THE LEFSCHETZ-HOPF THEOREM AND AXIOMS FOR THE LEFSCHETZ NUMBER

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The reduced Lefschetz number, that is, $L(\cdot) - 1$ where $L(\cdot)$ denotes the Lefschetz number, is proved to be the unique integer-valued function λ on self-maps of compact polyhedra which is constant on homotopy classes such that (1) $\lambda(fg) = \lambda(gf)$ for $f : X \to Y$ and $g : Y \to X$; (2) if (f_1, f_2, f_3) is a map of a cofiber sequence into itself, then $\lambda(f_1) = \lambda(f_1) + \lambda(f_3)$; (3) $\lambda(f) = -(\deg(p_1 f e_1) + \cdots + \deg(p_k f e_k))$, where f is a self-map of a wedge of k circles, e_r is the inclusion of a circle into the rth summand, and p_r is the projection onto the rth summand. If $f : X \to X$ is a self-map of a polyhedron and $I(f)$ is the fixed-point index of f on all of X, then we show that $I(\cdot) - 1$ satisfies the above axioms. This gives a new proof of the normalization theorem: if $f : X \to X$ is a self-map of a polyhedron, then $I(f)$ equals the Lefschetz number $L(f)$ of f. This result is equivalent to the Lefschetz-Hopf theorem: if $f : X \to X$ is a self-map of a finite simplicial complex with a finite number of fixed points, each lying in a maximal simplex, then the Lefschetz number of f is the sum of the indices of all the fixed points of f.

1. Introduction

Let X be a finite polyhedron and denote by $\widetilde{H}_*(X)$ its reduced homology with rational coefficients. Then the *reduced Euler characteristic* of X, denoted by $\tilde{\chi}(X)$, is defined by

$$\tilde{\chi}(X) = \sum_k (-1)^k \dim \widetilde{H}_k(X). \tag{1.1}$$

Clearly, $\tilde{\chi}(X)$ is just the Euler characteristic minus one. In 1962, Watts [13] characterized the reduced Euler characteristic as follows. Let ϵ be a function from the set of finite polyhedra with base points to the integers such that (i) $\epsilon(S^0) = 1$, where S^0 is the 0-sphere, and (ii) $\epsilon(X) = \epsilon(A) + \epsilon(X/A)$, where A is a subpolyhedron of X. Then $\epsilon(X) = \tilde{\chi}(X)$.

Let $\mathscr{C}$ be the collection of spaces X of the homotopy type of a finite, connected CW-complex. If $X \in \mathscr{C}$, we do not assume that X has a base point except when X is a sphere or a wedge of spheres. It is not assumed that maps between spaces with base points are based. A map $f : X \to X$, where $X \in \mathscr{C}$, induces trivial homomorphisms $f_{*k} : H_k(X) \to H_k(X)$

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of rational homology vector spaces for all $j > \dim X$. The *Lefschetz number* $L(f)$ of f is defined by

$$L(f) = \sum_k (-1)^k \operatorname{Tr} f_{*k}, \tag{1.2}$$

where Tr denotes the trace. The reduced Lefschetz number $\widetilde{L}$ is given by $\widetilde{L}(f) = L(f) - 1$ or, equivalently, by considering the rational, reduced homology homomorphism induced by f.

Since $\widetilde{L}(\mathrm{id}) = \tilde{\chi}(X)$, where $\mathrm{id} : X \to X$ is the identity map, Watts's Theorem suggests an axiomatization for the reduced Lefschetz number which we state below in Theorem 1.1.

For $k \geq 1$, denote by $\bigvee^k S^n$ the wedge of k copies of the n-sphere S^n, $n \geq 1$. If we write $\bigvee^k S^n$ as $S_1^n \vee S_2^n \vee \cdots \vee S_k^n$, where $S_j^n = S^n$, then we have inclusions $e_j : S_j^n \to \bigvee^k S^n$ into the jth summand and projections $p_j : \bigvee^k S^n \to S_j^n$ onto the jth summand, for $j = 1,\dots,k$. If $f : \bigvee^k S^n \to \bigvee^k S^n$ is a map, then $f_j : S_j^n \to S_j^n$ denotes the composition $p_j f e_j$. The degree of a map $f : S^n \to S^n$ is denoted by $\deg(f)$.

