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Some extragradient methods for common solutions of generalized equilibrium problems and fixed points of nonexpansive mappings

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Abstract

In this article, we introduce some new iterative schemes based on the extragradient method (and the hybrid method) for finding a common element of the set of solutions of a generalized equilibrium problem, and the set of fixed points of a family of infinitely nonexpansive mappings and the set of solutions of the variational inequality for a monotone, Lipschitz-continuous mapping in Hilbert spaces. We obtain some strong convergence theorems and weak convergence theorems. The results in this article generalize, improve, and unify some well-known convergence theorems in the literature.

Keywords: Generalized equilibrium problem, Extragradient method, Hybrid method, Nonex-pansive mapping, Strong convergence, Weak convergence

1. Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $||\cdot||$. Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R and let $B : C \to H$ be a nonlinear mapping, where R is the set of real numbers. Moudafi [1], Moudafi and Thera [2], Peng and Yao [3,4], Takahashi and Takahashi [5] considered the following generalized equilibrium problem:

$$\text{Find } x \in C \text{ Such that } F(x, y) + \langle Bx, y - x \rangle \geq 0, \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GEP(F, B)$. If $B = 0$, the generalized equilibrium problem (1.1) becomes the equilibrium problem for $F : C \times C \to R$, which is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(F)$.

The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others; see for instance [1-7].

Recall that a mapping $S : C \to C$ is nonexpansive if there holds that

$$||Sx - Sy|| \leq ||x - y|| \quad \text{for all } x, y \in C.$$

We denote the set of fixed points of S by $\mathrm{Fix}(S)$.

Let the mapping $A : C \to H$ be monotone and k-Lipschitz-continuous. The variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0$$

for all $y \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Several algorithms have been proposed for finding the solution of problem (1.1). Moudafi [1] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space, and proved a weak convergence theorem. Moudafi and Thera [2] introduced an auxiliary scheme for finding a solution of problem (1.1) in a Hilbert space and obtained a weak convergence theorem. Peng and Yao [3,4] introduced some iterative schemes for finding a common element of the set of solutions of problem (1.1), the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for a monotone, Lipschitz-continuous mapping and obtain both strong convergence theorems, and weak convergence theorems for the sequences generated by the corresponding processes in Hilbert spaces. Takahashi and Takahashi [5] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space, and proved a strong convergence theorem.

Some methods also have been proposed to solve the problem (1.2); see, for instance, [8-19] and the references therein. Takahashi and Takahashi [9] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a non-expansive mapping, and proved a strong convergence theorem in a Hilbert space. Su et al. [10] introduced and researched an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for an α-inverse-strongly monotone mapping in a Hilbert space. Tada and Takahashi [11] introduced two iterative schemes for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a nonexpansive mapping in a Hilbert space, and obtained both strong convergence and weak convergence theorems. Plubtieng and Punpaeng [12] introduced an iterative processes based on the extragradient method for finding the common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of variational inequality problem for an α-inverse-strongly monotone mapping. Chang et al. [13] introduced an iterative processes based on the extragradient method for finding the common element of the set of solutions of an equilibrium problem, the set of common fixed point for a family of infinitely nonexpansive mappings and the set of solutions of variational inequality problem for an α-inverse-strongly monotone mapping. Yao et al. [14] and Ceng and Yao [15] introduced some iterative viscosity approximation schemes for finding the common element of the set of solutions of problem (1.2) and the set of fixed points of a family of infinitely nonexpansive mappings in a Hilbert space. Colao et al. [16] introduced an iterative viscosity approximation scheme for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a family of finitely nonexpansive mappings in a Hilbert space. We observe that the algorithms in [13-16] involves the W-mapping generated by a family of infinitely (finitely) nonexpansive mappings which is an effective tool in nonlinear analysis (see [20,21]). However, the W-mapping generated by a family of infinitely (finitely) nonexpansive mappings is too completed to use for finding the common element of the set of solutions of problem (1.2) and the set of fixed points of a family of infinitely (finitely) nonexpansive mappings. It is natural to raise and to give an answer to the following question: Can one construct algorithms for finding a common element of the set of solutions of a generalized equilibrium problem (an equilibrium problem), the common set of fixed points of a family of infinitely nonexpansive mappings and the set of solutions of a variational inequality without the W-mapping generated by a family of infinitely (finitely) nonexpansive mappings? In this article, we will give a positive answer to this question.

