

# FIXED POINT THEOREMS IN CAT(0) SPACES AND $\mathbb{R}$ -TREES

W. A. KIRK

*Received 10 June 2004*

We show that if  $U$  is a bounded open set in a complete CAT(0) space  $X$ , and if  $f : \bar{U} \rightarrow X$  is nonexpansive, then  $f$  always has a fixed point if there exists  $p \in U$  such that  $x \notin [p, f(x))$  for all  $x \in \partial U$ . It is also shown that if  $K$  is a geodesically bounded closed convex subset of a complete  $\mathbb{R}$ -tree with  $\text{int}(K) \neq \emptyset$ , and if  $f : K \rightarrow X$  is a continuous mapping for which  $x \notin [p, f(x))$  for some  $p \in \text{int}(K)$  and all  $x \in \partial K$ , then  $f$  has a fixed point. It is also noted that a geodesically bounded complete  $\mathbb{R}$ -tree has the fixed point property for continuous mappings. These latter results are used to obtain variants of the classical fixed edge theorem in graph theory.

## 1. Introduction

A metric space  $X$  is said to be a CAT(0) space (the term is due to M. Gromov—see, e.g., [1, page 159]) if it is geodesically connected, and if every geodesic triangle in  $X$  is at least as “thin” as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include the classical hyperbolic spaces, Euclidean buildings (see [2]), the complex Hilbert ball with a hyperbolic metric (see [6]; also [12, inequality (4.3)] and subsequent comments), and many others. (On the other hand, if a Banach space is a CAT( $\kappa$ ) space for some  $\kappa \in \mathbb{R}$ , then it is necessarily a Hilbert space and CAT(0).) For a thorough discussion of these spaces and of the fundamental role they play in geometry, see Bridson and Haefliger [1]. Burago et al. [3] present a somewhat more elementary treatment, and Gromov [8] a deeper study.

In this paper, it is shown that if  $U$  is a bounded open set in a complete CAT(0) space  $X$ , and if  $f : \bar{U} \rightarrow X$  is nonexpansive, then  $f$  always has a fixed point if there exists  $p \in U$  such that  $x \notin [p, f(x))$  for all  $x \in \partial U$ . (In a Banach space, this condition is equivalent to the classical Leray-Schauder boundary condition:  $f(x) - p \neq \lambda(x - p)$  for  $x \in \partial U$  and  $\lambda > 1$ .) It is then shown that boundedness of  $U$  can be replaced with convexity and geodesic boundedness if  $X$  is an  $\mathbb{R}$ -tree. In fact this latter result holds for any continuous mapping. Three variants of the classical fixed edge theorem in graph theory are also obtained. Precise definitions are given below.

**2. Preliminary remarks**

Let  $(X, d)$  be a metric space. Recall that a *geodesic path* joining  $x \in X$  to  $y \in X$  (or, more briefly, a *geodesic* from  $x$  to  $y$ ) is a map  $c$  from a closed interval  $[0, l] \subset \mathbb{R}$  to  $X$  such that  $c(0) = x$ ,  $c(l) = y$ , and  $d(c(t), c(t')) = |t - t'|$  for all  $t, t' \in [0, l]$ . In particular,  $c$  is an isometry and  $d(x, y) = l$ . The image  $\alpha$  of  $c$  is called a *geodesic* (or *metric segment*) joining  $x$  and  $y$ . When unique, this geodesic is denoted  $[x, y]$ . The space  $(X, d)$  is said to be a *geodesic space* if every two points of  $X$  are joined by a geodesic, and  $X$  is said to be *uniquely geodesic* if there is exactly one geodesic joining  $x$  and  $y$  for each  $x, y \in X$ . A subset  $Y \subseteq X$  is said to be *convex* if  $Y$  includes every geodesic segment joining any two of its points.

For complete details and further discussion, see, for example, [1] or [3].

