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Convergence of the modified Mann's iterative method for asymptotically κ-strictly pseudocontractive mappings

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Abstract

Let E be a real uniformly convex Banach space which has the Fréchet differentiable norm, and K a nonempty, closed, and convex subset of E. Let $T : K \to K$ be an asymptotically κ-strictly pseudocontractive mapping with a nonempty fixed point set. We prove that $(I - T)$ is demiclosed at 0 and obtain a weak convergence theorem of the modified Mann's algorithm for T under suitable control conditions. Moreover, we also elicit a necessary and sufficient condition that guarantees strong convergence of the modified Mann's iterative sequence to a fixed point of T in a real Banach spaces with the Fréchet differentiable norm.

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1 Introduction

Let E and E^* be a real Banach space and the dual space of E, respectively. Let K be a nonempty subset of E. Let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{f \in E^* : \langle x, f\rangle = ||x||^2 = ||f||^2\}$, for all $x \in E$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*. In the sequel, we will denote the set of fixed points of a mapping $T : K \to K$ by $F(T) = \{x \in K : Tx = x\}$.

A mapping $T : K \to K$ is called asymptotically κ-strictly pseudocontractive with sequence $\{\kappa_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ such that $\lim_{n\to\infty}\kappa_n = 1$ (see, e.g., [1-3]) if for all $x, y \in K$, there exist a constant $\kappa \in [0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \le \kappa_n \| x - y\|^2 - \kappa \| x - y - (T^n x - T^n y)\|^2, \forall n \ge 1. \quad (1)$$

If I denotes the identity operator, then (1) can be written as

$$\langle (I-T^n)x-(I-T^n)y, j(x-y)\rangle \ge \kappa \| (I-T^n)x-(I-T^n)y\|^2 - (\kappa_n-1) \| x-y\|^2, \forall n \ge 1. \quad (2)$$

The class of asymptotically κ-strictly pseudocontractive mappings was first introduced in Hilbert spaces by Qihou [3]. In Hilbert spaces, j is the identity and it is shown by Osilike et al. [2] that (1) (and hence (2)) is equivalent to the inequality

$$\| T^n x - T^n y\|^2 \le \lambda_n \| x - y\|^2 + \lambda \| x - y - (T^n x - T^n y)\|^2,$$

where $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty}[1 + 2(\kappa_n - 1)] = 1, \lambda = (1 - 2\kappa) \in [0, 1)$.

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strictly pseudocontractive* of Browder-Petryshyn type [4], if for all $x, y \in D(T)$, there exists $\kappa \in [0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \| x - y\|^2 - \kappa \| x - y - (Tx - Ty)\|^2. \tag{3}$$

If I denotes the identity operator, then (3) can be written as

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \kappa \| (I - T)x - (I - T)y\|^2. \tag{4}$$

In Hilbert spaces, (3) (and hence (4)) is equivalent to the inequality

$$\| Tx - Ty\|^2 \leq \| x - y\|^2 + k \| x - y - (Tx - Ty)\|^2, k = (1 - 2\kappa) < 1,$$

It is shown in [5] that the class of asymptotically κ-strictly pseudocontractive mappings and the class of κ-strictly pseudocontractive mappings are independent.

A mapping T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\| T^n x - T^n y \| \leq L \| x - y \|, \ n \geq 1$$

for all $x, y \in K$ and is said to be demiclosed at a point p if whenever $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx = p$.

Kim and Xu [6] studied weak and strong convergence theorems for the class of asymptotically κ-strictly pseudocontractive mappings in Hilbert space. They obtained a weak convergence theorem of modified Mann iterative processes for this class of mappings. Moreover, a strong convergence theorem was also established in a real Hilbert space by hybrid projection method. They proved the following.

Theorem KX [6] Let K be a closed and convex subset of a Hilbert space H. Let $T : K \to K$ be an asymptotically κ-strictly pseudocontractive mapping for some $0 \leq \kappa < 1$ with sequence $\{\kappa_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (\kappa_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by the modified Mann's iteration method:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^n x_n, n \geq 1,$$

Assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen in such a way that $\kappa + \lambda \leq \alpha_n \leq 1 - \lambda$ for all n, where $\lambda \in (0, 1)$ is a small enough constant. Then, $\{x_n\}$ converges weakly to a fixed point of T.

The modified Mann's iteration scheme was introduced by Schu [7,8] and has been used by several authors (see, for example, [1-3,9-11]). One question is raised naturally: is the result in Theorem KX true in the framework of the much general Banach space?