We characterize the reduced Lefschetz number as follows.

Theorem 1.1. *The reduced Lefschetz number $\widetilde{L}$ is the unique function λ from the set of self-maps of spaces in $\mathscr{C}$ to the integers that satisfies the following conditions.*

(1) *(Homotopy axiom) If $f, g : X \to X$ are homotopic maps, then $\lambda(f) = \lambda(g)$.*

(2) *(Cofibration axiom) If A is a subpolyhedron of X, $A \to X \to X/A$ is the resulting cofiber sequence, and there exists a commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ A & \longrightarrow & X & \longrightarrow & X/A, \end{array} \tag{1.3}$$

then $\lambda(f) = \lambda(f') + \lambda(\bar{f})$.

(3) *(Commutativity axiom) If $f : X \to Y$ and $g : Y \to X$ are maps, then $\lambda(gf) = \lambda(fg)$.*

(4) *(Wedge of circles axiom) If $f : \bigvee^k S^1 \to \bigvee^k S^1$ is a map, $k \geq 1$, then*

$$\lambda(f) = -\left(\deg(f_1) + \cdots + \deg(f_k)\right), \tag{1.4}$$

where $f_j = p_j f e_j$.

In an unpublished dissertation [10], Hoang extended Watts's axioms to characterize the reduced Lefschetz number for basepoint-preserving self-maps of finite polyhedra. His list of axioms is different from, but similar to, those in Theorem 1.1.

One of the classical results of fixed-point theory is the following theorem.

Theorem 1.2 (Lefschetz-Hopf). *If $f : X \to X$ is a map of a finite polyhedron with a finite set of fixed points, each of which lies in a maximal simplex of X, then $L(f)$ is the sum of the indices of all the fixed points of f.*

The history of this result is described in [3], see also [8, page 458]. A proof that depends on a delicate argument due to Dold [4] can be found in [2] and, in a more condensed form, in [5]. In an appendix to his dissertation [12], McCord outlined a possibly more direct argument, but no details were published. The book of Granas and Dugundji [8, pages 441–450] presents an argument based on classical techniques of Hopf [11]. We use the characterization of the reduced Lefschetz number in Theorem 1.1 to prove the Lefschetz-Hopf theorem in a quite natural manner by showing that the fixed-point index satisfies the axioms of Theorem 1.1. That is, we prove the following theorem.

Theorem 1.3 (normalization property). *If $f : X \to X$ is any map of a finite polyhedron, then $L(f) = i(X, f, X)$, the fixed-point index of f on all of X.*

The Lefschetz-Hopf theorem follows from the normalization property by the additivity property of the fixed-point index. In fact, these two statements are equivalent. The Hopf construction [2, page 117] implies that a map f from a finite polyhedron to itself is homotopic to a map that satisfies the hypotheses of the Lefschetz-Hopf theorem. Thus, the homotopy and additivity properties of the fixed-point index imply that the normalization property follows from the Lefschetz-Hopf theorem.

2. Lefschetz numbers and exact sequences

In this section, all vector spaces are over a fixed field F, which will not be mentioned, and are finite dimensional. A graded vector space $V = \{V_n\}$ will always have the following properties: (1) each V_n is finite dimensional and (2) $V_n = 0$, for $n < 0$ and for $n > N$, for some nonnegative integer N. A *map* $f : V \to W$ of graded vector spaces $V = \{V_n\}$ and $W = \{W_n\}$ is a sequence of linear transformations $f_n : V_n \to W_n$. For a map $f : V \to V$, the *Lefschetz number* is defined by

$$L(f) = \sum_n (-1)^n \operatorname{Tr} f_n. \tag{2.1}$$

The proof of the following lemma is straightforward, and hence omitted.

Lemma 2.1. *Given a map of short exact sequences of vector spaces*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0, \end{array} \tag{2.2}$$

then $\operatorname{Tr} g = \operatorname{Tr} f + \operatorname{Tr} h$.

