ on $[0,s)$ as $\psi(t) = \gamma t$ for all $t$ in $[0,s)$. Then $\psi$ is a continuous self-map on $[0,s)$, $tk(t) \leq \psi(t) < t$ for all $t$ in $(0,s)$, and $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t$ in $[0,s)$. Hence Corollary 4.16 follows from Corollary 4.15. □

Corollary 4.17 [12, Theorem 5]. *Suppose that $(X,d)$ is complete, $F : X \to \mathrm{CB}(X)$, $k \in \mathfrak{I}$, $\tilde{k}(0) < 1$, and inequality (3.1) is true for all $x,y$ in $X$. Then $F$ has a fixed point in $X$.*

*Proof.* The proof follows from Corollary 4.16. □

*Remark 4.18.* Corollary 6 of Daffer et al. [6] states that if $(X,d)$ is complete, $F : X \to \mathrm{CB}(X)$ satisfies inequality (3.1) for all $x,y$ in $X$, where $k : \mathbb{R}^+ \to [0,1]$ with $k(t) < 1$ for all $t$ in $(0,\infty)$, and $k(t) \leq 1 - at^{b-1}$ for all $t$ in $(0,s_0)$ for some $a > 0$, $b \in (1,2)$, and $s_0 \in (0,a^{-1/(b-1)})$, then $F$ has a fixed point in $X$. Since this statement was given by them as a corollary of Theorem 5 of [6] (see Corollary 4.13), it should contain the additional hypothesis on $k$ that it is upper right semicontinuous. With this correction, in view of Lemma 3.11, it becomes a corollary of Corollary 4.15 on choosing $\psi(t) = t - at^b$ and $0 < s < \min\{s_0,(ab)^{-1/(b-1)}\}$.

Corollary 4.19 [5, Theorem 4.2]. *Suppose that $(X,d)$ is complete, $k \in \mathfrak{I}$, $\tilde{k}(0) < 1$, and for every $x,y \in X$, $u \in Fx$, there exists $v \in Fy$ such that*

$$d(u,v) \leq k(d(x,y))d(x,y). \tag{4.9}$$

*Then $F$ has a fixed point in $X$.*

*Proof.* As in the proof of Corollary 4.9, it can be seen here that inequality (3.1) is true for all $x,y$ in $X$. Hence Corollary 4.19 follows from Corollary 4.16. □

Corollary 4.20. *Suppose that $(X,d)$ is complete, either $S(X)$ or $T(X)$ is closed, $Fx \subseteq TX$ and $Gx \subseteq SX$ for all $x$ in $X$, $k \in \mathfrak{I}'$, $\hat{k}(t) < 1$ for all $t$ in $(0,s']$ for some positive real number $s' \geq \max\{\alpha_X, \beta_X\}$, $\tilde{k}(0) < 1$, and*

$$H(Fx,Gy) \leq \max\{k(d(Sx,Ty))d(Sx,Ty), k(A(x,y))A(x,y), k(B_{1/2}(x,y))B_{1/2}(x,y)\} \tag{4.10}$$

*for all $x,y$ in $X$. Then $\{Sx : x \in X \text{ and } Sx \in Fx\}$ and $\{Tx : x \in X \text{ and } Tx \in Gx\}$ are nonempty, closed sets and are equal.*

*Proof.* Let $\gamma \in (\tilde{k}(0), 1)$. Then there exists $s \in (0,\infty)$ such that $k(t) < \gamma$ for all $t$ in $(0,s]$. Hence $tk(t) \leq \gamma t$ for all $t$ in $[0,s]$. Define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(t) = \gamma t$ if $t \in [0,s)$ and $\varphi(t) = tk(t)$ if $t \in [s,\infty)$. Then $\varphi \in \Gamma_s$, $\hat{\varphi}(t) < t$ for all $t$ in $(0,s']$, and inequality (3.9) is true for all $x,y$ in $X$. Hence Corollary 4.20 follows from Corollary 4.3. □

*Remark 4.21.* Theorem 5 of [12] follows from Corollary 4.20 on taking $S = T = I$ and $G = F$.