Osilike et al. [5] proved the convergence theorems of modified Mann iteration method in the framework of q-uniformly smooth Banach spaces which are also uniformly convex. They also obtained that a modified Mann iterative process $\{x_n\}$ converges weakly to a fixed point of T under suitable control conditions. However, the control sequence $\{\alpha_n\} \subset [0,1]$ depended on the Lipschizian constant L and excluded the natural choice $\alpha_n = \frac{1}{n}, n \geq 1$. These are motivations for us to improve the results.

We prove the demiclosedness principle and weak convergence theorem of the modified Mann's algorithm for T in the framework of uniformly convex Banach spaces which have the Fréchet differentiable norm. Moreover, we also elicit a necessary and sufficient condition that guarantees strong convergence of the modified Mann's iterative sequence to a fixed point of T in a real Banach spaces with the Fréchet differentiable norm.

We will use the notation:

1. $\rightharpoonup$ for weak convergence.
2. $\omega_{\mathcal{W}}(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2 Preliminaries

Let E be a real Banach space. The space E is called *uniformly convex* if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for $x, y \in E$ with $||x|| \leq 1$, $||y|| \leq 1$, $||x - y|| \geq \varepsilon$, we have $\| \frac{1}{2}(x+y) \| \leq 1 - \delta$. The *modulus of convexity* of E is defined by

$$\delta_E(\varepsilon) = \inf\{1 - \| \frac{1}{2}(x+y) \| : \| x \| \leq 1,\ \| y \| \leq 1,\ \| x - y \| \geq \varepsilon, \} \ \forall x, y \in E$$

for all $\varepsilon \in [0,2]$. E is uniformly convex if $\delta_E(0) = 0$ and $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \in [0, \infty)$ defined by

$$\rho_E(\tau) = \sup\{\frac{1}{2}(\| x+y \| + \| x-y \|) - 1 : \| x \| \leq 1, \| y \| \leq \tau\},\ \forall x, y \in E.$$

E is *uniformly smooth* if and only if $\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$.

E is said to have a *Fréchet differentiable norm* if for all $x \in U = \{x \in E : ||x|| = 1\}$

$$\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}$$

exists and is attained uniformly in $y \in U$. In this case, there exists an increasing function $b : [0, \infty) \to [0, \infty)$ with $\lim_{t \to 0}[b(t)/t] = 0$ such that for all $x, h \in E$

$$\frac{1}{2}||x||^2 + \langle h, j(x) \rangle \leq \frac{1}{2}||x+h||^2 \leq \frac{1}{2}||x||^2 + \langle h, j(x) \rangle + b(||h||). \tag{5}$$

It is well known (see, for example, [[12], p. 107]) that uniformly smooth Banach space has a Fréchet differentiable norm.

Lemma 2.1 [2, p. 80] Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\delta_n\}_{n=1}^{\infty}$ be nonnegative sequences of real numbers satisfying the following inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n,\ \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 [2, p. 78] Let E be a real Banach space, K a nonempty subset of E, and $T : K \to K$ an asymptotically κ-strictly pseudocontractive mapping. Then, T is uniformly L-Lipschitzian.

Lemma 2.3 [[13], p. 29] Let K be a nonempty, closed, convex, and bounded subset of a uniformly convex Banach space E, and let $T : K \to E$ be a nonexpansive mappings. Let $\{x_n\}$ be a sequence in K such that $\{x_n\}$ converges weakly to some point $x \in K$.

Then, there exists an increasing continuous function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ depending on the diameter of K such that

$$h(\| x - Tx \|) \le \liminf_{n\to\infty} \| x_n - Tx_n \| .$$

Lemma 2.4 [[14], p. 9] Let E be a real Banach space with the Fréchet differentiable norm.

For $x \in E$, let $\beta^*(t)$ be defined for $0 < t < \infty$ by

$$\beta^*(t) = \sup_{y\in U} \left| \frac{\| x + ty\|^2 - \| x\|^2}{t} - 2\langle y, j(x)\rangle \right| .$$

Then, $\lim_{t\to 0^+} \beta^*(t) = 0$ and

$$\| x + h\|^2 \le \| x\|^2 + 2\langle h, j(x)\rangle + \| h \| \beta^*(\| h \|), \forall h \in E\backslash\{0\}. \tag{6}$$

Remark 2.5 In a real Hilbert space, we can see that $\beta^*(t) = t$ for $t > 0$. In our more general setting, throughout this article we will still assume that

$$\beta^*(t) \le 2t,$$

where β^* is a function appearing in (6).

