CONTINUATION THEORY FOR GENERAL CONTRACTIONS IN GAUGE SPACES

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A continuation principle of Leray-Schauder type is presented for contractions with respect to a gauge structure depending on the homotopy parameter. The result involves the most general notion of a contractive map on a gauge space and in particular yields homotopy invariance results for several types of generalized contractions.

1. Introduction

One of the most useful results in nonlinear functional analysis, the Banach contraction principle, states that every contraction on a complete metric space into itself has a unique fixed point which can be obtained by successive approximations starting from any element of the space.

Further extensions have tried to relax the metrical structure of the space, its completeness, or the contraction condition itself. Thus, there are known versions of the Banach fixed point theorem for contractions defined on subsets of locally convex spaces: Marinescu [18, page 181], in gauge spaces (spaces endowed with a family of pseudometrics): Colojoară [5] and Gheorghiu [11], in uniform spaces: Knill [16], and in syntopogenous spaces: Precup [21].

As concerns the completeness of the space, there are known results for a space endowed with two metrics (or, more generally, with two families of pseudometrics). The space is assumed to be complete with respect to one of them, while the contraction condition is expressed in terms of the second one. The first result in this direction is due to Maia [17]. The extensions of Maia's result to gauge spaces with two families of pseudometrics and to spaces with two syntopogenous structures were given by Gheorghiu [12] and Precup [22], respectively.

As regards the contraction condition, several results have been established for various types of generalized contractions on metric spaces. We only refer to the earlier papers of Kannan [15], Reich [27], Rus [29], and Ćirić [4], and to the survey article of Rhoades [28].

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We may say that almost every fixed point theorem for self-maps can be accompanied by a continuation result of Leray-Schauder type (or a homotopy invariance result). An elementary proof of the continuation principle for contractions on closed subsets of a Banach space (another proof is based on the degree theory) is due to Gatica and Kirk [10]. The homotopy invariance principle for contractions on complete metric spaces was established by Granas [14] (see also Frigon and Granas [8] and Andres and Górniewicz [2]), extended to spaces endowed with two metrics or two vector-valued metrics, and completed by an iterative procedure of discrete continuation along the fixed points curve by Precup [23, 24] (see also O'Regan and Precup [19] and Precup [26]). Continuation results for contractions on complete gauge spaces were given by Frigon [7] and for generalized contractions in the sense of Kannan-Reich-Rus and Ćirić, by Agarwal and O'Regan [1] and the first author [3].

However, until now, a unitary continuation theory for the most general notion of a contraction in gauge spaces has not been developed. The goal of this paper is to fill this gap solving this way a problem stated in Precup [25]. We are also motivated by a number of papers which have been published in the last decade, such as those of Frigon and Granas [9] and O'Regan and Precup [20], and also by the applications to integral and differential equations in locally convex spaces, see Gheorghiu and Turinici [13].

2. Preliminaries

2.1. Gauge spaces. Let X be any set. A map $p : X \times X \to \mathbb{R}_+$ is called a *pseudometric* (or a *gauge*) on X if $p(x,x) = 0$, $p(x,y) = p(y,x)$, and $p(x,y) \le p(x,z) + p(z,y)$ for every $x,y,z \in X$. A family $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$ of pseudometrics on X (or a *gauge structure* on X) is said to be *separating* if for each pair of points $x,y \in X$ with $x \neq y$, there is a $p_\alpha \in \mathcal{P}$ such that $p_\alpha(x,y) \neq 0$. A pair $(X,\mathcal{P})$ of a nonempty set X and a separating gauge structure $\mathcal{P}$ on X is called a *gauge space*.

It is well known (see Dugundji [6, pages 198–204]) that any family $\mathcal{P}$ of pseudometrics on a set X induces on X a structure $\mathcal{U}$ of uniform space and conversely, any uniform structure on X is induced by a family of pseudometrics on X. In addition, $\mathcal{U}$ is separating (or Hausdorff) if and only if $\mathcal{P}$ is separating. Hence we may identify the gauge spaces and the Hausdorff uniform spaces.

For the rest of this section we consider a gauge space $(X,\mathcal{P})$ with the gauge structure $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$. A sequence (x_n) of elements in X is said to be *Cauchy* if for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x_n,x_{n+k}) \le \varepsilon$ for all $n \ge N$ and $k \in \mathbb{N}$. The sequence (x_n) is called *convergent* if there exists an $x_0 \in X$ such that for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x_0,x_n) \le \varepsilon$ for all $n \ge N$. A gauge space is called *sequentially complete* if any Cauchy sequence is convergent. A subset of X is said to be *sequentially closed* if it contains the limit of any convergent sequence of its elements.