Corollary 4.22 [3, Theorem 9]. *Suppose that $(X,d)$ is complete, $F$ and $G$ are mappings from $X$ into $\mathrm{CB}(X)$, $k \in \mathfrak{I}$, $\tilde{k}(0) < 1$, and*

$$H(Fx,Gy) \leq k(m(x,y))m(x,y) \tag{4.11}$$

*for all $x,y$ in $X$. Then $\{x \in X : x \in Fx\}$ and $\{x \in X : x \in Gx\}$ are nonempty and equal.*

*Proof.* The proof follows from Corollary 4.20 on taking $S = T = I$. □

*Remark 4.23.* Unless $k$ is increasing on $\mathbb{R}^+$, the claim of Chang [3] that his Theorem 9 is a generalization of Theorem 5 of Mizoguchi and Takahashi [12] (see Corollary 4.17) may not be valid. The latter establishes Reich's conjecture under an additional hypothesis on the control function $k$, namely, $\tilde{k}(0) < 1$.

Corollary 4.24. *Suppose that $(X,d)$ is complete, $S = I$, $Fx \subseteq TX$ for all $x$ in $X$, $\varphi \in \Gamma^*$, $\hat{\varphi}(t) < t$ for all $t$ in $(0,s']$ for some positive real number $s' \geq \max\{\alpha_X, \beta_X\}$, and inequality (3.9) is true for all $x,y$ in $X$. Then $\{x \in X : x \in Fx\}$ and $\{Tx : x \in X \text{ and } Tx \in Gx\}$ are nonempty, closed sets and are equal. Furthermore, if $(G,T)$ is $w^*$-compatible, then there exists $z \in X$ such that $z \in Fz$ and $Tz \in Gz$.*

*Proof.* Since $I(X) = X$ is closed, the first conclusion follows from Corollary 4.3. In particular, there exist $z,w \in X$ such that $z = Tw$, $z \in Fz$, and $Tw \in Gw$. Suppose now that $(G,T)$ is $w^*$-compatible. Then $Tz = T^2w \in GTw = Gz$. □

Corollary 4.25 [3, Theorem 6]. *Suppose that $(X,d)$ is complete, $F,G : X \to \mathrm{CB}(X)$, $S = I$, $Fx \subseteq TX$ for all $x$ in $X$, $(G,T)$ is compatible, $\varphi \in \Gamma'$, and $H(Fx,Gy) \leq \varphi(B_{1/2}(x,y))$ for all $x,y$ in $X$. Then there exists $z \in X$ such that $z \in Fz$ and $Tz \in Gz$.*

*Proof.* The proof follows from Corollary 4.24 and Remark 3.32. □

Corollary 4.26. *Suppose that $(X,d)$ is complete, $S(X)$ is closed, $Fx \subseteq S(X)$ for all $x$ in $X$, $\varphi \in \Gamma^*$, $\hat{\varphi}(t) < t$ for all $t$ in $(0,s']$ for some positive real number $s' \geq \alpha_X$, and*

$$H(Fx,Fy) \leq \max\{\varphi(d(Sx,Sy)),\varphi(A_0(x,y)),\varphi(C_0(x,y))\} \tag{4.12}$$

*for all $x,y$ in $X$. Then $\{x \in X : Sx \in Fx\}$ is nonempty and closed.*

*Proof.* The proof follows from Corollary 4.3 on taking $T = S$ and $G = F$. □

Corollary 4.27. *Suppose that $(X,d)$ is complete, $Fx \subseteq S(X)$ for all $x$ in $X$, $S$ is continuous on $X$, $f$ is lower semicontinuous on $X$, $(F,S)$ is $w$-compatible, $\varphi \in \Gamma$, $\hat{\varphi}(t) < t$ for all $t$ in $(0,s']$ for some positive real number $s' \geq \alpha_X$, and for $x,y$ in $X$,*