Then, we prove the demiclosedness principle of T in the uniformly convex Banach space which has the Fréchet differentiable norm.

Lemma 2.6 Let E be a real uniformly convex Banach space which has the Fréchet differentiable norm. Let K be a nonempty, closed, and convex subset of E and $T : K \to K$ an asymptotically κ-strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Then, $(I - T)$ is demiclosed at 0.

Proof. Let $\{x_n\}$ be a sequence in K which converges weakly to $p \in K$ and $\{x_n - Tx_n\}$ converges strongly to 0. We prove that $(I - T)(p) = 0$. Let $x^* \in F(T)$. Then, there exists a constant $r > 0$ such that $||x_n - x^*|| \le r$, $\forall n \ge 1$. Let $\bar{B}_r = \{x \in E : ||x - x^*|| \le r\}$, and let $C = K \cap \bar{B}_r$. Then, C is nonempty, closed, convex, and bounded, and $\{x_n\} \subseteq C$. Choose any $\alpha \in (0, \kappa)$ and let $T_{\alpha,n} : K \to K$ be defined for all $x \in K$ by

$$T_{\alpha,n}x = (1 - \alpha)x + \alpha T^n x, n \ge 1,$$

Then for all $x, y \in K$,

$$\begin{aligned} ||T_{\alpha,n}x - T_{\alpha,n}y||^2 = & \ ||(x - y) - \alpha[(I - T^n)x - (I - T^n)y]||^2 \\ \le & \ ||x - y||^2 - 2\alpha\langle (I - T^n)x - (I - T^n)y, j(x - y)\rangle \\ & + \alpha||x - y - (T^n x - T^n y)||\beta^*[\alpha||x - y - (T^n x - T^n y)||] \\ & ||x - y||^2 - 2\alpha[\kappa||x - y - (T^n x - T^n y)||^2 - (\kappa_n - 1)||x - y||^2] \\ & + 2\alpha^2||x - y - (T^n x - T^n y)||^2 \\ = & \ [1 + 2\alpha(\kappa_n - 1)]||x - y||^2 - 2\alpha(\kappa - \alpha)||x - y - (T^n x - T^n y)||^2 \\ \le & \ \tau_n^2||x - y||^2, \end{aligned} \tag{7}$$

where $\tau_n = [1 + 2\alpha(\kappa_n - 1)]^{\frac{1}{2}}$. (In fact, in (7) the domain of $\beta^*(\cdot)$ requires $||x - y - (T^n x - T^n y)|| \neq 0$. But when $||x - y - (T^n x - T^n y)|| = 0$, we have $||T_{\alpha,n}x - T_{\alpha,n}y||^2 = ||x - y||^2$, which still satisfies the inequality $||T_{\alpha,n}x - T_{\alpha,n}y||^2 \le \tau_n^2||x - y||^2$. So we do not specially emphasize the situation that the argument of $\beta^*(\cdot)$ equals 0 in this inequality and the following proof of Theorem 3.1.) Define $G_{\alpha,m} : K \to E$ by

$$G_{\alpha,m}x = \frac{1}{\tau_m}T_{\alpha,m}x, \quad m \geq 1.$$

Then, $G_{\alpha,m}$ is nonexpansive and it follows from Lemma 2.3 that there exists an increasing continuous function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ depending on the diameter of K such that

$$h(||p - G_{\alpha,m}p||) \leq \liminf_{n\to\infty} ||x_n - G_{\alpha,m}x_n||. \tag{8}$$

Observe that

$$\begin{aligned} ||x_n - G_{\alpha,m}x_n|| &= ||x_n - \frac{1}{\tau_m}T_{\alpha,m}x_n|| \\ &\leq ||x_n - T_{\alpha,m}x_n|| + (1 - \frac{1}{\tau_m})(\tau_m||x_n - x^*|| + ||x^*||) \\ &\leq ||x_n - T_{\alpha,m}x_n|| + (1 - \frac{1}{\tau_m})(\tau_m r + ||x^*||), \end{aligned} \tag{9}$$

and as $n \to \infty$

$$||x_n - T_{\alpha,m}x_n|| = \alpha||x_n - T^m x_n|| \leq \sum_{j=1}^{m} ||T^{j-1}x_n - T^j x_n|| \leq [1 + L(m-1)]||x_n - Tx_n|| \to 0. \tag{10}$$