Theorem 2.2. *Let A, B, and C be graded vector spaces with maps $\alpha : A \to B$, $\beta : B \to C$ and self-maps $f : A \to A$, $g : B \to B$, and $h : C \to C$. If, for every n, there is a linear transformation*

$\partial_n : C_n \to A_{n-1}$ *such that the following diagram is commutative and has exact rows:*

$$\begin{CD}
0 @>>> A_N @>\alpha_N>> B_N @>\beta_N>> C_N @>\partial_N>> A_{N-1} @>\alpha_{N-1}>> \cdots \\
@. @Vf_NVV @Vg_NVV @Vh_NVV @Vf_{N-1}VV \\
0 @>>> A_N @>\alpha_N>> B_N @>\beta_N>> C_N @>\partial_N>> A_{N-1} @>\alpha_{N-1}>> \cdots
\end{CD}$$

$$\begin{CD}
\cdots @>\partial_1>> A_0 @>\alpha_0>> B_0 @>\beta_0>> C_0 @>>> 0 \\
@. @Vf_0VV @Vg_0VV @Vh_0VV \\
\cdots @>\partial_1>> A_0 @>\alpha_0>> B_0 @>\beta_0>> C_0 @>>> 0,
\end{CD} \tag{2.3}$$

then

$$L(g) = L(f) + L(h). \tag{2.4}$$

Proof. Let Im denote the image of a linear transformation and consider the commutative diagram

$$\begin{CD}
0 @>>> \mathrm{Im} @>>> C_n @>>> \mathrm{Im}\,\partial_n @>>> 0 \\
@. @Vh_n|\mathrm{Im}\beta_nVV @Vh_nVV @Vf_{n-1}|\mathrm{Im}\,\partial_nVV \\
0 @>>> \mathrm{Im}\beta_n @>>> C_n @>>> \mathrm{Im}\,\partial_n @>>> 0.
\end{CD} \tag{2.5}$$

By Lemma 2.1, $\mathrm{Tr}(h_n) = \mathrm{Tr}(h_n \,|\, \mathrm{Im}\beta_n) + \mathrm{Tr}(f_{n-1} \,|\, \mathrm{Im}\,\partial_n)$. Similarly, the commutative diagram

$$\begin{CD}
0 @>>> \mathrm{Im}\,\partial_n @>>> A_{n-1} @>>> \mathrm{Im}\,\alpha_{n-1} @>>> 0 \\
@. @Vf_{n-1}|\mathrm{Im}\,\partial_nVV @Vf_{n-1}VV @Vg_{n-1}|\mathrm{Im}\,\alpha_{n-1}VV \\
0 @>>> \mathrm{Im}\,\partial_n @>>> A_{n-1} @>>> \mathrm{Im}\,\alpha_{n-1} @>>> 0
\end{CD} \tag{2.6}$$

yields $\mathrm{Tr}(f_{n-1} \,|\, \mathrm{Im}\,\partial_n) = \mathrm{Tr}(f_{n-1}) - \mathrm{Tr}(g_{n-1} \,|\, \mathrm{Im}\,\alpha_{n-1})$. Therefore,

$$\mathrm{Tr}\,(h_n) = \mathrm{Tr}\,(h_n \,|\, \mathrm{Im}\beta_n) + \mathrm{Tr}\,(f_{n-1}) - \mathrm{Tr}\,(g_{n-1} \,|\, \mathrm{Im}\,\alpha_{n-1}). \tag{2.7}$$

Now consider

$$\begin{CD}
0 @>>> \mathrm{Im}\,\alpha_{n-1} @>>> B_{n-1} @>>> \mathrm{Im}\beta_{n-1} @>>> 0 \\
@. @Vg_{n-1}|\mathrm{Im}\,\alpha_{n-1}VV @Vg_{n-1}VV @Vh_{n-1}|\mathrm{Im}\beta_{n-1}VV \\
0 @>>> \mathrm{Im}\,\alpha_{n-1} @>>> B_{n-1} @>>> \mathrm{Im}\beta_{n-1} @>>> 0.
\end{CD} \tag{2.8}$$