$$d(Sx,Fx) \leq \varphi(d(Sx,Sy)) \quad \text{if } Sx \in Fy. \tag{4.13}$$

*Then $\{x \in X : Sx \in Fx\}$ is nonempty.*

*Proof.* The proof follows from Corollary 4.2 on taking $T = S$ and $G = F$. □

Corollary 4.28 [2, Theorem 1]. *Suppose that $(X,d)$ is complete, $S$ is continuous on $X$, $h$ is a continuous self-map on $X$, $h(X) \subseteq S(X)$, $S$ and $h$ are weakly commuting, $\varphi$ is increasing on $\mathbb{R}^+$, $\varphi(0) = 0$, $0 < \varphi(t) < t$ for all $t \in (0,\infty)$, the function $\chi : (0,\infty) \to (0,\infty)$ defined as $\chi(t) = t/(t-\varphi(t))$ is decreasing on $(0,\infty)$ and $\int_0^s \chi(t)dt < +\infty$ for all $s \in (0,\infty)$, and that*

$$d(hx,hy) \leq \varphi\Big(\max\Big\{d(Sx,Sy),d(Sx,hx),d(Sy,hy),\frac{1}{2}[d(Sx,hy)+d(Sy,hx)]\Big\}\Big) \tag{4.14}$$

*for all $x,y$ in $X$. Then $S$ and $h$ have a unique common fixed point in $X$.*

*Proof.* Since $S$ and $h$ are continuous on $X$, the function which maps $x \in X$ to $d(Sx,hx)$ is continuous on $X$. From Proposition 3.21 and inequality (4.14) we have $d(Sx,hx) \leq \varphi(d(Sx,Sy))$ if $Sx = hy$. Since $S$ and $h$ are weakly commutative, they are compatible and hence $w$-compatible. Hence, from Corollary 4.27 and Lemma 3.13, it follows that $\{x \in X : Sx = hx\}$ is nonempty. Let $u \in X$ be such that $Su = hu$ $(= w$, say). Since $S$ and $h$ are compatible, we have $Sw = hw$. Hence, on taking $x = u$ and $y = w$ in inequality (4.14) and on using the fact that $\varphi(t) < t$ for all $t$ in $(0,\infty)$, we see that $Sw = hw = w$. From inequality (4.14) it is evident that $S$ and $h$ have at most one common fixed point in $X$. □

*Remark 4.29.* In Corollary 4.28, the condition $0 < \varphi(t)$ for all $t \in (0,\infty)$ is redundant.

Corollary 4.30. *Suppose that $(X,d)$ is complete, $\varphi \in \Gamma^*$, $\hat{\varphi}(t) < t$ for all $t$ in $(0,s']$ for some real number $s' \geq \gamma_X$, and*

$$H(Fx,Fy) \leq \max\{\varphi(d(x,y)),\varphi(A_1(x,y)),\varphi(C_1(x,y))\} \tag{4.15}$$

*for all $x,y$ in $X$. Then $\{x \in X : x \in Fx\}$ is nonempty and closed.*

*Proof.* The proof follows from Corollary 4.26 on taking $S = I$. □

Corollary 4.31 [3, Theorem 7]. *Suppose that* $(X,d)$ *is complete,* $F : X \to \mathrm{CB}(X)$, $\varphi \in \Gamma'$, *and*

$$H(Fx,Fy) \le \varphi(C_1(x,y)) \tag{4.16}$$

*for all* $x,y$ *in* $X$. *Then* $F$ *has a fixed point in* $X$.

*Proof.* The proof follows from Corollary 4.30. □

*Remark 4.32.* Unless $\varphi$ is increasing on $\mathbb{R}^+$, the truth of inequality (4.5) may not imply the truth of inequality (4.16) as was assumed by Daffer et al. in proving Theorem 5 of [6] by using Theorem 7 of Chang [3]. Daffer et al. reiterated in [5] that Theorem 5 of [6] was proved utilizing Theorem 7 of Chang [3]. However, Theorem 5 of [6] is correct (see Corollary 4.13). The declaration of Daffer et al. made in [5] that Theorem 7 of Chang [3] generalizes Theorem 5 of Mizoguchi and Takahashi [12] is false.