Thus, it follows from (9) and (10) that

$$\limsup_{n\to\infty} ||x_n - G_{\alpha,m}x_n|| \leq (1 - \frac{1}{\tau_m})(\tau_m r + ||x^*||),$$

so that (8) implies that

$$h(||p - G_{\alpha,m}p||) \leq (1 - \frac{1}{\tau_m})(\tau_m r + ||x^*||).$$

Observe that

$$\begin{aligned} ||p - G_{\alpha,m}p|| &\geq ||p - T_{\alpha,m}p|| - (1 - \frac{1}{\tau_m})||T_{\alpha,m}p|| \\ &\geq ||p - T_{\alpha,m}p|| - (1 - \frac{1}{\tau_m})(\tau_m r + ||x^*||), \end{aligned}$$

so that

$$\begin{aligned} ||p - T_{\alpha,m}p|| &\leq ||p - G_{\alpha,m}p|| + (1 - \frac{1}{\tau_m})(\tau_m r + ||x^*||) \\ &\leq h^{-1}[(1 - \frac{1}{\tau_m})(\tau_m r + ||x^*||)] + (1 - \frac{1}{\tau_m})(\tau_m r + ||x^*||) \to 0, \textit{as } m \to \infty. \end{aligned}$$

Since T is continuous, we have $(I - T)(p) = 0$, completing the proof of Lemma 2.6. □

Lemma 2.7 Let E be a real uniformly convex Banach space which has the Fréchet differentiable norm, and let K be a nonempty, closed, and convex subset of E. Let $T : K \to K$ be an asymptotically κ-strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence satisfying the following conditions:

(*a*) $\lim_{n\to\infty} ||x_n - p||$ *exists for every* $p \in F(T)$;

(*b*) $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$;

(*c*) $\lim_{n\to\infty} ||tx_n + (1-t)p_1 - p_2||$ *exists for all* $t \in [0, 1]$ *and for all* $p_1, p_2 \in F(T)$.

Then, the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Proof. Since $\lim_{n\to\infty} ||x_n - p||$ exists, then $\{x_n\}$ is bounded. By (b) and Lemma 2.6, we have $\omega_{\mathcal{W}}(x_n) \subset F(T)$. Assume that $p_1, p_2 \in \omega_{\mathcal{W}}(x_n)$ and that $\{x_{n_i}\}$ and $\{x_{m_j}\}$ are subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p_1$ and $x_{m_j} \rightharpoonup p_2$, respectively. Since E has the Fréchet differentiable norm, by setting $x = p_1 - p_2$, $h = t(x_n - p_1)$ in (5) we obtain

$$\begin{aligned}\frac{1}{2}||p_1 - p_2||^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle &\le \frac{1}{2}||tx_n + (1-t)p_1 - p_2||^2 \\ &\le \frac{1}{2}||p_1 - p_2||^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle + b(t||x_n - p_1||),\end{aligned}$$

where b is an increasing function. Since $||x_n - p_1|| \le M$, $\forall n \ge 1$, for some $M > 0$, then

$$\begin{aligned}\frac{1}{2}||p_1 - p_2||^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle &\le \frac{1}{2}||tx_n + (1-t)p_1 - p_2||^2 \\ &\le \frac{1}{2}||p_1 - p_2||^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle + b(tM).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2}||p_1 - p_2||^2 + t\limsup_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle &\le \frac{1}{2}\lim_{n\to\infty}||tx_n + (1-t)p_1 - p_2||^2 \\ &\le \frac{1}{2}||p_1 - p_2||^2 + t\liminf_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle + b(tM).\end{aligned}$$

Hence, $\limsup_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle \le \liminf_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle + \frac{b(tM)}{t}$. Since $\lim_{t\to 0^+}\frac{b(tM)}{t} = 0$, then $\lim_{n\to\infty} \langle x_n - p_1, j(p_1 - p_2)\rangle$ exists. Since $\lim_{n\to\infty} \langle x_n - p_1, j(p_1 - p_2)\rangle = \langle p - p_1, j(p_1 - p_2)\rangle$, for all $p \in \omega_{\mathcal{W}}(x_n)$. Set $p = p_2$. We have $\langle p_2 - p_1, j(p_1 - p_2)\rangle = 0$, that is, $p_2 = p_1$. Hence, $\omega_{\mathcal{W}}(x_n)$ is singleton, so that $\{x_n\}$ converges weakly to a fixed point of T. □