So $\mathrm{Tr}(g_{n-1} \mid \mathrm{Im}\,\alpha_{n-1}) = \mathrm{Tr}(g_{n-1}) - \mathrm{Tr}(h_{n-1} \mid \mathrm{Im}\,\beta_{n-1})$. Putting this all together, we obtain

$$\mathrm{Tr}(h_n) = \mathrm{Tr}(h_n \mid \mathrm{Im}\,\beta_n) + \mathrm{Tr}(f_{n-1}) - \mathrm{Tr}(g_{n-1}) + \mathrm{Tr}(h_{n-1} \mid \mathrm{Im}\,\beta_{n-1}). \tag{2.9}$$

We next look at the left end of diagram (2.3) and get

$$0 = \mathrm{Tr}(h_{N+1}) = \mathrm{Tr}(f_N) - \mathrm{Tr}(g_N) + \mathrm{Tr}(h_N \mid \mathrm{Im}\,\beta_N), \tag{2.10}$$

and at the right end which gives

$$\mathrm{Tr}(h_1) = \mathrm{Tr}(h_1 \mid \mathrm{Im}\,\beta_1) + \mathrm{Tr}(f_0) - \mathrm{Tr}(g_0) + \mathrm{Tr}(h_0). \tag{2.11}$$

A simple calculation now yields (where a homomorphism with a negative subscript is the zero homomorphism)

$$\begin{aligned}
&\sum_{n=0}^{N}(-1)^n \mathrm{Tr}(h_n) \\
&\quad= \sum_{n=0}^{N+1}(-1)^n\big(\mathrm{Tr}(h_n \mid \mathrm{Im}\,\beta_n) + \mathrm{Tr}(f_{n-1}) - \mathrm{Tr}(g_{n-1}) + \mathrm{Tr}(h_{n-1} \mid \mathrm{Im}\,\beta_{n-1})\big) \\
&\quad= -\sum_{n=0}^{N}(-1)^n \mathrm{Tr}(f_n) + \sum_{n=0}^{N}(-1)^n \mathrm{Tr}(g_n).
\end{aligned} \tag{2.12}$$

Therefore, $L(h) = -L(f) + L(g)$. □

A more condensed version of this argument has recently been published, see [8, page 420].

We next give some simple consequences of Theorem 2.2.

If $f : (X,A) \to (X,A)$ is a self-map of a pair, where $X, A \in \mathscr{C}$, then f determines $f_X : X \to X$ and $f_A : A \to A$. The map f induces homomorphisms $f_{*k} : H_k(X,A) \to H_k(X,A)$ of relative homology with coefficients in F. The *relative Lefschetz number* $L(f;X,A)$ is defined by

$$L(f;X,A) = \sum_k (-1)^k \mathrm{Tr}\, f_{*k}. \tag{2.13}$$

Applying Theorem 2.2 to the homology exact sequence of the pair (X,A), we obtain the following corollary.

Corollary 2.3. *If $f : (X,A) \to (X,A)$ is a map of pairs, where $X, A \in \mathscr{C}$, then*

$$L(f;X,A) = L(f_X) - L(f_A). \tag{2.14}$$

This result was obtained by Bowszyc [1].

Corollary 2.4. *Suppose $X = P \cup Q$, where $X,P,Q \in \mathcal{C}$ and $(X;P,Q)$ is a proper triad* [6, page 34]. *If $f : X \to X$ is a map such that $f(P) \subseteq P$ and $f(Q) \subseteq Q$, then, for f_P, f_Q, and $f_{P\cap Q}$ being the restrictions of f to P, Q, and $P \cap Q$, respectively, there exists*

$$L(f) = L(f_P) + L(f_Q) - L(f_{P\cap Q}). \tag{2.15}$$

Proof. The map f and its restrictions induce a map of the Mayer-Vietoris homology sequence [6, page 39] to itself, so the result follows from Theorem 2.2. □

A similar result was obtained by Ferrario [7, Theorem 3.2.1].

Our final consequence of Theorem 2.2 will be used in the characterization of the reduced Lefschetz number.