Recently, OHaraa et al. [22] introduced and researched an iterative approach for finding a nearest point of infinitely many nonexpansive mappings in a Hilbert spaces without using the W-mapping generated by a family of infinitely (finitely) nonexpansive mappings. Inspired by the ideas in [1-6,8-16,22] and the references therein, we introduce some new iterative schemes based on the extragradient method (and the hybrid method) for finding a common element of the set of solutions of a generalized equilibrium problem, the set of fixed points of a family of infinitely nonexpansive mappings, and the set of solutions of the variational inequality for a monotone, Lipschitz–continuous mapping without using the W-mapping generated by a family of infinitely (finitely) nonexpansive mappings. We obtain both strong convergence theorems and weak convergence theorems for the sequences generated by the corresponding processes. The results in this article generalize, improve, and unify some well-known convergence theorems in the literature.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $||\cdot||$. Let C be a nonempty closed convex subset of H. Let symbols $\rightarrow$ and $\rightharpoonup$ denote strong and weak convergences, respectively. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

For any $x \in H$, there exists the unique nearest point in C, denoted by $P_C(x)$, such that $||x - P_C(x)|| \leq ||x - y||$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. It is also known that $P_C x \in C$ and

$$\langle x - P_C(x), P_C(x) - y\rangle \geq 0 \tag{2.1}$$

for all $x \in H$ and $y \in C$.

It is easy to see that (2.1) is equivalent to

$$\|x-y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \tag{2.2}$$

for all $x \in H$ and $y \in C$.

A mapping A of C into H is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0$$

for all $x, y \in C$. A mapping A of C into H is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \left\|Ax - Ay\right\|^2$$

for all $x, y \in C$. A mapping $A : C \to H$ is called k-Lipschitz-continuous if there exists a positive real number k such that

$$\left\|Ax - Ay\right\| \leq k \left\|x - y\right\|$$

for all $x, y \in C$. It is easy to see that if A is α-inverse-strongly monotone, then A is monotone and Lipschitz-continuous. The converse is not true in general. The class of α-inverse-strongly monotone mappings does not contain some important classes of mappings even in a finite-dimensional case. For example, if the matrix in the corresponding linear complementarity problem is positively semidefinite, but not positively definite, then the mapping A will be monotone and Lipschitz-continuous, but not α-inverse-strongly monotone (see [23]).

Let A be a monotone mapping of C into H. In the context of the variational inequality problem, the characterization of projection (2.1) implies the following:

$$u \in VI(C, A) \Rightarrow u = P_C(u - \lambda Au),\ \lambda > 0.$$

and

$$u = P_C(u - \lambda Au) \text{ for some } \lambda > 0 \Rightarrow u \in VI(C, A).$$

It is also known that H satisfies the Opial's condition [24], i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \left\|x_n - y\right\|$$

holds for every $y \in H$ with $x \neq y$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone, k-Lipschitz-continuous mapping of C into H and $N_C v$ be normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ (see [25]).

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following condition:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

We recall some lemmas which will be needed in the rest of this article.

Lemma 2.1.[7] Let C be a nonempty closed convex subset of H, let F be a bifunction from $C \times C$ to R satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \text{for all } y \in C.$$

Lemma 2.2.[8] Let C be a nonempty closed convex subset of H, let F be a bi-function from $C \times C$ to R satisfying (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $Tr : H \to C$ as follows:

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $x \in H$. Then, the following statements hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\left\|T_r(x) - T_r(y)\right\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

3. The main results

We first show a strong convergence of an iterative algorithm based on extragradient and hybrid methods which solves the problem of finding a common element of the set of solutions of a generalized equilibrium problem, the set of fixed points of a family of infinitely nonexpansive mappings, and the set of solutions of the variational inequality for a monotone, Lipschitz-continuous mapping in a Hilbert space.