2.2. General contractions on gauge spaces. We now recall the notion of contraction on a gauge space introduced by Gheorghiu [11]. Let $(X,\mathcal{P})$ be a gauge space with $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$. A map $F : D \subset X \to X$ is a *contraction* if there exists a function $\varphi : A \to A$ and $a \in \mathbb{R}_+^A$, $a = \{a_\alpha\}_{\alpha\in A}$ such that

$$p_\alpha(F(x),F(y)) \le a_\alpha p_{\varphi(\alpha)}(x,y) \quad \forall \alpha \in A,\ x,y \in D, \tag{2.1}$$

$$\sum_{n=1}^{\infty} a_\alpha a_{\varphi(\alpha)} a_{\varphi^2(\alpha)} \cdots a_{\varphi^{n-1}(\alpha)} p_{\varphi^n(\alpha)}(x,y) < \infty \tag{2.2}$$

for every $\alpha \in A$ and $x, y \in D$. Here, φ^n is the nth iteration of φ.

Notice that a sufficient condition for (2.2) is that

$$\sum_{n=1}^{\infty} a_\alpha a_{\varphi(\alpha)} a_{\varphi^2(\alpha)} \cdots a_{\varphi^{n-1}(\alpha)} < \infty, \tag{2.3}$$

$$\sup\{p_{\varphi^n(\alpha)}(x,y) : n = 0,1,\ldots\} < \infty \quad \forall \alpha \in A,\ x,y \in D. \tag{2.4}$$

The above definition contains as particular cases the notion of contraction on a subset of a locally convex space introduced by Marinescu [18], for which $\varphi^2 = \varphi$, and the most worked notion of contraction on a gauge space as defined in Tarafdar [30], which corresponds to $\varphi(\alpha) = \alpha$ and $a_\alpha < 1$ for all $\alpha \in A$.

Given a space X endowed with two gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha \in A}$ and $\mathcal{Q} = \{q_\beta\}_{\beta \in B}$, in order to precise the gauge structure with respect to which a topological-type notion is considered, we will indicate the corresponding gauge structure in light of that notion. So, we will speak about $\mathcal{P}$-Cauchy, $\mathcal{Q}$-Cauchy, $\mathcal{P}$-convergent, and $\mathcal{Q}$-convergent sequences; $\mathcal{P}$-sequentially closed and $\mathcal{Q}$-sequentially closed sets; $\mathcal{P}$-contractions and $\mathcal{Q}$-contractions, and so forth. Also we say that a map $F : X \to X$ is $(\mathcal{P},\mathcal{Q})$*-sequentially continuous* if for every $\mathcal{P}$-convergent sequence (x_n) with the limit x, the sequence $(F(x_n))$ is $\mathcal{Q}$-convergent to $F(x)$.

We now state Gheorghiu's fixed point theorem of Maia type for self-maps of gauge spaces [12].

Theorem 2.1 (Gheorghiu). *Let X be a nonempty set endowed with two separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha \in A}$ and $\mathcal{Q} = \{q_\beta\}_{\beta \in B}$ and let $F : X \to X$ be a map. Assume that the following conditions are satisfied:*

(i) *there is a function $\psi : A \to B$ and $c \in (0,\infty)^A$, $c = \{c_\alpha\}_{\alpha \in A}$ such that*

$$p_\alpha(x,y) \le c_\alpha q_{\psi(\alpha)}(x,y) \quad \forall \alpha \in A,\ x,y \in X; \tag{2.5}$$

(ii) *$(X,\mathcal{P})$ is a sequentially complete gauge space;*
(iii) *F is $(\mathcal{P},\mathcal{Q})$-sequentially continuous;*
(iv) *F is a $\mathcal{Q}$-contraction.*

Then F has a unique fixed point which can be obtained by successive approximations starting from any element of X.

The following slight extension of Gheorghiu's theorem will be used in the sequel.

Theorem 2.2. *Let X be a set endowed with two separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha \in A}$ and $\mathcal{Q} = \{q_\beta\}_{\beta \in B}$, let D_0 and D be two nonempty subsets of X with $D_0 \subset D$, and let $F : D \to X$ be a map. Assume that $F(D_0) \subset D_0$ and D is $\mathcal{P}$-closed. In addition, assume that the following conditions are satisfied:*

(i) *there is a function* $\psi : A \to B$ *and* $c \in (0,\infty)^A$, $c = \{c_\alpha\}_{\alpha\in A}$ *such that*

$$p_\alpha(x,y) \le c_\alpha q_{\psi(\alpha)}(x,y) \quad \forall \alpha \in A,\ x,y \in X; \tag{2.6}$$

(ii) $(X,\mathcal{P})$ *is a sequentially complete gauge space;*

(iii) *if* $x_0 \in D_0$, $x_n = F(x_{n-1})$ *for* $n = 1,2,\dots$, *and* $\mathcal{P}\text{-}\lim_{n\to\infty} x_n = x$ *for some* $x \in D$, *then* $F(x) = x$;

(iv) F *is a* $\mathcal{Q}$*-contraction on* D.