3 Main results

Theorem 3.1 Let E be a real uniformly convex Banach space which has the Fréchet differentiable norm, and let K be a nonempty, closed, and convex subset of E. Let $T : K \to K$ be an asymptotically κ-strictly pseudocontractive mapping for some $0 \le \kappa < 1$ with sequence $\{\kappa_n\}_{n=1}^{\infty} \subset [1, \infty)$, such that $\sum_{n=1}^{\infty}(\kappa_n - 1) < \infty$, and let $F(T) \neq \emptyset$. Assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen so that

(i*) $0 < \alpha_n < \kappa$, $n \ge 1$;

(ii*) $\sum_{n=1}^{\infty}\alpha_n(\kappa - \alpha_n) = \infty$. (11)

Given $x_1 \in K$, then the sequence $\{x_n\}_{n=1}^{\infty}$ is generated by the modified Mann's algorithm:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \tag{12}$$

converges weakly to a fixed point of T.

Proof. Pick a $p \in F(T)$. We firstly show that $\lim_{n\to\infty} ||x_n - p||$ exists. To see this, using (2) and (6), we obtain

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||(x_n - p) - \alpha_n(x_n - T^n x_n)||^2 \\ &\le ||x_n - p||^2 - 2\alpha_n \langle x_n - T^n x_n, j(x_n - p)\rangle + \alpha_n ||x_n - T^n x_n||\beta^*(\alpha_n||x_n - T^n x_n||) \\ &\le ||x_n - p||^2 - 2\alpha_n[\kappa||x_n - T^n x_n||^2 - (\kappa_n - 1)||x_n - p||^2] + 2\alpha_n^2||x_n - T^n x_n||^2 \\ &= [1 + 2\alpha_n(\kappa_n - 1)]||x_n - p||^2 - 2\alpha_n(\kappa - \alpha_n)||x_n - T^n x_n||^2. \end{aligned} \tag{13}$$

Obviously,

$$||x_{n+1} - p||^2 \le [1 + 2\alpha_n(\kappa_n - 1)]||x_n - p||^2. \tag{14}$$

Let $\delta_n = 1 + 2\alpha_n(\kappa_n - 1)$. Since $\sum_{n=1}^{\infty}(\kappa_n - 1) < \infty$, we have

$$\sum_{n=1}^{\infty}(\delta_n - 1) \le 2\sum_{n=1}^{\infty}(\kappa_n - 1) < \infty,$$

then (14) implies $\lim_{n\to\infty} ||x_n - p||$ exists by Lemma 2.1 (and hence the sequence $\{||x_n - p||\}$ is bounded, that is, there exists a constant $M > 0$ such that $||x_n - p|| < M$).

Then, we prove $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. In fact, it follows from (13) that

$$\begin{aligned} \sum_{n=1}^{j} 2\alpha_n(\kappa - \alpha_n)||x_n - T^n x_n||^2 &\le \sum_{n=1}^{j}(||x_n - p||^2 - ||x_{n+1} - p||^2) + \sum_{n=1}^{j}[2\alpha_n(\kappa_n - 1)]||x_n - p||^2 \\ &\le \sum_{n=1}^{j}(||x_n - p||^2 - ||x_{n+1} - p||^2) + \sum_{n=1}^{j}(\delta_n - 1)M^2. \end{aligned}$$

Then,

$$\sum_{n=1}^{\infty} 2\alpha_n(\kappa - \alpha_n)||x_n - T^n x_n||^2 < ||x_1 - p||^2 + M^2\sum_{n=1}^{\infty}(\delta_n - 1) < \infty. \tag{15}$$

Since $\sum_{n=1}^{\infty}\alpha_n(\kappa - \alpha_n) = \infty$, then (15) implies that $\liminf_{n\to\infty} ||x_n - T^n x_n|| = 0$. Thus $\lim_{n\to\infty} ||x_n - T^n x_n|| = 0$.