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A4). Let A be a monotone and k-Lipschitz-continuous mapping of C into H and B be an α-inverse-strongly monotone mapping of C into H. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Omega = \cap_{i=1}^{\infty}\mathrm{Fix}(S_i) \cap VI(C, A) \cap \mathrm{GEP}(F, B) \neq \emptyset$. Assume that for all $i \in \{1, 2, \ldots\}$ and for any bounded subset K of C, thenthere holds

$$\lim_{n\to\infty}\sup_{x\in K} ||S_n x - S_i(S_n x)|| = 0. \qquad (\star)$$

Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = (1-\gamma_n)u_n + \gamma_n P_C(u_n - \lambda_n A u_n), \\ z_n = (1 - \alpha_n - \beta_n)x_n + \alpha_n y_n + \beta_n S_n P_C(u_n - \lambda_n A y_n), \\ C_n = \{z \in C : ||z_n - z||^2 \leq ||x_n - z||^2 + (3 - 3\gamma_n + \alpha_n)b^2||Au_n||^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases} \tag{3.1}$$

for every $n = 1, 2,...$ where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{4k})$, $\{r_n\} \subset [d, e]$ for some $d, e \in (0, 2\alpha)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \leq 1$ for all $n \in N$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\liminf_{n\to\infty} \beta_n > 0$;
(iv) $\lim_{n\to\infty} \gamma_n = 1$ and $\gamma_n > \frac{3}{4}$ for all $n \in N$;

Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $w = P_\Omega(x)$.

Proof. It is obvious that C_n is closed, and Q_n is closed and convex for every $n = 1, 2,...$. Since

$$C_n = \{z \in H : \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z\rangle \leq (3 - 3\gamma_n + \alpha_n)b^2\|Au_n\|^2\},$$

we also have that C_n is convex for every $n = 1, 2,...$. It is easy to see that $\langle x_n - z, x - x_n\rangle \geq 0$ for all $z \in Q_n$ and by (2.1), $x_n = P_{Q_n}x$. Let $t_n = P_C(u_n - \lambda_n Ay_n)$ for every $n = 1, 2,...$. Let $u \in \Omega$ and let $\{T_{r_n}\}$ >be a sequence of mappings defined as in Lemma 2.2. Then $u = P_C(u - \lambda_n Au) = T_{r_n}(u - r_n Bu)$. From $u_n = T_{r_n}(x_n - r_n Bx_n) \in C$ and the α-inverse strongly monotonicity of B, we have

$$\begin{aligned}
\|u_n - u\|^2 &= \left\|T_{r_n}(x_n - r_n Bx_n) - T_{r_n}(u - r_n Bu)\right\|^2 \\
&\leq \left\|x_n - r_n Bx_n - (u - r_n Bu)\right\|^2 \\
&\leq \|x_n - u\|^2 - 2r_n\langle x_n - u, Bx_n - Bu\rangle + r_n^2\|Bx_n - Bu\|^2 \\
&\leq \|x_n - u\|^2 - 2r_n\alpha\|Bx_n - Bu\|^2 + r_n^2\|Bx_n - Bu\|^2 \\
&= \|x_n - u\|^2 + r_n(r_n - 2\alpha)\|Bx_n - Bu\|^2 \\
&\leq \|x_n - u\|^2.
\end{aligned} \tag{3.2}$$

From (2.2), the monotonicity of A, and $u \in VI(C, A)$, we have

$$\begin{aligned}
\|t_n - u\|^2 &\leq \left\|u_n - \lambda_n Ay_n - u\right\|^2 - \left\|u_n - \lambda_n Ay_n - t_n\right\|^2 \\
&= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\langle Ay_n, u - t_n\rangle \\
&= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\big(\langle Ay_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle Ay_n, y_n - t_n\rangle\big) \\
&\leq \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&\leq \|u_n - u\|^2 - \left\|u_n - y_n\right\|^2 - 2\langle u_n - y_n, y_n - t_n\rangle - \left\|y_n - t_n\right\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&= \|u_n - u\|^2 - \left\|u_n - y_n\right\|^2 - \left\|y_n - t_n\right\|^2 + 2\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle.
\end{aligned}$$

Further, Since $y_n = (1 - \gamma_n)u_n + \gamma_n P_C(u_n - \lambda_n Au_n)$ and A is k-Lipschitz-continuous, we have

$$\begin{aligned}
&\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle \\
&= \langle u_n - \lambda_n Au_n - y_n, t_n - y_n\rangle + \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n\rangle \\
&\leq \langle u_n - \lambda_n Au_n - (1 - \gamma_n)u_n - \gamma_n P_C(u_n - \lambda_n Au_n), t_n - y_n\rangle + \lambda_n \left\|Au_n - Ay_n\right\| \left\|t_n - y_n\right\| \\
&\leq \gamma_n\langle u_n - \lambda_n Au_n - P_C(u_n - \lambda_n Au_n), t_n - y_n\rangle - (1 - \gamma_n)\lambda_n\langle Au_n, t_n - y_n\rangle + \lambda_n k \left\|u_n - y_n\right\| \left\|t_n - y_n\right\|.
\end{aligned}$$