Then F *has a unique fixed point which can be obtained by successive approximations starting from any element of* D_0.

Proof. Take any $x_0 \in D_0$ and consider the sequence (x_n) of successive approximations, $x_n = F(x_{n-1})$, $n = 1,2,\dots$. Since $F(D_0) \subset D_0$, one has $x_n \in D_0$ for all $n \in \mathbb{N}$. By (iv), (x_n) is $\mathcal{Q}$-Cauchy. Next, (i) implies that (x_n) is also $\mathcal{P}$-Cauchy, hence it is $\mathcal{P}$-convergent to some $x \in D$, in virtue of (ii). Now, (iii) guarantees that $F(x) = x$. The uniqueness is a consequence of (iv). □

2.3. Generalized contractions on metric spaces. It is worth noting that a number of fixed point results for generalized contractions on complete metric spaces appear as direct consequences of Theorem 2.2. Here are two examples.

Let (X,p) be a complete metric space and $F : X \to X$ a map.

(1) Assume that F satisfies

$$p(F(x),F(y)) \le a\big[p(x,F(x)) + p(y,F(y))\big] + bp(x,y) \tag{2.7}$$

for all $x,y \in X$, where $a,b \in \mathbb{R}_+$, $a > 0$, and $2a+b < 1$.

We associate to F a family of pseudometrics q_k, $k \in \mathbb{N}$, given by

$$q_k(x,y) = \begin{cases} a\dfrac{r^k - b^k}{r^k(r-b)}\big[p(x,F(x)) + p(y,F(y))\big] + \left(\dfrac{b}{r}\right)^k p(x,y) & \text{for } x \ne y, \\ 0 & \text{for } x = y. \end{cases} \tag{2.8}$$

Here, $r = (a+b)/(1-a)$ and $b < r < 1$. By induction, we can see that

$$q_k(F(x),F(y)) \le rq_{k+1}(x,y) \quad \forall k \in \mathbb{N},\ x,y \in X. \tag{2.9}$$

It is clear that $\mathcal{Q} = \{q_k\}_{k\in\mathbb{N}}$ is a separating gauge structure on X and from (2.9) we have that F is a $\mathcal{Q}$-contraction on X. In this case, $\varphi : \mathbf{N} \to \mathbf{N}$ is given by $\varphi(k) = k+1$ and $a_k = r$ for all $k \in \mathbb{N}$. Also, for any $k \in \mathbb{N}$, (2.2) means $\sum_{n=1}^{\infty} r^n q_{k+n}(x,y) < \infty$, which according to (2.8) is true since $0 \le b < 1$ and $b < r < 1$.

Corollary 2.3 (Reich-Rus). *If* (X,p) *is a complete metric space and* $F : X \to X$ *satisfies (2.7), then* F *has a unique fixed point.*

Proof. Let $\mathcal{P} = \{p\}$ and $\mathcal{Q} = \{q_k\}_{k\in\mathbb{N}}$. Here, $A = \{1\}$ and $B = \mathbb{N}$. In Theorem 2.2, condition (i) holds because $q_0 = p$, (ii) reduces to the completeness of (X,p), and (iv) was explained above. Now we check (iii). Assume $x_0 \in X$, $x_n = F(x_{n-1})$ for $n = 1,2,\dots$, and

$\mathcal{P}$-$\lim_{n\to\infty} x_n = x$, that is, $p(x,x_n) \to 0$ as $n \to \infty$. From (2.7), we have

$$\begin{aligned} p(x_n, F(x)) &= p(F(x_{n-1}), F(x)) \\ &\leq a[p(x_{n-1}, x_n) + p(x, F(x))] + bp(x_{n-1}, x). \end{aligned} \tag{2.10}$$

Passing to the limit, we obtain $p(x,F(x)) \leq ap(x,F(x))$, whence $p(x,F(x)) = 0$, that is, $F(x) = x$. Now the conclusion follows from Theorem 2.2. □

(2) Assume that F satisfies

$$p(F(x),F(y)) \leq a\max\{p(x,y), p(x,F(x)), p(y,F(y)), p(x,F(y)), p(y,F(x))\} \tag{2.11}$$

for all $x, y \in X$ and some $a \in [0,1)$. Then F is a $\mathcal{Q}$-contraction, where $\mathcal{Q} = \{q_k\}_{k\in\mathbb{N}}$ and

$$q_k(x,y) = \begin{cases} \max\{p(F^i(x),F^j(x)), p(F^i(y),F^j(y)), \\ \qquad p(F^i(x),F^j(y)) : i,j = 0,1,\ldots,k\} & \text{for } x \neq y, \\ 0 & \text{for } x = y. \end{cases} \tag{2.12}$$