By Lemma 2.2 we know that T is uniformly L-Lipschitzian, then there exists a constant $L > 0$, such that

$$\begin{aligned} ||x_n - Tx_n|| &\le ||x_n - T^n x_n|| + ||T^n x_n - Tx_n|| \le ||x_n - T^n x_n|| + L||T^{n-1}x_n - x_n|| \\ &\le ||x_n - T^n x_n|| + L||T^{n-1}x_n - T^{n-1}x_{n-1}|| + L||T^{n-1}x_{n-1} - x_n|| \\ &\le ||x_n - T^n x_n|| + L^2||x_n - x_{n-1}|| + L||T^{n-1}x_{n-1} - x_{n-1}|| + L||x_n - x_{n-1}|| \\ &\le ||x_n - T^n x_n|| + L(2 + L)||T^{n-1}x_{n-1} - x_{n-1}|| \end{aligned}$$

Hence, $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$.

Now we prove that for all $p_1, p_2 \in F(T)$, $\lim_{n\to\infty} ||tx_n + (1 - t)p_1 - p_2||$ exists for all $t \in [0, 1]$. Let $\sigma_n(t) = ||tx_n + (1 - t)p_1 - p_2||$. It is obvious that $\lim_{n\to\infty} \sigma_n(0) = ||p_1 - p_2||$ and $\lim_{n\to\infty} \sigma_n(1) = \lim_{n\to\infty} ||x_n - p_2||$ exist. So, we only need to consider the case of $t \in (0, 1)$.

Define $T_n : K \to K$ by

$$T_n x = (1 - \alpha_n)x + \alpha_n T^n x, \quad x \in K.$$

Then for all $x, y \in K$,

$$\begin{aligned}||T_n x - T_n y||^2 &= ||(x-y) - \alpha_n[(I-T^n)x - (I-T^n)y]||^2 \\ &\le ||x-y||^2 - 2\alpha_n\langle (I-T^n)x - (I-T^n)y, j(x-y)\rangle \\ &\quad + \alpha_n||x-y-(T^n x - T^n y)||\beta^*[\alpha_n||x-y-(T^n x - T^n y)||] \\ &\le ||x-y||^2 - 2\alpha_n[\kappa||x-y-(T^n x - T^n y)||^2 - (\kappa_n - 1)||x-y||^2] \\ &\quad + 2\alpha_n^2||x-y-(T^n x - T^n y)||^2 \\ &= [1 + 2\alpha_n(\kappa_n - 1)]||x-y||^2 - 2\alpha_n(\kappa - \alpha_n)||x-y-(T^n x - T^n y)||^2.\end{aligned}$$

By the choice of α_n, we have $||T_n x - T_n y||^2 \le [1 + 2\alpha_n(\kappa_n - 1)]||x - y||^2$. For the convenience of the following discussing, set $\lambda_n = [1 + 2\alpha_n(\kappa_n - 1)]^{\frac{1}{2}}$, then $||T_n x - T_n y|| \le \lambda_n||x - y||$.

Set $S_{n,m} = T_{n+m-1}T_{n+m-2} \cdots T_n$, $m \ge 1$. We have

$$||S_{n,m}x - S_{n,m}y|| \le \left(\prod_{j=n}^{n+m-1}\lambda_j\right)||x-y|| \textit{for all } x, y \in K,$$

and

$$S_{n,m}x_n = x_{n+m}, \quad S_{n,m}p = p \textit{ for all } p \in F(T).$$

Set $b_{n,m} = ||S_{n,m}(tx_n + (1 - t)p_1) - tS_{n,m}x_n - (1 - t)S_{n,m}p_1||$. If $||x_n - p_1|| = 0$ for some n_0, then $x_n = p_1$ for any $n \ge n_0$ so that $\lim_{n\to\infty} ||x_n - p_1|| = 0$, in fact $\{x_n\}$ converges strongly to $p_1 \in F(T)$. Thus, we may assume $||x_n - p_1|| > 0$ for any $n \ge 1$. Let δ denote the *modulus of convexity* of E. It is well known (see, for example, [[15], p. 108]) that

$$\begin{aligned}||tx + (1-t)y|| &\le 1 - 2\min\{t, (1-t)\}\delta(||x-y||) \\ &\le 1 - 2t(1-t)\delta(||x-y||)\end{aligned} \tag{16}$$

for all $t \in [0, 1]$ and for all $x, y \in E$ such that $||x|| \le 1$, $||y|| \le 1$. Set

$$\begin{aligned}w_{n,m} &= \frac{S_{n,m}p_1 - S_{n,m}(tx_n + (1-t)p_1)}{t\left(\prod_{j=n}^{n+m-1}\lambda_j\right)||x_n - p_1||} \\ z_{n,m} &= \frac{S_{n,m}(tx_n + (1-t)p_1) - S_{n,m}x_n}{(1-t)\left(\prod_{j=n}^{n+m-1}\lambda_j\right)||x_n - p_1||}\end{aligned}$$