In addition, from the definition of P_C, we have

$$\begin{aligned}
&\langle u_n - \lambda_n Au_n - P_C(u_n - \lambda_n Au_n), t_n - y_n\rangle \\
&= \langle u_n - \lambda_n Au_n - P_C(u_n - \lambda_n Au_n), t_n - (1-\gamma_n)u_n - \gamma_n P_C(u_n - \lambda_n Au_n)\rangle \\
&= (1-\gamma_n)\langle u_n - \lambda_n Au_n - P_C(u_n - \lambda_n Au_n), t_n - u_n\rangle \\
&+\gamma_n\langle u_n - \lambda_n Au_n - P_C(u_n - \lambda_n Au_n), t_n - P_C(u_n - \lambda_n Au_n)\rangle \\
&\le (1-\gamma_n)\left\|u_n - \lambda_n Au_n - P_C(u_n - \lambda_n Au_n)\right\| \|t_n - u_n\| \\
&\le (1-\gamma_n)\lambda_n \|u_n - Au_n - u_n\| \left(\|t_n - y_n\| + \|y_n - u_n\|\right) \\
&\le (1-\gamma_n)\lambda_n \|Au_n\| \left(\|t_n - y_n\| + \|y_n - u_n\|\right).
\end{aligned}$$

It follows from $b < \frac{1}{4k}$, $\gamma_n > \dfrac{3}{4}$ and (3.2) that

$$\begin{aligned}
\|t_n - u\|^2 &\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\gamma_n(1-\gamma_n)b\|Au_n\|\left(\|t_n - y_n\| + \|y_n - u_n\|\right) \\
&\quad +2(1-\gamma_n)b\|Au_n\|\|t_n - y_n\| + 2bk\|u_n - y_n\|\|t_n - y_n\| \\
&\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + (1-\gamma_n)(2b^2\|Au_n\|^2 + \|t_n - y_n\|^2 + \|y_n - u_n\|^2) \\
&\quad +(1-\gamma_n)(b^2\|Au_n\|^2 + \|t_n - y_n\|^2) + bk(\|u_n - y_n\|^2 + \|t_n - y_n\|^2) \\
&= \|u_n - u\|^2 - (\gamma_n - bk)\|u_n - y_n\|^2 + (1 - 2\gamma_n + bk)\|t_n - y_n\|^2 + 3(1-\gamma_n)b^2\|Au_n\|^2 \\
&\le \|u_n - u\|^2 + 3(1-\gamma_n)b^2\|Au_n\|^2 \\
&\le \|x_n - u\|^2 + 3(1-\gamma_n)b^2\|Au_n\|^2.
\end{aligned} \tag{3.3}$$

In addition, from $u \in VI(C, A)$ and (3.2), we have

$$\begin{aligned}
\|y_n - u\|^2 &= \left\|(1-\gamma_n)(u_n - u) + \gamma_n(P_C(u_n - \lambda_n Au_n) - u)\right\|^2 \\
&\le (1-\gamma_n)\|u_n - u\|^2 + \gamma_n\left\|P_C(u_n - \lambda_n Au_n) - P_C(u)\right\|^2 \\
&\le (1-\gamma_n)\|u_n - u\|^2 + \gamma_n\|u_n - \lambda_n Au_n - u\|^2 \\
&\le (1-\gamma_n)\|u_n - u\|^2 + \gamma_n[\|u_n - u\|^2 - 2\lambda_n\langle Au_n, u_n - u\rangle + \lambda_n^2\|Au_n\|^2] \\
&\le \|u_n - u\|^2 + b^2\|Au_n\|^2 \\
&\le \|x_n - u\|^2 + b^2\|Au_n\|^2.
\end{aligned} \tag{3.4}$$