Corollary 4.33 [9, Corollary 2]. *Suppose that* $(X,d)$ *is complete,* $\varphi$ *is upper semicontinuous from the right on* $\mathbb{R}^+$, $\varphi(t) < t$ *for all* $t \in (0,\infty)$, $\varphi$ *is strictly increasing on* $[0,s]$ *and* $\sum_{n=1}^{\infty} \varphi^n(s) < +\infty$ *for some positive real number* $s$, *and that inequality (4.16) is true for all* $x,y$ *in* $X$. *Then* $F$ *has a fixed point.*

*Proof.* The proof follows from Corollary 4.30. □

Corollary 4.34 [4, Theorem 3.3]. *Suppose that* $(X,d)$ *is complete,* $F : X \to \mathrm{CB}(X)$, $\rho$ *is lower semicontinuous on* $X$, $k$ *is a constant in* $[0,1)$, *and that*

$$H(Fx,Fy) \le k, C_1(x,y) \tag{4.17}$$

*for all* $x,y$ *in* $X$. *Then there exists* $z \in X$ *such that* $z \in Fz$.

*Proof.* Define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(t) = kt$ for all $t$ in $\mathbb{R}^+$. Then $\varphi$ is continuous on $\mathbb{R}^+$, $\varphi(t) < t$ for all $t$ in $(0,\infty)$, $\varphi \in \Gamma_\infty$, and inequality (4.15) reduces to inequality (4.17). Hence Corollary 4.34 follows from Corollary 4.30. □

*Remark 4.35.* Corollary 4.34 shows that in Corollary 4.30 the lower semicontinuity condition on the function $\rho$ is redundant and that $\mathrm{CB}(X)$ can be replaced by $C(X)$. Daffer and Kaneko [4] claimed that Corollary 4.34 remains valid if $F$ maps $X$ into $B(X)$ instead of $\mathrm{CB}(X)$ and if inequality (4.17) is replaced by the following inequality:

$$H(Fx,Fy) \le k_1\big(\max\{d(x,y),d(x,Fx),d(y,Fy),k_2[d(x,Fy)+d(y,Fx)]\}\big), \tag{4.18}$$

where $k_1$, $k_2$ are constants such that $0 \le k_1 < 1$ and $0 < k_2 < 1/(2k_1+\delta)$ for some $\delta > 0$. (Vide [4, Theorem 3.4].) Example 4.36 shows that their claim is false. If we choose $k = \max\{k_1, 2k_1/(2k_1+\delta)\}$, then the truth of inequality (4.18) implies that of inequality (4.17). Hence, [4, Theorem 3.4] would be correct even when the lower semicontinuity of the function $\rho$ is dropped, provided that $B(X)$ in it is replaced by $\mathrm{CB}(X)$ or $C(X)$.

*Example 4.36.* Let $X = [0,1]$ with the usual metric. Let $A$ denote the set of all rational numbers in $[0,1]$ and $B$ the complement of $A$ in $X$. Clearly, $H(A,B) = 0$. Define $F : X \to B(X)$ as $Fx = A$ if $x \in B$ and $Fx = B$ if $x \in A$. Then $H(Fx,Fy) = 0$ for all $x,y$ in $X$ and $d(x,Fx) = 0$ for all $x$ in $X$. Hence inequality (4.18) is satisfied for all $x,y$ in $X$ with $k_1 = 0$, and the function $x \to d(x,Fx)$ is continuous on $X$. Clearly, there is no $x \in X$ such that $x \in Fx$.

Corollary 4.37. *Suppose that $(X,d)$ is complete and*

$$H(Fx,Fy) \leq \left(1 - \frac{\theta(x,y)}{d(x,y)}\right) C_1(x,y) \tag{4.19}$$

*for all distinct $x,y$ in $X$, where $\theta$ is a compactly positive function on $X \times X$ satisfying inequality (3.3) for some positive real number $\epsilon$ and such that $\theta(x,y) \leq d(x,y)$ for all $x,y$ in $X$. Then $\{x \in X : x \in Fx\}$ is nonempty and closed.*