Corollary 2.5. *If A is a subpolyhedron of X, $A \to X \to X/A$ is the resulting cofiber sequence of spaces in $\mathcal{C}$ and there exists a commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ A & \longrightarrow & X & \longrightarrow & X/A, \end{array} \tag{2.16}$$

then

$$L(f) = L(f') + L(\bar{f}) - 1. \tag{2.17}$$

Proof. We apply Theorem 2.2 to the homology cofiber sequence. The "minus one" on the right-hand side arises because such sequence ends with

$$\longrightarrow H_0(A) \longrightarrow H_0(X) \longrightarrow \tilde{H}_0(X/A) \longrightarrow 0. \tag{2.18}$$

□

3. Characterization of the Lefschetz number

Throughout this section, all spaces are assumed to lie in $\mathcal{C}$.

We let λ be a function from the set of self-maps of spaces in $\mathcal{C}$ to the integers that satisfies the homotopy axiom, cofibration axiom, commutativity axiom, and wedge of circles axiom of Theorem 1.1 as stated in the introduction.

We draw a few simple consequences of these axioms. From the commutativity and homotopy axioms, we obtain the following lemma.

Lemma 3.1. *If $f : X \to X$ is a map and $h : X \to Y$ is a homotopy equivalence with homotopy inverse $k : Y \to X$, then $\lambda(f) = \lambda(hfk)$.*

Lemma 3.2. *If $f : X \to X$ is homotopic to a constant map, then $\lambda(f) = 0$.*

Proof. Let $*$ be a one-point space and $* : * \to *$ the unique map. From the map of cofiber sequences

$$\begin{array}{ccccc} * & \longrightarrow & * & \longrightarrow & * \\ {\scriptstyle *}\downarrow & & {\scriptstyle *}\downarrow & & {\scriptstyle *}\downarrow \\ * & \longrightarrow & * & \longrightarrow & * \end{array} \tag{3.1}$$

and the cofibration axiom, we have $\lambda(*) = \lambda(*) + \lambda(*)$, and therefore $\lambda(*) = 0$. Write any constant map $c : X \to X$ as $c(x) = *$, for some $* \in X$, let $e : * \to X$ be inclusion and $p : X \to *$ projection. Then $c = ep$ and $pe = *$, and so $\lambda(c) = 0$ by the commutativity axiom. The lemma follows from the homotopy axiom. □

If X is a based space with base point $*$, that is, a sphere or wedge of spheres, then the cone and suspension of X are defined by $CX = X \times I/(X \times 1 \cup * \times I)$ and $\Sigma X = CX/(X \times 0)$, respectively.

Lemma 3.3. *If X is a based space, $f : X \to X$ is a based map, and $\Sigma f : \Sigma X \to \Sigma X$ is the suspension of f, then $\lambda(\Sigma f) = -\lambda(f)$.*

Proof. Consider the maps of cofiber sequences

$$\begin{array}{ccccc} X & \longrightarrow & CX & \longrightarrow & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle Cf}\downarrow & & {\scriptstyle \Sigma f}\downarrow \\ X & \longrightarrow & CX & \longrightarrow & \Sigma X. \end{array} \tag{3.2}$$

Since CX is contractible, Cf is homotopic to a constant map. Therefore, by Lemma 3.2 and the cofibration axiom,

$$0 = \lambda(Cf) = \lambda(\Sigma f) + \lambda(f). \tag{3.3}$$

□

Lemma 3.4. *For any $k \geq 1$ and $n \geq 1$, if $f : \bigvee^k S^n \to \bigvee^k S^n$ is a map, then*

$$\lambda(f) = (-1)^n \left(\deg(f_1) + \cdots + \deg(f_k)\right), \tag{3.4}$$

where $e_j : S^n \to \bigvee^k S^n$ and $p_j : \bigvee^k S^n \to S^n$, for $j = 1, \ldots, k$, are the inclusions and projections, respectively, and $f_j = p_j f e_j$.