Therefore, from (3.2) to (3.4) and $z_n = (1 - \alpha_n - \beta_n)x_n + \alpha_n y_n + \beta_n S_n t_n$ and $u = S_n u$, we have

$$\begin{aligned}
\|z_n - u\|^2 &= \left\|(1-\alpha_n-\beta_n)x_n + \alpha_n y_n + \beta_n S_n t_n - u\right\|^2 \\
&\le (1-\alpha_n-\beta_n)\|x_n - u\|^2 + \alpha_n\|y_n - u\|^2 + \beta_n\|S_n t_n - u\|^2 \\
&\le (1-\alpha_n-\beta_n)\|x_n - u\|^2 + \alpha_n\|y_n - u\|^2 + \beta_n\|t_n - u\|^2 \\
&\le (1-\alpha_n-\beta_n)\|x_n - u\|^2 + \alpha_n[\|u_n - u\|^2 + b^2\|Au_n\|^2] \\
&\quad +\beta_n[\|u_n - u\|^2 + 3(1-\gamma_n)b^2\|Au_n\|^2] \\
&\le \|x_n - u\|^2 + (3 - 3\gamma_n + \alpha_n)b^2\|Au_n\|^2],
\end{aligned} \tag{3.5}$$

for every $n = 1, 2,...$ and hence $u \in C_n$. So, $\Omega \subset C_n$ for every $n = 1, 2,...$. Next, let us show by mathematical induction that x_n is well defined and $\Omega \subset C_n \cap Q_n$ for every $n = 1, 2,...$. For $n = 1$ we have $x_1 = x \in C$ and $Q_1 = C$. Hence, we obtain $\Omega \subset C_1 \cap Q_1$. Suppose that x_k is given and $\Omega \subset C_k \cap Q_k$ for some $k \in N$. Since Ω is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of H. Hence, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}x$. It is also obvious that there holds $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for every $z \in C_k \cap Q_k$. Since $\Omega \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for every $z \in \Omega$ and hence $\Omega \subset Q_{k+1}$. Therefore, we obtain $\Omega \subset C_{k+1} \cap Q_{k+1}$.

Let $l_0 = P_\Omega x$. From $x_{n+1} = P_{C_n \cap Q_n} x$ and l_0 v $\Omega \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \leq \|l_0 - x\| \tag{3.6}$$

for every $n = 1, 2,...$. Therefore, $\{x_n\}$ is bounded. From (3.2) to (3.5) and the lipschitz continuity of A, we also obtain that $\{u_n\}$, $\{y_n\}$, $\{Au_n\}$, $\{t_n\}$ and $\{z_n\}$ are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset C_n$ and $x_n = P_{Q_n} x$, we have

$$\|x_n - x\| \leq \|x_{n+1} - x\|$$

for every $n = 1, 2,...$. It follows from (3.6) that $\lim_{n\to\infty} ||x_n - x||$ exists.

Since $x_n = P_{Q_n} x$ and $x_{n+1} \in Q_n$, using (2.2), we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2$$

for every $n = 1, 2,...$. This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have $||z_n - x_{n+1}||^2 \leq ||x_n - x_{n+1}||^2 + (3 - 3\gamma_n + \alpha_n)b^2||Au_n||^2$ and hence it follows from $\lim_{n\to\infty} \gamma_n = 1$ and $\lim_{n\to\infty} \alpha_n = 0$ that $\lim_{n\to\infty} ||z_n - x_{n+1}|| = 0$. Since

$$||x_n - z_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - z_n||$$

for every $n = 1, 2,...$, we have $||x_n - z_n|| \to 0$.

For $u \in \Omega$, from (3.5), we obtain

$$\begin{aligned} &||z_n - u||^2 - ||x_n - u||^2 \\ &\leq (-\alpha_n - \beta_n)||x_n - u||^2 + \alpha_n||y_n - u||^2 + \beta_n||S_n t_n - u||^2 \\ &\leq (3 - 3\gamma_n + \alpha_n)b^2||Au_n||^2. \end{aligned}$$

Since $\lim_{n\to\infty} \gamma_n = 1$ and $\lim_{n\to\infty} \alpha_n = 0$, $\{x_n\}$, $\{y_n\}$, $\{Au_n\}$, and $\{z_n\}$ are bounded, we have

$$\lim_{n\to\infty} \beta_n(||S_n t_n - u||^2 - ||x_n - u||^2) = 0.$$

By $\liminf_{n\to\infty} \beta_n > 0$, we get

$$\lim_{n\to\infty} ||S_n t_n - u||^2 - ||x_n - u||^2 = 0.$$

From (3.3) and $u = S_n u$, we have

$$\begin{aligned} \lim_{n\to\infty} ||S_n t_n - u||^2 - ||x_n - u||^2 &\leq \lim_{n\to\infty} ||t_n - u||^2 - ||x_n - u||^2 \\ &\leq \lim_{n\to\infty} 3(1 - \gamma_n)b^2||Au_n||^2 = 0. \end{aligned}$$

Thus, $\lim_{n\to\infty} ||t_n - u||^2 - ||x_n - u||^2 = 0$.