*Proof.* Let $\psi$ and $\varphi$ be as stated in Lemma 3.15. Then, for distinct $x,y$ in $X$, we have $(1 - \theta(x,y)/d(x,y)) \leq \psi(d(x,y)) \leq \psi(C_1(x,y))$. Hence, from inequality (4.19), we have $H(Fx,Fy) \leq \varphi(C_1(x,y))$ for all $x,y$ in $X$. Now Corollary 4.37 is evident from Corollary 4.30 and Lemma 3.15. □

*Remark 4.38.* If $\sup\{\rho(x) : x \in X\}$ is finite, then the function $\psi$ defined on $X \times X$ as $\psi(x,y) = d(x,y)/C_1(x,y)$ if $x \neq y$ and $\psi(x,y) = 0$ if $x = y$ is compactly positive. If $\psi$ is compactly positive, then Corollary 4.37 remains valid if the right-hand side of inequality (4.19) is replaced by $C_1(x,y) - \theta(x,y)$, since the function $\psi\theta$ is compactly positive and $\psi\theta \leq d$.

Theorem 4.39 [4, Theorem 2.3]. *Suppose that $(X,d)$ is complete and $F : X \to \mathrm{CB}(X)$ satisfies inequality (3.13) for all $x,y$ in $X$, where $\theta$ is a compactly positive function on $X \times X$ such that*

$$\liminf_{b \to 0} \frac{\lambda(a,b)}{b} > 0 \quad (0 < a \leq b), \tag{4.20}$$

*where $\lambda(a,b) = \inf\{\theta(x,y) : x,y \in X \text{ and } a \leq d(x,y) \leq b\}$ for any positive real numbers $a$ and $b$ with $a \leq b$. Then $F$ has a fixed point in $X$.*

*Remark 4.40.* Unless $a$ is a function of $b$, inequality (4.20) is vague. Theorem 2.3 of [4] becomes a corollary of Corollary 4.37 when $a$ is interpreted as a function of $b$ in inequality (4.20).

*Definition 4.41.* Let $v_0 \in X$. By an orbit of $F$ with respect to $v_0$ we mean a sequence $\{v_n\}_{n=0}^{\infty}$ in $X$ such that $v_n \in Fv_{n-1}$ for all $n \in \mathbb{N}$. We say that $X$ is $F$-orbitally complete if any Cauchy subsequence of any orbit of $F$ is convergent in $X$. A real-valued function $h$ on $X$ is said to be $F$-orbitally lower semicontinuous on $X$ if for any $z \in X$, $h(z) \leq \liminf_{k\to\infty} h(v_{n_k})$ whenever $\{v_{n_k}\}_{k=1}^{\infty}$ is a convergent subsequence of an orbit of $F$ and $\lim_{k\to\infty} v_{n_k} = z$.

*Remark 4.42.* We note that $|d(x,Fx)-d(y,Fy)| \le d(x,y)+H(Fx,Fy)$ for all $x,y$ in $X$. Hence, if $F$ satisfies inequality (4.5) for all $x,y$ in $X$ and if $\hat{\varphi}(0)=0$, then the function $\rho$ is uniformly continuous on $X$.

THEOREM 4.43. *Suppose that $(X,d)$ is $F$-orbitally complete, $\rho$ is $F$-orbitally lower semicontinuous on $X$, $\varphi \in \Gamma$, $\hat{\varphi}(t) < t$ for all $t$ in $(0,s']$ for some positive real number $s' \ge \gamma_X$, and that*

$$d(y,Fy) \le \varphi(d(x,y)) \tag{4.21}$$

*whenever $x \in X$ and $y \in Fx$. Then there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of $F$, which converges to a fixed point of $F$.*

*Proof.* The proof is similar to that of Theorem 4.1. □

*Remark 4.44.* Examples 4.45 and 4.46 show that Theorem 4.43 fails if the condition on the convergence of the sequence of iterates of the control function $\varphi$ is dropped, even if $F$ is single-valued and $\varphi$ is a strictly increasing continuous function on $\mathbb{R}^+$ with $\varphi(t) < t$ for all $t$ in $(0,\infty)$. While in Example 4.45 the metric space is unbounded, in Example 4.46, it is bounded.