We have $q_0 = p$ and from (2.11) we obtain

$$\begin{aligned} p(F^i(x),F^j(x)) &\leq aq_k(x,y), \\ p(F^i(y),F^j(y)) &\leq aq_k(x,y), \\ p(F^i(x),F^j(y)) &\leq aq_k(x,y), \end{aligned} \tag{2.13}$$

for all $i, j \in \{0,1,\ldots,k\}$ and $x, y \in X$. It follows that

$$q_k(F(x),F(y)) \leq aq_{k+1}(x,y) \tag{2.14}$$

and also

$$q_k(x,y) = \max\{p(x,F^i(x)), p(y,F^i(y)), p(x,F^i(y)), p(y,F^i(x))\}, \tag{2.15}$$

where the maximum is taken over $i \in \{0,1,\ldots,k\}$. If, for example, $q_k(x,y) = p(x,F^i(x))$ for some $i \in \{1,2,\ldots,k\}$, then

$$q_k(x,y) \leq p(x,F(x)) + p(F(x),F^i(x)) \leq p(x,F(x)) + aq_k(x,y). \tag{2.16}$$

Hence

$$q_k(x,y) \leq \frac{1}{1-a}p(x,F(x)) \leq \frac{1}{1-a}q_1(x,y). \tag{2.17}$$

Generally, we can prove similarly that

$$q_k(x,y) \leq \frac{1}{1-a}q_1(x,y) \tag{2.18}$$

for all $k \in \mathbb{N}$ and $x, y \in X$. This shows that (2.3) holds for the gauge structure $\mathcal{Q} = \{q_k\}_{k\in\mathbb{N}}$ and $a_k = a$ for every $k \in \mathbb{N}$.

Corollary 2.4 (Ćirić). *If (X,p) is a complete metric space and $F : X \to X$ satisfies (2.11), then F has a unique fixed point.*

Proof. Here again $\mathcal{P} = \{p\}$, $\mathcal{Q} = \{q_k\}_{k\in\mathbb{N}}$, and $q_0 = p$. To check (iii), assume $x_0 \in X$, $x_n = F(x_{n-1})$ for $n = 1,2,\ldots$, and $\mathcal{P}$-$\lim_{n\to\infty} x_n = x$, that is, $p(x,x_n) \to 0$ as $n \to \infty$. From (2.11), we obtain

$$\begin{aligned} p(x_n,F(x)) &= p(F(x_{n-1}),F(x)) \\ &\le a\max\{p(x_{n-1},x),p(x_{n-1},F(x_{n-1})),p(x,F(x)), \\ &\qquad p(x_{n-1},F(x)),p(x,F(x_{n-1}))\}. \end{aligned} \tag{2.19}$$

Passing to the limit, we obtain $p(x,F(x)) \le ap(x,F(x))$, whence $p(x,F(x)) = 0$, that is, $F(x) = x$. Thus we may apply Theorem 2.2. □

3. Continuation theorems in gauge spaces

For a map $H : D\times[0,1] \to X$, where $D \subset X$, we will use the following notations:

$$\begin{aligned} \Sigma &= \{(x,\lambda) \in D\times[0,1] : H(x,\lambda) = x\}, \\ \mathcal{S} &= \{x \in D : H(x,\lambda) = x \text{ for some } \lambda \in [0,1]\}, \\ \Lambda &= \{\lambda \in [0,1] : H(x,\lambda) = x \text{ for some } x \in D\}. \end{aligned} \tag{3.1}$$

Now we state and prove the main result of this paper: a continuation principle for contractions on spaces with a gauge structure depending on the homotopy parameter.

Theorem 3.1. *Let X be a set endowed with the separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$ and $\mathcal{Q}^\lambda = \{q_\beta^\lambda\}_{\beta\in B}$ for $\lambda \in [0,1]$. Let $D \subset X$ be $\mathcal{P}$-sequentially closed, $H : D\times[0,1] \to X$ a map, and assume that the following conditions are satisfied:*

(i) *for each $\lambda \in [0,1]$, there exists a function $\varphi_\lambda : B \to B$ and $a^\lambda \in [0,1)^B$, $a^\lambda = \{a_\beta^\lambda\}_{\beta\in B}$ such that*

$$\begin{gathered} q_\beta^\lambda(H(x,\lambda),H(y,\lambda)) \le a_\beta^\lambda q_{\varphi_\lambda(\beta)}^\lambda(x,y), \\ \sum_{n=1}^{\infty} a_\beta^\lambda a_{\varphi_\lambda(\beta)}^\lambda a_{\varphi_\lambda^2(\beta)}^\lambda \cdots a_{\varphi_\lambda^{n-1}(\beta)}^\lambda q_{\varphi_\lambda^n(\beta)}^\lambda(x,y) < \infty \end{gathered} \tag{3.2}$$

for every $\beta \in B$ and $x,y \in D$;