Denote by  $M_\kappa^2$  the following classical metric spaces:

- (1) if  $\kappa = 0$  then  $M_0^2$  is the Euclidean plane  $\mathbb{E}^2$ ;
- (2) if  $\kappa > 0$  then  $M_\kappa^2$  is obtained from the classical sphere  $S^2$  by multiplying the spherical distance by  $1/\sqrt{\kappa}$ ;
- (3) if  $\kappa < 0$  then  $M_\kappa^2$  is obtained from the classical hyperbolic plane  $\mathbb{H}^2$  by multiplying the hyperbolic distance by  $1/\sqrt{-\kappa}$ .

A *geodesic triangle*  $\Delta(x_1, x_2, x_3)$  in a geodesic metric space  $(X, d)$  consists of three points in  $X$  (the *vertices* of  $\Delta$ ) and a geodesic segment between each pair of vertices (the *edges* of  $\Delta$ ). A *comparison triangle* for geodesic triangle  $\Delta(x_1, x_2, x_3)$  in  $(X, d)$  is a triangle  $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$  in  $M_\kappa^2$  such that  $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$  for  $i, j \in \{1, 2, 3\}$ . If  $\kappa > 0$  it is further assumed that the perimeter of  $\Delta(x_1, x_2, x_3)$  is less than  $2D_\kappa$ , where  $D_\kappa$  denotes the diameter of  $M_\kappa^2$ . The triangle inequality assures that comparison triangles always exist.

A geodesic metric space is said to be a  $CAT(\kappa)$  space if all geodesic triangles of appropriate size satisfy the following  $CAT(\kappa)$  comparison axiom.

$CAT(\kappa)$ . Let  $\Delta$  be a geodesic triangle in  $X$  and let  $\bar{\Delta} \subset M_\kappa^2$  be a comparison triangle for  $\Delta$ .

Then  $\Delta$  is said to satisfy the  $CAT(\kappa)$  *inequality* if for all  $x, y \in \Delta$  and all comparison points  $\bar{x}, \bar{y} \in \bar{\Delta}$ ,

$$d(x, y) \leq d(\bar{x}, \bar{y}). \tag{2.1}$$

Of particular interest in this note are the complete  $CAT(0)$  spaces, often called *Hadamard spaces*. These spaces are uniquely geodesic and they include, as a very special case, the following class of spaces.

*Definition 2.1.* An  $\mathbb{R}$ -tree is a metric space  $T$  such that

- (i) there is a unique geodesic segment (denoted by  $[x, y]$ ) joining each pair of points  $x, y \in T$ ;
- (ii) if  $[y, x] \cap [x, z] = \{x\}$ , then  $[y, x] \cup [x, z] = [y, z]$ .

**PROPOSITION 2.2.** *The following relations hold.*

- (1) If  $X$  is a  $CAT(\kappa)$  space, then it is a  $CAT(\kappa')$  space for every  $\kappa' \geq \kappa$ .
- (2)  $X$  is a  $CAT(\kappa)$  space for all  $\kappa$  if and only if  $X$  is an  $\mathbb{R}$ -tree.

One consequence of (1) and (2) is that any result proved for CAT(0) spaces automatically carries over to any CAT( $\kappa$ ) spaces for  $\kappa < 0$ , and, in particular, to  $\mathbb{R}$ -trees.

### 3. Fixed point theorems

We use the following continuation principle due to Granas in the proof of our first result.

**THEOREM 3.1** [7]. *Let  $U$  be a domain in a complete metric space  $X$ , let  $f, g : \bar{U} \rightarrow X$  be two contraction mappings, and suppose there exists  $H : \bar{U} \times [0, 1] \rightarrow X$  such that*

- (a)  $H(\cdot, 1) = f, H(\cdot, 0) = g$ ;
- (b)  $H(x, t) \neq x$  for every  $x \in \partial U$  and  $t \in [0, 1]$ ;
- (c) there exists  $\alpha < 1$  such that  $d(H(x, t), H(y, t)) \leq \alpha d(x, y)$  for every  $x, y \in \bar{U}$  and  $t \in [0, 1]$ ;
- (d) there exists a constant  $M \geq 0$  such that for every  $x \in \bar{U}$  and  $t, s \in [0, 1]$ ,

$$d(H(x, t), H(x, s)) \leq M|s - t|. \tag{3.1}$$

Then  $f$  has a fixed point if and only if  $g$  has a fixed point.