Proof. The proof is by induction on the dimension n of the spheres. The case $n = 1$ is the wedge of circles axiom. If $n \geq 2$, then the map $f : \bigvee^k S^n \to \bigvee^k S^n$ is homotopic to a based map $f' : \bigvee^k S^n \to \bigvee^k S^n$. Then f' is homotopic to Σg, for some map $g : \bigvee^k S^{n-1} \to \bigvee^k S^{n-1}$. Note that if $g_j : S_j^{n-1} \to S_j^{n-1}$, then Σg_j is homotopic to $f_j : S_j^n \to S_j^n$. Therefore, by

Lemma 3.3 and the induction hypothesis,

$$\begin{aligned} \lambda(f) = \lambda(f') = -\lambda(g) &= -(-1)^{n-1}\big(\deg(g_1) + \cdots + \deg(g_k)\big) \\ &= (-1)^n\big(\deg(f_1) + \cdots + \deg(f_k)\big). \end{aligned} \tag{3.5}$$

□

Proof of Theorem 1.1. Since $\tilde{L}(f) = L(f) - 1$, Corollary 2.5 implies that $\tilde{L}$ satisfies the cofibration axiom. We next show that $\tilde{L}$ satisfies the wedge of circles axiom. There is an isomorphism $\theta : \bigoplus^k H_1(S^1) \to H_1(\bigvee^k S^1)$ defined by $\theta(x_1,\dots,x_k) = e_{1*}(x_1) + \cdots + e_{k*}(x_k)$, where $x_i \in H_1(S^1)$. The inverse $\theta^{-1} : H_1(\bigvee^k S^1) \to \bigoplus^k H_1(S^1)$ is given by $\theta^{-1}(y) = (p_{1*}(y),\dots,p_{k*}(y))$. If $u \in H_1(S^1)$ is a generator, then a basis for $H_1(\bigvee^k S^1)$ is $e_{1*}(u),\dots, e_{k*}(u)$. By calculating the trace of $f_{*1} : H_1(\bigvee^k S^1) \to H_1(\bigvee^k S^1)$ with respect to this basis, we obtain $\tilde{L}(f) = -(\deg(f_1) + \cdots + \deg(f_k))$. The remaining axioms are obviously satisfied by $\tilde{L}$. Thus $\tilde{L}$ satisfies the axioms of Theorem 1.1.

Now suppose λ is a function from the self-maps of spaces in $\mathscr{C}$ to the integers that satisfies the axioms. We regard X as a connected, finite CW-complex and proceed by induction on the dimension of X. If X is 1-dimensional, then it is the homotopy type of a wedge of circles. By Lemma 3.1, we can regard f as a self-map of $\bigvee^k S^1$, and so the wedge of circles axiom gives

$$\lambda(f) = -\big(\deg(f_1) + \cdots + \deg(f_k)\big) = \tilde{L}(f). \tag{3.6}$$

Now suppose that X is n-dimensional and let X^{n-1} denote the $(n-1)$-skeleton of X. Then f is homotopic to a cellular map $g : X \to X$ by the cellular approximation theorem [9, Theorem 4.8, page 349]. Thus $g(X^{n-1}) \subseteq X^{n-1}$, and so we have a commutative diagram

$$\begin{array}{ccccc} X^{n-1} & \longrightarrow & X & \longrightarrow & X/X^{n-1} = \bigvee^k S^n \\ \Big\downarrow{\scriptstyle g'} & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle \bar{g}} \\ X^{n-1} & \longrightarrow & X & \longrightarrow & X/X^{n-1} = \bigvee^k S^n. \end{array} \tag{3.7}$$

Then, by the cofibration axiom, $\lambda(g) = \lambda(g') + \lambda(\bar{g})$. Lemma 3.4 implies that $\lambda(\bar{g}) = \tilde{L}(\bar{g})$. So, applying the induction hypothesis to g', we have $\lambda(g) = \tilde{L}(g') + \tilde{L}(\bar{g})$. Since we have seen that the reduced Lefschetz number satisfies the cofibration axiom, we conclude that $\lambda(g) = \tilde{L}(g)$. By the homotopy axiom, $\lambda(f) = \tilde{L}(f)$. □

4. The normalization property

Let X be a finite polyhedron and $f : X \to X$ a map. Denote by $I(f)$ the fixed-point index of f on all of X, that is, $I(f) = i(X, f, X)$ in the notation of [2] and let $\tilde{I}(f) = I(f) - 1$.