(ii) *there exists $\rho > 0$ such that for each $(x,\lambda) \in \Sigma$, there is a $\beta \in B$ with*

$$\inf\{q_\beta^\lambda(x,y) : y \in X\setminus D\} > \rho; \tag{3.3}$$

(iii) *for each $\lambda \in [0,1]$, there is a function $\psi : A \to B$ and $c \in (0,\infty)^A$, $c = \{c_\alpha\}_{\alpha\in A}$ such that*

$$p_\alpha(x,y) \le c_\alpha q_{\psi(\alpha)}^\lambda(x,y) \quad \forall \alpha \in A,\ x,y \in X; \tag{3.4}$$

(iv) $(X,\mathcal{P})$ *is a sequentially complete gauge space;*
(v) *if* $\lambda \in [0,1]$, $x_0 \in D$, $x_n = H(x_{n-1},\lambda)$ *for* $n = 1,2,\ldots$, *and* $\mathcal{P}$-$\lim_{n\to\infty} x_n = x$, *then* $H(x,\lambda) = x$;
(vi) *for every* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon) > 0$ *with*

$$q^{\lambda}_{\varphi^n_\lambda(\beta)}(x,H(x,\lambda)) \le \left(1 - a^{\lambda}_{\varphi^n_\lambda(\beta)}\right)\varepsilon \tag{3.5}$$

for $(x,\mu) \in \Sigma$, $|\lambda - \mu| \le \delta$, *all* $\beta \in B$, *and* $n \in \mathbb{N}$.

In addition, assume that $H_0 := H(\cdot,0)$ *has a fixed point. Then, for each* $\lambda \in [0,1]$, *the map* $H_\lambda := H(\cdot,\lambda)$ *has a unique fixed point.*

Proof. We prove that there exists a number $h > 0$ such that if $\mu \in \Lambda$, then $\lambda \in \Lambda$ for every λ satisfying $|\lambda - \mu| \le h$. This, together with $0 \in \Lambda$, clearly implies $\Lambda = [0,1]$.

First we note that from (ii) it follows that for each $(x,\lambda) \in \Sigma$, there exists $\beta \in B$ such that the set

$$B(x,\lambda,\beta) =: \left\{ y \in X : q^{\lambda}_{\varphi^n_\lambda(\beta)}(x,y) \le \rho \ \forall n \in \mathbb{N} \right\} \tag{3.6}$$

is included in D.

Let $\mu \in \Lambda$ and let $H(x,\mu) = x$. From (vi), there is $h = h(\rho) > 0$, independent of μ and x, such that

$$q^{\lambda}_{\varphi^n_\lambda(\beta)}(x,H(x,\lambda)) = q^{\lambda}_{\varphi^n_\lambda(\beta)}(H(x,\mu),H(x,\lambda)) \le \left(1 - a^{\lambda}_{\varphi^n_\lambda(\beta)}\right)\rho \tag{3.7}$$

for $|\lambda - \mu| \le h$ and all $n \in \mathbb{N}$. Consequently, if $|\lambda - \mu| \le h$ and $y \in B(x,\lambda,\beta)$, then

$$\begin{aligned} q^{\lambda}_{\varphi^n_\lambda(\beta)}(x,H(y,\lambda)) &\le q^{\lambda}_{\varphi^n_\lambda(\beta)}(x,H(x,\lambda)) + q^{\lambda}_{\varphi^n_\lambda(\beta)}(H(x,\lambda),H(y,\lambda)) \\ &\le \left(1 - a^{\lambda}_{\varphi^n_\lambda(\beta)}\right)\rho + a^{\lambda}_{\varphi^n_\lambda(\beta)} q^{\lambda}_{\varphi^{n+1}_\lambda(\beta)}(x,y) \\ &\le \left(1 - a^{\lambda}_{\varphi^n_\lambda(\beta)}\right)\rho + a^{\lambda}_{\varphi^n_\lambda(\beta)}\rho = \rho. \end{aligned} \tag{3.8}$$

Hence, for $|\lambda - \mu| \le h$, H_λ is a self-map of $D_0 := B(x,\lambda,\beta)$. Now Theorem 2.2 guarantees that $\lambda \in \Lambda$ for $|\lambda - \mu| \le h$. □

Assuming a continuity property of H, we derive from Theorem 3.1 the following result.