We will also need the following lemma due to Crandall and Pazy.

**LEMMA 3.2** [4]. *Let  $\{z_n\}$  be a subset of a Hilbert space  $H$  and let  $\{r_n\}$  be a sequence of positive numbers. Suppose*

$$\langle z_n - z_m, r_n z_n - r_m z_m \rangle \leq 0, \quad \text{for } m = 1, 2, \dots \tag{3.2}$$

Then if  $r_n$  is strictly decreasing,  $\|z_n\|$  is increasing. If  $\|z_n\|$  is bounded,  $\lim_{n \rightarrow \infty} z_n$  exists.

**THEOREM 3.3.** *Let  $U$  be a bounded open set in a complete CAT(0) space  $X$ , and suppose  $f : \bar{U} \rightarrow X$  is nonexpansive. Suppose there exists  $p \in U$  such that  $x \notin [p, f(x)]$  for all  $x \in \partial U$ . Then  $f$  has a fixed point in  $\bar{U}$ .*

When  $X$  is a Hilbert space, Theorem 3.3 holds under the even weaker assumption that  $f$  is a Lipschitzian pseudocontractive mapping. This has been known for some time (see [13]). Our proof is patterned after Precup’s Hilbert space proof [10] for nonexpansive mappings. We observe here that the CAT(0) inequality is sufficient.

*Proof of Theorem 3.3.* Let  $t \in (0, 1)$  and for  $u \in U$  let  $f_t(u)$  be the point of the segment  $[p, f(u)]$  with distance  $td(p, f(u))$  from  $p$ . Let  $x, y \in U$  and consider the comparison triangle  $\bar{\Delta} = \Delta(\bar{p}, \bar{x}, \bar{y})$  of  $\Delta(p, x, y)$  in  $\mathbb{E}^2$ . If  $\bar{f}_t(x)$  and  $\bar{f}_t(y)$  denote the respective comparison points of  $f_t(x)$  and  $f_t(y)$  in  $\bar{\Delta}$ , then by the CAT(0) inequality,

$$d(f_t(x), f_t(y)) \leq \|\bar{f}_t(x) - \bar{f}_t(y)\| = t\|\bar{x} - \bar{y}\| = td(x, y). \tag{3.3}$$

Therefore  $f_t$  is a contraction mapping of  $U \rightarrow X$ . Moreover, if  $B(p; r) \subset U$ , then  $f_t : U \rightarrow B(p; r)$  for  $t$  sufficiently small. Thus  $f_t$  has a fixed point for  $t$  sufficiently small. Now let  $\lambda \in (0, 1)$ . We apply Theorem 3.1 to show that  $f_\lambda$  has a fixed point. Define the homotopy  $H : \bar{U} \times [0, 1] \rightarrow X$  by setting  $H(x, t) = f_{\lambda t}(x)$ . Then  $H(\cdot, 1) = f_\lambda$  and  $H(\cdot, 0)$  is a constant map. If  $H(x, t) = x$  for some  $x \in \partial U$  and  $t \in [0, 1]$ , then  $f_{\lambda t}(x) = x$  and  $x \in [p, f(x)]$ .

Since this is not possible, condition (b) of Theorem 3.1 holds. Condition (c) holds upon taking  $\alpha$  to be  $\lambda$ . Finally,

$$d(H(x, t), H(x, s)) \leq |s - t|d(p, f(x)), \tag{3.4}$$

for all  $t, s \in [0, 1]$ , and since  $U$  is bounded, condition (d) holds. Therefore, by Theorem 3.1,  $f_\lambda$  has a unique fixed point, and it follows that  $f_t$  has a unique fixed point  $x_t$  for each  $t \in (0, 1)$ .