*Example 4.45.* For $n \in \mathbb{N}$, let $a_n = \sum_{k=1}^{n}(1/k)$. Let $X = \{a_n : n \in \mathbb{N}\}$ with the usual metric. Clearly, $X$ is complete. Let $x_0 = a_1$. Define $F : X \to X$ as $Fa_n = a_{n+1}$ for all $n \in \mathbb{N}$. Define $\varphi$ on $\mathbb{R}^+$ as $\varphi(t) = t/(1+t)$ for all $t$ in $\mathbb{R}^+$. Then $\varphi$ is a strictly increasing nonnegative continuous function on $\mathbb{R}^+$, $\varphi(t) < t$ for all $t$ in $(0,\infty)$, $\varphi^n(t) = t/(1+nt)$ for all $t$ in $\mathbb{R}^+$ and for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty}\varphi^n(t) = \sum_{n=1}^{\infty}(t/(1+nt)) = +\infty$ for all $t$ in $(0,\infty)$, and

$$|Fx - F^2x| = \varphi(|x - Fx|) \tag{4.22}$$

for all $x$ in $X$. Clearly, $F$ has no fixed point in $X$.

*Example 4.46.* For $k \in \mathbb{N}$, let $e_k = \{\delta_{km}\}_{m=1}^{\infty}$, where $\delta_{km}$ is Kronecker's delta. For $n \in \mathbb{N}$, let $u_n = (1/2n)\sum_{k=1}^{n} e_k$. Let $X = \{u_n : n \in \mathbb{N}\}$. Then $X$ is a closed bounded subset of the Banach space $l^1$. We note that $\|u_n\|_1 = 1/2$ for all $n$ in $\mathbb{N}$ and for $n,m$ in $\mathbb{N}$ with $m > n$, $\|u_n - u_m\|_1 = (m-n)/m$. Define $F : X \to X$ as $Fu_n = u_{n+1}$ for all $n \in \mathbb{N}$, and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ as in Example 4.45. Then

$$\|Fx - F^2x\|_1 = \varphi(\|x - Fx\|_1) \tag{4.23}$$

for all $x$ in $X$. Clearly, $F$ has no fixed point in $X$.

From Theorem 4.43 we have the following corollary.

COROLLARY 4.47. *Suppose that $(X,d)$ is $F$-orbitally complete, $\rho$ is $F$-orbitally lower semicontinuous on $X$, $k : \mathbb{R}^+ \to \mathbb{R}^+$, $\hat{k}(t) < 1$ for all $t$ in $(0,s']$ for some positive real number $s' \ge \alpha'$, there exist an $s \in (0,\infty)$ and an increasing map $\psi : [0,s) \to [0,s)$ such that $tk(t) \le \psi(t)$ and $\psi(t+) < t$ for all $t$ in $(0,s)$ and $\sum_{n=1}^{\infty}\psi^n(t_0) < +\infty$ for some $t_0 \in (0,s)$, and that*

$$d(y,Fy) \le k(d(x,y))d(x,y) \tag{4.24}$$

*whenever $x \in X$ and $y \in Fx$. Then there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of $F$, which converges to a fixed point of $F$.*

*Remark 4.48.* Except for the condition on $k$ involving $\psi$, the hypothesis of Corollary 4.47 is considerably weaker than that of the conjecture of Reich.

Corollary 4.49 [9, Corollary 1]. *Suppose that $(X,d)$ is complete, $\varphi$ is upper semicontinuous from the right on $\mathbb{R}^+$, $\varphi(t) < t$ for all $t \in (0,\infty)$, $\varphi$ is strictly increasing on $[0,s]$ and $\sum_{n=1}^{\infty} \varphi^n(s) < +\infty$ for some positive real number $s$, and that inequality (4.5) is true for all $x,y$ in $X$. Then $F$ has a fixed point.*

*Proof.* The proof follows from Theorem 4.43 as well as Corollary 4.8. □

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S. V. R. Naidu: Department of Applied Mathematics, Andhra University, Visakhapatnam 530003, India
*E-mail address*: svrnaidu@hotmail.com