In this section, we prove Theorem 1.3 by showing that, with rational coefficients, $I(f) = L(f)$.

Proof of Theorem 1.3. We will prove that $\tilde{I}$ satisfies the axioms, and therefore, by Theorem 1.1, $\tilde{I}(f) = \tilde{L}(f)$. The homotopy and commutativity axioms are well-known properties of the fixed-point index (see [2, pages 59–62]).

To show that $\tilde{I}$ satisfies the cofibration axiom, it suffices to consider A a subpolyhedron of X and $f(A) \subseteq A$. Let $f' : A \to A$ denote the restriction of f and $\bar{f} : X/A \to X/A$ the map induced on quotient spaces. Let $r : U \to A$ be a deformation retraction of a neighborhood of A in X onto A and let L be a subpolyhedron of a barycentric subdivision of X such that $A \subseteq \operatorname{int} L \subseteq L \subseteq U$. By the homotopy extension theorem, there is a homotopy $H : X \times I \to X$ such that $H(x,0) = f(x)$ for all $x \in X$, $H(a,t) = f(a)$ for all $a \in A$, and $H(x,1) = fr(x)$ for all $x \in L$. If we set $g(x) = H(x,1)$, then, since there are no fixed points of g on $L - A$, the additivity property implies that

$$I(g) = i(X,g,\operatorname{int} L) + i(X,g,X - L). \tag{4.1}$$

We discuss each summand of (4.1) separately. We begin with $i(X,g,\operatorname{int} L)$. Since $g(L) \subseteq A \subseteq L$, it follows from the definition of the index (see [2, page 56]) that $i(X,g,\operatorname{int} L) = i(L,g,\operatorname{int} L)$. Moreover, $i(L,g,\operatorname{int} L) = i(L,g,L)$ since there are no fixed points on $L - \operatorname{int} L$ (the excision property of the index). Let $e : A \to L$ be inclusion, then, by the commutativity property [2, page 62], we have

$$i(L,g,L) = i(L,eg,L) = i(A,ge,A) = I(f') \tag{4.2}$$

because $f(a) = g(a)$ for all $a \in A$.

Next we consider the summand $i(X,g,X - L)$ of (4.1). Let $\pi : X \to X/A$ be the quotient map, set $\pi(A) = *$, and note that $\pi^{-1}(*) = A$. If $\bar{g} : X/A \to X/A$ is induced by g, the restriction of $\bar{g}$ to the neighborhood $\pi(\operatorname{int} L)$ of $*$ in X/A is constant, so $i(X/A,\bar{g},\pi(\operatorname{int} L)) = 1$. If we denote the set of fixed points of $\bar{g}$ with $*$ deleted by $\operatorname{Fix}_* \bar{g}$, then $\operatorname{Fix}_* \bar{g}$ is in the open subset $X/A - \pi(L)$ of X/A. Let W be an open subset of X/A such that $\operatorname{Fix}_* \bar{g} \subseteq W \subseteq X/A - \pi(L)$ with the property $\bar{g}(W) \cap \pi(L) = \emptyset$. By the additivity property, we have

$$I(\bar{g}) = i(X/A,\bar{g},\pi(\operatorname{int} L)) + i(X/A,\bar{g},W) = 1 + i(X/A,\bar{g},W). \tag{4.3}$$

Now, identifying $X - L$ with the corresponding subset $\pi(X - L)$ of X/A and identifying the restrictions of $\bar{g}$ and g to those subsets, we have $i(X/A,\bar{g},W) = i(X,g,\pi^{-1}(W))$. The excision property of the index implies that $i(X,g,\pi^{-1}(W)) = i(X,g,X - L)$. Thus we have determined the second summand of (4.1): $i(X,g,X - L) = I(\bar{g}) - 1$.