Now denote by  $x_n, n \in \mathbb{N}$ , the point  $x_t$  for  $t = 1 - 1/n$ . For  $m, n \in \mathbb{N}, m, n > 1$ , consider the comparison triangle  $\bar{\Delta} = \Delta(0, \bar{f}(x_m), \bar{f}(x_n))$  of  $\Delta(p, f(x_m), f(x_n))$  in  $\mathbb{E}^2$ , and let  $\bar{x}_m, \bar{x}_n$  denote the respective comparison points of  $x_m, x_n$ . Then, using the fact that  $f$  is nonexpansive in conjunction with the CAT(0) inequality,

$$\|\bar{f}(x_m) - \bar{f}(x_n)\| = d(f(x_n), f(x_m)) \leq d(x_n, x_m) \leq \|\bar{x}_n - \bar{x}_m\|. \tag{3.5}$$

Consequently, if  $r_m = (m - 1)^{-1}$  and  $r_n = (n - 1)^{-1}$ ,

$$\langle r_n \bar{x}_n - r_m \bar{x}_m, \bar{x}_n - \bar{x}_m \rangle = \langle \bar{f}(x_n) - \bar{f}(x_m), \bar{x}_n - \bar{x}_m \rangle - \|\bar{x}_n - \bar{x}_m\|^2 \leq 0. \tag{3.6}$$

Since  $\{r_n\}$  is strictly decreasing,  $\{\bar{x}_n\}$  converges by Lemma 3.2. Since  $d(x_n, x_m) \leq d(\bar{x}_n, \bar{x}_m)$ ,  $\{x_n\}$  converges as well, necessarily to a fixed point of  $f$ .  $\square$

It is noteworthy that in the preceding result the domain  $U$  is *not* assumed to be convex.

An entirely different approach yields a stronger result if  $X$  is an  $\mathbb{R}$ -tree. For this result we do require convexity of the domain, but the boundedness assumption is relaxed, and the result holds for continuous mappings. We begin with the following fact, which illustrates that compactness is not necessary for continuous mappings to have fixed points in complete  $\mathbb{R}$ -trees. This fact may be known, perhaps as a special case of more abstract theory, but it does not seem to be readily found in the literature.

**THEOREM 3.4.** *Suppose  $X$  is a geodesically bounded complete  $\mathbb{R}$ -tree. Then every continuous mapping  $f : X \rightarrow X$  has a fixed point.*

*Proof.* For  $u, v \in X$  we let  $[u, v]$  denote the (unique) metric segment joining  $u$  and  $v$  and let  $[u, v) = [u, v] \setminus \{v\}$ . We associate with each point  $x \in X$  a point  $\varphi(x)$  as follows. For each  $t \in [x, f(x)]$ , let  $\xi(t)$  be the point of  $X$  for which

$$[x, f(x)] \cap [x, f(t)] = [x, \xi(t)]. \tag{3.7}$$

(It follows from the definition of an  $\mathbb{R}$ -tree that such a point always exists.) If  $\xi(f(x)) = f(x)$  take  $\varphi(x) = f(x)$ . Otherwise it must be the case that  $\xi(f(x)) \in [x, f(x))$ . Let

$$\begin{aligned} A &= \{t \in [x, f(x)] : \xi(t) \in [x, t]\}; \\ B &= \{t \in [x, f(x)] : \xi(t) \in [t, f(x)]\}. \end{aligned} \tag{3.8}$$

Clearly  $A \cup B = [x, f(x)]$ . Since  $\xi$  is continuous, both  $A$  and  $B$  are closed. Also  $A \neq \emptyset$  as  $f(x) \in A$ . However, the fact that  $f(t) \rightarrow f(x)$  as  $t \rightarrow x$  implies  $B \neq \emptyset$  (because  $t \in A$  implies  $d(f(t), f(x)) \geq d(t, x)$ ). Therefore there exists a point  $\varphi(x) \in A \cap B$ . If  $\varphi(x) = x$ , then  $f(x) = x$  and we are done. Otherwise  $x \neq \varphi(x)$  and

$$[x, f(x)] \cap [x, f(\varphi(x))] = [x, \varphi(x)]. \tag{3.9}$$

Now let  $x_0 \in X$  and let  $x_n = \varphi^n(x_0)$ . Assuming that the process does not terminate upon reaching a fixed point of  $f$ , by construction, the points  $\{x_0, x_1, x_2, \dots\}$  are linear and thus lie on a subset of  $X$  which is isometric with a subset of the real line, that is, on a geodesic. Since  $X$  does not contain a geodesic of infinite length, it must be the case that