Then, $||w_{n,m}|| \le 1$ and $||z_{n,m}|| \le 1$ so that it follows from (16) that

$$2t(1-t)\delta(||w_{n,m} - z_{n,m}||) \le 1 - ||tw_{n,m} + (1-t)z_{n,m}||. \tag{17}$$

Observe that

$$||w_{n,m} - z_{n,m}|| = \frac{b_{n,m}}{t(1-t)(\prod_{j=n}^{n+m-1}\lambda_j)||x_n - p_1||}$$

and

$$||tw_{n,m} + (1-t)z_{n,m}|| = \frac{||S_{n,m}x_n - S_{n,m}p_1||}{(\prod_{j=n}^{n+m-1}\lambda_j)||x_n - p_1||},$$

it follows from (17) that

$$\begin{aligned}&2t(1-t)\left(\prod_{j=n}^{n+m-1}\lambda_j\right)||x_n-p_1||\delta\left(\frac{b_{n,m}}{t(1-t)(\prod_{j=n}^{n+m-1}\lambda_j)||x_n-p_1||}\right)\\&\le\left(\prod_{j=n}^{n+m-1}\lambda_j\right)||x_n-p_1||-||S_{n,m}x_n-S_{n,m}p_1||=\left(\prod_{j=n}^{n+m-1}\lambda_j\right)||x_n-p_1||-||x_{n+m}-p_1||.\end{aligned}\tag{18}$$

Since E is uniformly convex, then $\frac{\delta(s)}{s}$ is nondecreasing, and since $(\prod_{j=n}^{n+m-1}\lambda_j)||x_n-p_1||\le(\prod_{j=n}^{n+m-1}\lambda_j)\lambda_{n-1}||x_{n-1}-p_1||\le\cdots\le(\prod_{j=n}^{n+m-1}\lambda_j)(\prod_{j=1}^{n-1}\lambda_j)||x_1-p_1||=(\prod_{j=1}^{n+m-1}\lambda_j)||x_1-p_1||$, hence it follows from (18) that

$$\frac{\left(\prod_{j=1}^{n+m-1}\lambda_j\right)||x_1-p_1||}{2}\delta\left(\frac{4}{\left(\prod_{j=1}^{n+m-1}\lambda_j\right)||x_1-p_1||}b_{n,m}\right)\le\left(\prod_{j=n}^{n+m-1}\lambda_j\right)||x_n-p_1||-||x_{n+m}-p_1||$$
$$\left(\text{since } t(1-t)\le\frac{1}{4}\text{for all } t\in[0,1]\right).$$

Since $\lim_{n\to\infty}\prod_{j=1}^{n+m-1}\lambda_j=1$ and since $\delta(0)=0$ and $\lim_{n\to\infty}||x_n-p_1||$ exists, then the continuity of δ yields $\lim_{n\to\infty}\ b_{n,m}=0$ uniformly for all $m\ge1$. Observe that

$$\begin{aligned}\sigma_{n+m}(t)\le\ &||tx_{n+m}+(1-t)p_1-p_2+(S_{n,m}(tx_n+(1-t)p_1)-tS_{n,m}x_n-(1-t)S_{n,m}p_1)||\\&+||S_{n,m}(tx_n+(1-t)p_1)-tS_{n,m}x_n-(1-t)S_{n,m}p_1||\\=\ &||S_{n,m}(tx_n+(1-t)p_1)-S_{n,m}p_2||+b_{n,m}\\\le\ &\left(\prod_{j=n}^{n+m-1}\lambda_j\right)||tx_n+(1-t)p_1-p_2||+b_{n,m}\\=\ &\left(\prod_{j=n}^{n+m-1}\lambda_j\right)\sigma_n(t)+b_{n,m}.\end{aligned}$$

Hence, $\limsup_{n\to\infty}\ \sigma_n(t)\le\liminf_{n\to\infty}\ \sigma_n(t)$, this ensures that $\lim_{n\to\infty}\ \sigma_n(t)$ exists for all $t\in(0,1)$.