Theorem 3.2. *Let* X *be a set endowed with the separating gauge structures* $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$ *and* $\mathcal{Q}^\lambda = \{q^\lambda_\beta\}_{\beta\in B}$ *for* $\lambda \in [0,1]$. *Let* $D \subset X$ *be* $\mathcal{P}$*-sequentially closed,* $H : D \times [0,1] \to X$ *a map, and assume that the following conditions are satisfied:*

(a) *for each* $\lambda \in [0,1]$, *there exists a function* $\varphi_\lambda : B \to B$ *and* $a^\lambda \in [0,1)^B$, $a^\lambda = \{a^\lambda_\beta\}_{\beta\in B}$ *such that*

$$q^\lambda_\beta(H(x,\lambda),H(y,\lambda)) \le a^\lambda_\beta q^\lambda_{\varphi_\lambda(\beta)}(x,y), \tag{3.9}$$

$$\sup\left\{q^\lambda_{\varphi^n_\lambda(\beta)}(x,y) : n \in \mathbb{N}\right\} < \infty, \tag{3.10}$$

$$\sup\left\{\sum_{n=1}^{\infty} a^\lambda_\beta a^\lambda_{\varphi_\lambda(\beta)} a^\lambda_{\varphi^2_\lambda(\beta)} \cdots a^\lambda_{\varphi^{n-1}_\lambda(\beta)} : \lambda \in [0,1]\right\} < \infty, \tag{3.11}$$

for all $\beta \in B$ *and* $x, y \in D$;

(b) *there exists a set* $U \subset D$ *such that* $H(x,\lambda) \ne x$ *for all* $x \in D \setminus U$ *and* $\lambda \in [0,1]$; *and for each* $(x,\mu) \in \Sigma$, *there is* $\beta \in B$, $\delta > 0$, *and* $\gamma > 0$ *such that for every* $\lambda \in [0,1]$ *with* $|\lambda - \mu| \le \gamma$,

$$\left\{y \in X : q^\lambda_\beta(x,y) < \delta\right\} \subset U; \tag{3.12}$$

(c) *for each* $\lambda \in [0,1]$, *there is a function* $\psi : A \to B$ *and* $c \in (0,\infty)^A$, $c = \{c_\alpha\}_{\alpha\in A}$ *such that*

$$p_\alpha(x,y) \le c_\alpha q^\lambda_{\psi(\alpha)}(x,y) \quad \forall \alpha \in A,\ x,y \in X; \tag{3.13}$$

(d) $(X,\mathcal{P})$ *is a sequentially complete gauge space;*

(e) H *is* $(\mathcal{P},\mathcal{P})$*-sequentially continuous;*

(f) *for every* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon) > 0$ *with*

$$q^\lambda_{\varphi^n_\lambda(\beta)}(x,H(x,\lambda)) \le \left(1 - a^\lambda_{\varphi^n_\lambda(\beta)}\right)\varepsilon \tag{3.14}$$

for $(x,\mu) \in \Sigma$, $|\lambda - \mu| \le \delta$, *and all* $\beta \in B$ *and* $n \in \mathbb{N}$.

In addition, assume that $H_0 := H(\cdot,0)$ *has a fixed point. Then, for each* $\lambda \in [0,1]$, *the map* $H_\lambda := H(\cdot,\lambda)$ *has a unique fixed point.*

Proof. Conditions (i), (iii), (iv), and (vi) in Theorem 3.1 are obviously satisfied. Assume (ii) is false. Then, for each $n \in \mathbb{N} \setminus \{0\}$, there is $(x_n,\lambda_n) \in \Sigma$ and $y_{n\beta} \in X \setminus D$ with

$$q^{\lambda_n}_\beta(x_n,y_{n\beta}) \le \frac{1}{n} \quad \text{for every } \beta \in B. \tag{3.15}$$

Clearly we may assume that $\lambda_n \to \lambda$.

Fix an arbitrary $\beta \in B$. From (f) we see that for a given $\varepsilon > 0$, there is a number $N = N(\varepsilon) > 0$ such that

$$q^\lambda_{\varphi^i_\lambda(\beta)}(x_n,H(x_n,\lambda)) \le \frac{\varepsilon}{4C} \tag{3.16}$$

for all $n \ge N$ and $i \in \mathbb{N}$, where C is any positive number with

$$1 + \sum_{i=1}^{\infty} a^\lambda_\beta a^\lambda_{\varphi_\lambda(\beta)} \cdots a^\lambda_{\varphi^{i-1}_\lambda(\beta)} \le C < \infty, \quad \lambda \in [0,1]. \tag{3.17}$$

Now, for $n, m \geq N$, using (a), we obtain

$$\begin{aligned} q_\beta^\lambda(x_n, x_m) &= q_\beta^\lambda(H(x_n,\lambda_n), H(x_m,\lambda_m)) \leq q_\beta^\lambda(H(x_n,\lambda_n), H(x_n,\lambda)) \\ &\quad + q_\beta^\lambda(H(x_m,\lambda_m), H(x_m,\lambda)) + q_\beta^\lambda(H(x_n,\lambda), H(x_m,\lambda)) \\ &\leq q_\beta^\lambda(H(x_n,\lambda_n), H(x_n,\lambda)) + q_\beta^\lambda(H(x_m,\lambda_m), H(x_m,\lambda)) \\ &\quad + a_\beta^\lambda q_{\varphi_\lambda(\beta)}^\lambda(x_n, x_m) \\ &\leq \frac{\varepsilon}{2C} + a_\beta^\lambda q_{\varphi_\lambda(\beta)}^\lambda(x_n, x_m). \end{aligned} \tag{3.18}$$