$$\sum_{i=0}^{\infty} d(x_i, x_{i+1}) < \infty, \tag{3.10}$$

and hence that  $\{x_n\}$  is a Cauchy sequence. Suppose  $\lim_{n \rightarrow \infty} x_n = z$ . Then

$$\lim_{n \rightarrow \infty} f(x_n) = f(z) \tag{3.11}$$

by continuity, and in particular  $\{f(x_n)\}$  is a Cauchy sequence. However, by construction,

$$d(f(x_n), f(x_{n+1})) = d(f(x_n), x_{n+1}) + d(x_{n+1}, f(x_{n+1})). \tag{3.12}$$

Since  $\lim_{n \rightarrow \infty} d(f(x_n), f(x_{n+1})) = 0$ , it follows that  $\lim_{n \rightarrow \infty} d(f(x_n), x_{n+1}) = d(f(z), z) = 0$  and  $f(z) = z$ . □

**THEOREM 3.5.** *Let  $(X, d)$  be a complete  $\mathbb{R}$ -tree, suppose  $K$  is a closed convex subset of  $X$  which does not contain a geodesic ray, suppose  $\text{int}(K) \neq \emptyset$ , and suppose  $f : K \rightarrow X$  is continuous. Suppose there exists  $p_0 \in \text{int}(K)$  such that  $x \notin \text{seg}[p_0, f(x)]$  for every  $x \in \partial K$ . Then  $f$  has a fixed point in  $K$ .*

*Proof.* Since  $K$  is a closed convex subset of a CAT(0) space, the nearest point projection  $P$  of  $X$  onto  $K$  is nonexpansive (see, e.g., [1, page 176]). Therefore the mapping  $P \circ f : K \rightarrow K$  is continuous and has a fixed point by Theorem 3.4. If  $x \in K$  then it must be the case that  $f(x) = x$  because  $P \circ f(x) \in \partial K$ . On the other hand, if  $x \in \partial K$ , then  $P \circ f(x) = x$  implies that  $x$  is on the segment joining  $p_0$  and  $f(x)$ , which is a contradiction. □

Another consequence of Theorem 3.4 is an analog of Ky Fan’s best approximation theorem for geodesically bounded  $\mathbb{R}$ -trees. This theorem actually includes Theorem 3.4; however Theorem 3.4 is used in the proof.

**THEOREM 3.6.** *Let  $(X, d)$  be a complete  $\mathbb{R}$ -tree, suppose  $K$  is a closed convex subset of  $X$  which does not contain a geodesic ray, and suppose  $f : K \rightarrow X$  is continuous. Then there exists  $x_0 \in K$  such that*

$$d(x_0, f(x_0)) \leq d(x, f(x)) \tag{3.13}$$

for every  $x \in K$ .

*Proof.* If  $P$  is the nearest point projection of  $X$  onto  $K$ , then any point  $x_0$  for which  $P \circ f(x_0) = x_0$  satisfies the conclusion.  $\square$

*Remark 3.7.* The analog of Theorem 3.4 does not hold for CAT(0) spaces, even for nonexpansive mappings. Ray [11] has shown that a closed convex subset of a Hilbert space has the fixed point property for nonexpansive mappings if and only if it is linearly bounded.

#### 4. Applications in graph theory

A *graph* is an ordered pair  $(V, E)$  where  $V$  is a set and  $E$  is a binary relation on  $V$  ( $E \subseteq V \times V$ ). Elements of  $E$  are called *edges*. We are concerned here with (undirected) graphs that have a “loop” at every vertex (i.e.,  $(a, a) \in E$  for each  $a \in V$ ) and no “multiple” edges. Such graphs are called *reflexive*. In this case  $E \subseteq V \times V$  corresponds to a reflexive (and symmetric) binary relation on  $V$ .

Given a graph  $G = (V, E)$ , a path of  $G$  is a sequence  $a_0, a_1, \dots, a_{n-1}, \dots$  with  $(a_{i+1}, a_i) \in E$  for each  $i = 0, 1, 2, \dots$ . A *cycle* is a finite path  $(a_0, a_1, \dots, a_{n-1})$  with  $(a_0, a_{n-1}) \in E$ . A graph is *connected* if there is a finite path joining any two of its vertices. A finite path  $(a_0, a_1, \dots, a_{n-1})$  is said to have *length*  $n$ . Finally, a *tree* is a connected graph with no cycles.