Now, apply Lemma 2.7 to conclude that $\{x_n\}$ converges weakly to a fixed point of T. □

Theorem 3.2 Let E be a real Banach space with the Fréchet differentiable norm, and let K be a nonempty, closed, and convex subset of E. Let $T:K\to K$ be an asymptotically κ-strictly pseudocontractive mapping for some $0\le\kappa<1$ with sequence $\{\kappa_n\}\subset[1,\infty)$ such that $\sum_{n=1}^{\infty}(\kappa_n-1)<\infty$, let $F(T)\neq\emptyset$. Let $\{\alpha_n\}$ be a real sequence satisfying the condition (11). Given $x_1\in K$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by the modified Mann's algorithm (12). Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if

$$\liminf_{n\to\infty}d(x_n,F(T))=0,$$

where $d(x_n,F(T))=\inf_{p\in F(T)}||x_n-p||$.

Proof. In the real Banach space E with the Fréchet differentiable norm, we still have

$$||x_{n+1} - p||^2 \le \delta_n ||x_n - p||^2. \tag{19}$$

as we have already proved in Theorem 3.1. Thus, $[d(x_{n+1} - p)]^2 \le \delta_n[d(x_n - p)]^2$ and it follows from Lemma 2.1 that $\lim_{n\text{-}\infty} d(x_n, F(T))$ exists.

Now if $\{x_n\}$ converges strongly to a fixed point p of T, then $\lim_{n\to\infty} ||x_n - p|| = 0$. Since

$$0 \le d(x_n, F(T)) \le ||x_n - p||,$$

we have $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Conversely, suppose $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, then the existence of $\lim_{n\to\infty} d(x_n, F(T))$ implies that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus, for arbitrary $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_n, F(T)) < \frac{\varepsilon}{2}$ for any $n \ge n_0$.

From (19), we have

$$||x_{n+1} - p||^2 \le ||x_n - p||^2 + M^2(\delta_n - 1),\ n \ge 1,$$

and for some $M > 0$, $||x_n - p|| < M$. Now, an induction yields

$$\begin{aligned} ||x_n - p||^2 &\le ||x_{n-1} - p||^2 + M^2(\delta_{n-1} - 1) \\ &\le ||x_{n-2} - p||^2 + M^2(\delta_{n-2} - 1) + M^2(\delta_{n-1} - 1) \\ &\le \cdots \le ||x_l - p||^2 + M^2 \sum_{j=l}^{n-1} (\delta_j - 1), \quad n - 1 \ge l \ge 1, \end{aligned}$$

Since $\sum_{n=1}^{\infty} (\delta_n - 1) < \infty$, then there exists a positive integer n_1 such that $\sum_{j=n}^{\infty} (\delta_j - 1) < (\frac{\varepsilon}{2M})^2$, $\forall n \ge n_1$. Choose $N = \max\{n_0, n_1\}$, then for all n, $m \ge N + 1$ and for all $p \in F(T)$ we have

$$\begin{aligned} ||x_n - x_m|| &\le ||x_n - p|| + ||x_m - p|| \\ &\le [||x_N - p||^2 + M^2 \sum_{j=N}^{n-1} (\delta_j - 1)]^{\frac{1}{2}} + [||x_N - p||^2 + M^2 \sum_{j=N}^{m-1} (\delta_j - 1)]^{\frac{1}{2}} \\ &\le [||x_N - p||^2 + M^2 \sum_{j=N}^{\infty} (\delta_j - 1)]^{\frac{1}{2}} + [||x_N - p||^2 + M^2 \sum_{j=N}^{\infty} (\delta_j - 1)]^{\frac{1}{2}}. \end{aligned}$$

Taking infimum over all $p \in F(T)$, we obtain

$$\begin{aligned} ||x_n - x_m|| &\le \{[d(x_N, F(T))]^2 + M^2 \sum_{j=N}^{\infty} (\delta_j - 1)\}^{\frac{1}{2}} + \{[d(x_N, F(T))]^2 + M^2 \sum_{j=N}^{\infty} (\delta_j - 1)\}^{\frac{1}{2}} \\ &< 2[(\frac{\varepsilon}{2})^2 + M^2(\frac{\varepsilon}{2M})^2]^{\frac{1}{2}} < 2\varepsilon. \end{aligned}$$

Thus, $\{x_n\}_{n=0}^{\infty}$ is Cauchy. We can also prove $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ as we have done in Theorem 3.1. Suppose $\lim_{n\to\infty} x_n = u$. Then,

$$0 \le ||u - Tu|| \le ||u - x_n|| + ||x_n - Tx_n|| + L||x_n - u|| \to 0, as\ n \to \infty.$$

Thus, $u \in F(T)$. □

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Authors' contributions
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