Similarly,

$$q_{\varphi_\lambda(\beta)}^\lambda(x_n, x_m) \leq \frac{\varepsilon}{2C} + a_{\varphi_\lambda(\beta)}^\lambda q_{\varphi_\lambda^2(\beta)}^\lambda(x_n, x_m) \tag{3.19}$$

and, in general,

$$q_{\varphi_\lambda^i(\beta)}^\lambda(x_n, x_m) \leq \frac{\varepsilon}{2C} + a_{\varphi_\lambda^i(\beta)}^\lambda q_{\varphi_\lambda^{i+1}(\beta)}^\lambda(x_n, x_m) \tag{3.20}$$

for all $i \in \mathbb{N}$. It follows that for all $n, m \geq N$ and every $l \in \mathbb{N}$, we have

$$\begin{aligned} q_\beta^\lambda(x_n, x_m) &\leq \frac{\varepsilon}{2C}\left(1 + \sum_{i=1}^{l} a_\beta^\lambda a_{\varphi_\lambda(\beta)}^\lambda \cdots a_{\varphi_\lambda^{i-1}(\beta)}^\lambda\right) \\ &\quad + a_\beta^\lambda a_{\varphi_\lambda(\beta)}^\lambda \cdots a_{\varphi_\lambda^l(\beta)}^\lambda q_{\varphi_\pi^{l+1}(\beta)}^\lambda(x_n, x_m) \\ &\leq \frac{\varepsilon}{2} + a_\beta^\lambda a_{\varphi_\lambda(\beta)}^\lambda \cdots a_{\varphi_\lambda^l(\beta)}^\lambda M(\lambda, \beta, x_n, x_m). \end{aligned} \tag{3.21}$$

Here, $M(\lambda,\beta,x,y) := \sup\{q_{\varphi_\lambda^n(\beta)}^\lambda(x,y) : n \in \mathbb{N}\}$. According to (3.11), for each couple $[n,m]$ with $n, m \geq N$, we may find an l such that

$$a_\beta^\lambda a_{\varphi_\lambda(\beta)}^\lambda \cdots a_{\varphi_\lambda^l(\beta)}^\lambda M(\lambda, \beta, x_n, x_m) \leq \frac{\varepsilon}{2}. \tag{3.22}$$

Hence

$$q_\beta^\lambda(x_n, x_m) \leq \varepsilon \quad \forall n, m \geq N. \tag{3.23}$$

Thus the sequence (x_n) is $\mathcal{Q}^\lambda$-Cauchy. Now (c) guarantees that (x_n) is $\mathcal{P}$-Cauchy. Furthermore, (d) implies that (x_n) is $\mathcal{P}$-convergent. Let $x = \mathcal{P}\text{-}\lim_{n\to\infty} x_n$. Clearly $x \in D$. Then, from (e), $\mathcal{P}\text{-}\lim_{n\to\infty} H(x_n,\lambda_n) = H(x,\lambda)$. Hence $H(x,\lambda) = x$.

Now we claim that

$$q_\beta^{\lambda_n}(x, x_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.24}$$

Indeed, since $(x,\lambda) \in \Sigma$ and $\lambda_n \to \lambda$, from (f) it follows that for a given $\varepsilon > 0$, there is a number $N_0 = N_0(\varepsilon) > 0$ such that

$$q_{\varphi_{\lambda_n}^i(\beta)}^{\lambda_n}(x, H(x,\lambda_n)) \leq \frac{\varepsilon}{2C} \tag{3.25}$$

for all $n \geq N_0$ and $i \in \mathbb{N}$. Then, for $n \geq N_0$, we have

$$\begin{aligned} q_\beta^{\lambda_n}(x,x_n) &= q_\beta^{\lambda_n}(x,H(x_n,\lambda_n)) \\ &\leq q_\beta^{\lambda_n}(x,H(x,\lambda_n)) + q_\beta^{\lambda_n}(H(x,\lambda_n),H(x_n,\lambda_n)) \\ &\leq \frac{\varepsilon}{2C} + a_\beta^{\lambda_n} q_{\varphi_{\lambda_n}(\beta)}^{\lambda_n}(x,x_n). \end{aligned} \tag{3.26}$$

Furthermore, as above, we deduce that

$$q_\beta^{\lambda_n}(x,x_n) \leq \varepsilon \quad \forall n \geq N_0. \tag{3.27}$$

This proves our claim.