Let  $G = (V, E)$  be a graph and  $G_1 = (V_1, E_1)$  a subgraph of  $G$ . A mapping  $f : V_1 \rightarrow V$  is said to be *edge preserving* if  $(a, b) \in E_1$  implies  $(f(a), f(b)) \in E$ . For such a mapping we simply write  $f : G_1 \rightarrow G$ . There is a standard way of *metrizing* connected graphs; let each edge have length one and take distance  $d(a, b)$  between two vertices  $a$  and  $b$  to be the length of the shortest path joining them. With this metric, the edge-preserving mappings become *nonexpansive* mappings. (In a reflexive graph an edge-preserving map may collapse edges between distinct points since loops are allowed.)

The classical fixed edge theorem in graph theory due to Nowakowski and Rival [9] asserts that an edge preserving mapping defined on a connected graph which has no cycles or infinite paths always leaves some edge of the graph fixed. Although the focus in this area has shifted to other fixed structures in graphs other than trees, it seems worthwhile to illustrate how the preceding result can be applied in graph theory. In [5] the fixed edge theorem is extended as follows.

**THEOREM 4.1** [5]. *Let  $G$  be a reflexive graph which is connected, contains no cycles, and contains no infinite paths. Suppose  $\mathfrak{F}$  is a commuting family of edge-preserving mappings of  $G$  into itself. Then, either*

- (a) *there is a unique edge in  $G$  that is left fixed by each member of  $\mathfrak{F}$ , or*
- (b) *some vertex of  $G$  is left fixed by each member of  $\mathfrak{F}$ .*

We now use Theorem 3.5 to give another variant of the fixed edge theorem. Let  $G$  be a graph and let  $G_1$  be a subgraph of  $G$ . A vertex  $p_0 \in G_1$  is said to be *interior* if given any vertex  $x \in G$ , the path joining  $p_0$  and  $x$  contains a point  $p_1 \in G_1$  for which  $p_1 \neq p_0$ . A point  $p \in G_1$  is said to be a *boundary point* of  $G_1$  if  $p$  is not an interior vertex.

**PROPOSITION 4.2.** *Let  $G$  be a connected reflexive graph which contains no cycles, and let  $G_1$  be a connected subgraph of  $G$  which contains an interior vertex  $p_0$ , and which contains no infinite path. Let  $f : G_1 \rightarrow G$  be an edge-preserving mapping, and suppose  $p$  does not lie*

on the path joining  $p_0$  and  $f(p)$  for any boundary point  $p \in G_1$ . Then  $f$  either leaves some vertex of  $G_1$  fixed or leaves a unique edge of  $G_1$  fixed.

*Proof.* Since a connected graph with no cycles is a tree, one can construct from the graph  $G$  an  $\mathbb{R}$ -tree  $T$  by identifying each (nontrivial) edge with a unit interval of the real line and assigning the shortest path distance to any two points of  $T$ . It is easy to see that with this metric  $T$  is complete and that  $G_1$  induces a subtree  $T_1$  in  $T$ . It is now possible to extend  $f$  affinely on each edge to the corresponding unit interval of  $T_1$ , and the resulting mapping  $\tilde{f}$  is a nonexpansive mapping of  $T_1 \rightarrow T$ . Also  $p \in \partial T_1$  if and only if  $p$  is a boundary point of  $G_1$ , and by assumption  $p \notin [p_0, \tilde{f}(p)]$  for such a point  $p$ . Therefore  $\tilde{f}$  has a fixed point  $z$  by Theorem 3.5. Either  $z$  is a vertex of  $G$ , or  $z$  lies properly on the unit interval joining the vertices of some edge  $(a, b)$  of  $G$ . In this case (since the fixed point set of  $\tilde{f}$  is convex) the only way  $f$  can fail to leave some vertex of  $G$  fixed is for  $z$  to be the midpoint of the metric interval  $[a, b]$ , with  $f(a) = b$  and  $f(b) = a$ . In this case  $(a, b)$  is the unique fixed edge of  $f$ .  $\square$

Theorem 3.4 similarly leads to an extension of the fixed edge theorem.