Therefore, from (4.1) we obtain $I(g) = I(f') + I(\bar{g}) - 1$. The homotopy property then tells us that

$$I(f) = I(f') + I(\bar{f}) - 1 \tag{4.4}$$

since f is homotopic to g and $\bar{f}$ is homotopic to $\bar{g}$. We conclude that $\tilde{I}$ satisfies the cofibration axiom.

It remains to verify the wedge of circles axiom. Let $X = \bigvee^k S^1 = S^1_1 \vee \cdots \vee S^1_k$ be a wedge of circles with basepoint $*$ and $f : X \to X$ a map. We first verify the axiom in the case $k = 1$. We have $f : S^1 \to S^1$ and we denote its degree by $\deg(f) = d$. We regard $S^1 \subseteq \mathbb{C}$, the complex numbers. Then f is homotopic to g_d, where $g_d(z) = z^d$ has $|d - 1|$ fixed points for $d \neq 1$. The fixed-point index of g_d in a neighborhood of a fixed point that contains no other fixed point of g_d is -1 if $d \geq 2$ and is 1 if $d \leq 0$. Since g_1 is homotopic to a map without fixed points, we see that $I(g_d) = -d+1$ for all integers d. We have shown that $I(f) = -\deg(f)+1$.

Now suppose $k \geq 2$. If $f(*) = *$, then, by the homotopy extension theorem, f is homotopic to a map which does not fix $*$. Thus we may assume, without loss of generality, that $f(*) \in S_1^1 - \{*\}$. Let V be a neighborhood of $f(*)$ in $S_1^1 - \{*\}$ such that there exists a neighborhood U of $*$ in X, disjoint from V, with $f(\bar{U}) \subseteq V$. Since $\bar{U}$ contains no fixed point of f and the open subsets $S_j^1 - \bar{U}$ of X are disjoint, the additivity property implies

$$I(f) = i(X,f,S_1^1 - \bar{U}) + \sum_{j=2}^{k} i(X,f,S_j^1 - \bar{U}). \tag{4.5}$$

The additivity property also implies that

$$I(f_j) = i(S_j^1, f_j, S_j^1 - \bar{U}) + i(S_j^1, f_j, S_j^1 \cap U). \tag{4.6}$$

There is a neighborhood W_j of $(\operatorname{Fix} f) \cap S_j^1$ in S_j^1 such that $f(\overline{W}_j) \subseteq S_j^1$. Thus $f_j(x) = f(x)$ for $x \in W_j$, and therefore, by the excision property,

$$i(S_j^1, f_j, S_j^1 - \overline{U}) = i(S_j^1, f_j, W_j) = i(X, f, W_j) = i(X, f, S_j^1 - \overline{U}). \tag{4.7}$$

Since $f(\overline{U}) \subseteq S_1^1$, then $f_1(x) = f(x)$ for all $x \in \overline{U} \cap S_1^1$. There are no fixed points of f in $\overline{U}$, so $i(S_1^1, f_1, S_1^1 \cap U) = 0$, and thus, $I(f_1) = i(X, f, S_1^1 - \overline{U})$ by (4.6) and (4.7).

For $j \geq 2$, the fact that $f_j(U) = *$ gives us $i(S_j^1, f_j, S_j^1 \cap U) = 1$, so $I(f_j) = i(X, f, S_j^1 - \overline{U}) + 1$ by (4.6) and (4.7). Since $f_j : S_j^1 \to S_j^1$, the $k = 1$ case of the argument tells us that $I(f_j) = -\deg(f_j) + 1$ for $j = 1, 2, \ldots, k$. In particular, $i(X, f, S_1^1 - \overline{U}) = -\deg(f_1) + 1$, whereas, for $j \geq 2$, we have $i(X, f, S_j^1 - \overline{U}) = -\deg(f_j)$. Therefore, by (4.5),

$$I(f) = i(X, f, S_1^1 - \overline{U}) + \sum_{j=2}^{k} i(X, f, S_j^1 - \overline{U}) = -\sum_{j=1}^{k} \deg(f_j) + 1. \tag{4.8}$$

This completes the proof of Theorem 1.3. □

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