Also (b) guarantees

$$q_\beta^{\lambda_n}(x,y_{n\beta}) \geq \delta \tag{3.28}$$

for a sufficiently large n and some $\beta \in B$. Now, from

$$0 < \delta \leq q_\beta^{\lambda_n}(x,y_{n\beta}) \leq q_\beta^{\lambda_n}(x,x_n) + q_\beta^{\lambda_n}(x_n,y_{n\beta}), \tag{3.29}$$

we derive a contradiction. This contradiction shows that (ii) holds. Also (v) immediately follows from (e). Thus Theorem 3.1 applies. □

Remark 3.3. In particular, if the gauge structures reduce to metric structures, that is, $\mathcal{P} = \{p\}$ and $\mathcal{Q}^\lambda = \mathcal{Q} = \{q\}$, p and q being two metrics on X, Theorem 3.2 becomes the first part of Theorem 2.2 of Precup [23] (with the additional assumption that there is a constant $c > 0$ with $p(x,y) \leq cq(x,y)$ for all $x,y \in X$).

4. Homotopy results for generalized contractions on metric spaces

In this section, we test Theorem 3.1 on generalized contractions on complete metric spaces. We begin with a continuation result for generalized contractions of Reich-Rus type.

Theorem 4.1. *Let (X,p) be a complete metric space, D a closed subset of X, and $H : D \times [0,1] \to X$ a map. Assume that the following conditions are satisfied:*

(A) *there exist $a,b \in \mathbb{R}_+$ with $a > 0$ and $2a + b < 1$ such that*

$$p(H_\lambda(x),H_\lambda(y)) \leq a\big[p(x,H_\lambda(x)) + pd(y,H_\lambda(y))\big] + bp(x,y) \tag{4.1}$$

for all $x,y \in D$ and $\lambda \in [0,1]$;

(B) $\inf\{p(x,y) : x \in \mathcal{S},\ y \in X \setminus D\} > 0$;

(C) *for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$p(H(x,\lambda),H(x,\mu)) \leq \varepsilon \quad \textit{for } |\lambda - \mu| \leq \delta, \textit{ all } x \in D. \tag{4.2}$$

In addition, assume that $H_0 := H(\cdot,0)$ has a fixed point. Then, for each $\lambda \in [0,1]$, the map $H_\lambda := H(\cdot,\lambda)$ has a unique fixed point.

Proof. We will apply Theorem 3.1. Let $\mathcal{P} = \{p\}$ and $\mathcal{Q}^\lambda = \{q_k^\lambda\}_{k\in\mathbb{N}}$, where q_k^λ are defined as (2.8) shows, with F replaced by H_λ. In this case $A = \{1\}$ and $B = \mathbb{N}$.

Condition (i) in Theorem 3.1 is satisfied as the reasoning in Section 2.3 shows.

Next (B) guarantees (ii) with $\beta = 0$, since $q_0^\lambda = p$. Now take $\psi(1) = 0$ to see that (iii) holds trivially. Since the space (X, p) is complete, we have (iv), while (v) can be checked as in the proof of Corollary 2.3. Thus it remains to check (vi), that is, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$q_m^\lambda(x, H(x,\lambda)) \leq (1-r)\varepsilon \tag{4.3}$$

for every $(x,\mu) \in \Sigma$, $|\lambda - \mu| \leq \delta$, and $m \in \mathbb{N}$. This can be proved by using (C) if we observe that $q_m^\lambda(x, H(x,\lambda))$ depends only on $p(x, H_\lambda(x)) = p(H(x,\mu), H(x,\lambda))$, and

$$a\frac{r^m - b^m}{r^m(r-b)} \leq \frac{a}{r-b}, \qquad \left(\frac{b}{r}\right)^m \leq 1. \tag{4.4}$$

□

Similarly, Theorem 3.1 yields a continuation result for generalized contractions in the sense of Ćirić.

THEOREM 4.2. *Let (X, p) be a complete metric space, D a closed subset of X, and $H : D \times [0,1] \to X$ a map. Assume that the following conditions are satisfied:*

(A) *there exists $a < 1$ such that*

$$\begin{aligned} &p(H_\lambda(x), H_\lambda(y)) \\ &\quad \leq a\max\{p(x,y), p(x,H_\lambda(x)), p(y,H_\lambda(y)), p(x,H_\lambda(y)), p(y,H_\lambda(x))\} \end{aligned} \tag{4.5}$$

for all $x, y \in D$ and $\lambda \in [0,1]$;

(B) $\inf\{p(x,y) : x \in \mathcal{S},\ y \in X \setminus D\} > 0$;

(C) *for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $p(H(x,\lambda), H(x,\mu)) \leq \varepsilon$ for $|\lambda - \mu| \leq \delta$ and all $x \in D$.*

In addition, assume that $H_0 := H(\cdot, 0)$ has a fixed point. Then, for each $\lambda \in [0,1]$, the map $H_\lambda := H(\cdot,\lambda)$ has a unique fixed point.

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