**PROPOSITION 4.3.** *Let  $G = (V, E)$  be a reflexive graph which is connected, contains no cycles, and contains no infinite paths. Suppose  $f : V \rightarrow 2^V$  has the property that  $f(a)$  is a path in  $G$  for each  $a \in V$ , and also  $f(a) \cup f(b)$  is a path for each  $(a, b) \in E$ . Then some edge  $(a, b) \in E$  lies in the path  $f(a) \cup f(b)$ .*

*Proof.* Construct an  $\mathbb{R}$ -tree  $T$  as in the preceding proof and define  $\tilde{f}$  by choosing  $\tilde{f}(a)$  and  $\tilde{f}(b)$  to be endpoints of  $f(a) \cup f(b)$  with  $\tilde{f}(a) \in f(a)$  and  $\tilde{f}(b) \in f(b)$ , and extend  $\tilde{f}$  affinely to obtain a mapping  $\tilde{f} : [a, b] \rightarrow [\tilde{f}(a), \tilde{f}(b)]$ . Then  $\tilde{f}$  is continuous and by Theorem 3.4  $\tilde{f}$  has a fixed point which lies on some edge  $(a, b)$  of  $G$ . Clearly this implies that  $(a, b) \subset f(a) \cup f(b)$ .  $\square$

The preceding result reduces to the classical fixed edge theorem if  $f : V \rightarrow V$ .

Finally, from Theorem 3.6 one can obtain the following.

**PROPOSITION 4.4.** *Let  $G$  be a connected reflexive graph which contains no cycles, let  $G_1$  be a connected subgraph of  $G$  which contains no infinite path, and let  $f : G_1 \rightarrow G$  be an edge-preserving mapping. Then either some edge of  $G_1$  is fixed, or there exists a vertex  $a$  in  $G_1$  such that  $a$  lies on the path joining  $b$  and  $f(a)$  for each  $b \in G_1$ .*

## References

- [1] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der Mathematischen Wissenschaften, vol. 319, Springer-Verlag, Berlin, 1999.
- [2] K. S. Brown, *Buildings*, Springer-Verlag, New York, 1989.
- [3] D. Burago, Y. Burago, and S. Ivanov, *A Course in Metric Geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Rhode Island, 2001.
- [4] M. G. Crandall and A. Pazy, *Semi-groups of nonlinear contractions and dissipative sets*, J. Functional Analysis **3** (1969), 376–418.
- [5] R. Espínola and W. A. Kirk, *Fixed point theorems in  $\mathbb{R}$ -trees with applications to graph theory*, preprint, 2004.

- [6] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 83, Marcel Dekker, New York, 1984.
- [7] A. Granas, *Continuation method for contractive maps*, Topol. Methods Nonlinear Anal. **3** (1994), no. 2, 375–379.
- [8] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston, Massachusetts, 1999.
- [9] R. Nowakowski and I. Rival, *Fixed-edge theorem for graphs with loops*, J. Graph Theory **3** (1979), no. 4, 339–350.
- [10] R. Precup, *Continuation results for mappings of contractive type*, Semin. Fixed Point Theory Cluj-Napoca **2** (2001), 23–40.
- [11] W. O. Ray, *The fixed point property and unbounded sets in Hilbert space*, Trans. Amer. Math. Soc. **258** (1980), no. 2, 531–537.
- [12] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal. **15** (1990), no. 6, 537–558.
- [13] J. Reinermann and R. Schöneberg, *Some results and problems in the fixed point theory for non-expansive and pseudocontractive mappings in Hilbert-space*, Fixed Point Theory and Its Applications (Proc. Sem., Dalhousie Univ., Halifax, Nova Scotia, 1975) (S. Swaminathan, ed.), Academic Press, New York, 1976, pp. 187–196.

W. A. Kirk: Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419, USA  
E-mail address: kirk@math.uiowa.edu