From (3.3) and (3.2), we have

$$\begin{aligned} &(\gamma_n - bk)||u_n - y_n||^2 + (2\gamma_n - 1 - bk)||t_n - y_n||^2 \\ &\leq ||x_n - u||^2 - ||t_n - u||^2 + 3(1 - \gamma_n)b^2||Au_n||^2. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} (\gamma_n - bk)||u_n - y_n||^2 + (2\gamma_n - 1 - bk)||t_n - y_n||^2 = 0.$$

The assumptions on γ_n and λ_n imply that $\gamma_n - bk > \frac{1}{2}$ and $2\gamma_n - 1 - bk > \frac{1}{4}$. Consequently, $\lim_{n\to\infty} ||u_n - y_n|| = \lim_{n\to\infty} ||t_n - y_n|| = 0$. Since A is Lipschitz-continuous, we have $\lim_{n\to\infty} ||At_n - Ay_n|| = 0$. It follows from $||u_n - t_n|| \le ||u_n - y_n|| + ||t_n - y_n||$ that $\lim_{n\to\infty} ||u_n - t_n|| = 0$.

We rewrite the definition of z_n as

$$z_n - x_n = \alpha_n(y_n - x_n) + \beta_n(S_n t_n - x_n).$$

From $\lim_{n\to\infty} ||z_n - x_n|| = 0$, $\lim_{n\to\infty} \alpha_n = 0$, the boundedness of $\{x_n\}$, $\{y_n\}$ and $\liminf_{n\to\infty} \beta_n > 0$ we infer that $\lim_{n\to\infty} ||S_n t_n - x_n|| = 0$.

By (3.2)-(3.5), we have

$$\begin{aligned}
||z_n - u||^2 &\le (1-\alpha_n-\beta_n)||x_n-u||^2 + \alpha_n[||u_n-u||^2 + b^2||Au_n||^2] + \beta_n[||u_n-u||^2 + 3(1-\gamma_n)b^2||Au_n||^2] \\
&\le (1-\alpha_n-\beta_n)||x_n-u||^2 + \alpha_n[||x_n-u||^2 + r_n(r_n-2\alpha)||Bx_n-Bu||^2 + b^2||Au_n||^2] \\
&\quad + \beta_n[||x_n-u||^2 + r_n(r_n-2\alpha)||Bx_n-Bu||^2 + 3(1-\gamma_n)b^2||Au_n||^2] \\
&\le ||x_n-u||^2 + (\alpha_n+\beta_n)r_n(r_n-2\alpha)||Bx_n-Bu||^2 + (3\beta_n - 3\beta_n\gamma_n + \alpha_n)b^2||Au_n||^2].
\end{aligned} \tag{3.7}$$

Hence, we have

$$\begin{aligned}
&(\alpha_n+\beta_n)d(2\alpha-e)||Bx_n-Bu||^2 \\
&\le (\alpha_n+\beta_n)r_n(2\alpha-r_n)||Bx_n-Bu||^2 \\
&\le ||x_n-u||^2 - ||z_n-u||^2 + (3\beta_n - 3\beta_n\gamma_n + \alpha_n)b^2||Au_n||^2 \\
&\le (||x_n-u|| + ||z_n-u||)||x_n-z_n|| + (3\beta_n - 3\beta_n\gamma_n + \alpha_n)b^2||Au_n||^2.
\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 1$, $\liminf_{n\to\infty} \beta_n > 0$, $\lim_{n\to\infty} \gamma_n = 1$, $||x_n - z_n|| \to 0$ and the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $||Bx_n - B_u|| \to 0$.

For $u \in \Omega$, we have, from Lemma 2.2,

$$\begin{aligned}
||u_n-u||^2 &= ||T_{r_n}(x_n - r_nBx_n) - T_{r_n}(u - r_nBu)||^2 \\
&\le \langle T_{r_n}(x_n - r_nBx_n) - T_{r_n}(u - r_nBu), x_n - r_nBx_n - (u - r_nBu)\rangle \\
&= \frac{1}{2}\{||u_n-u||^2 + ||x_n - r_nBx_n - (u - r_nBu)||^2 - ||x_n - r_nBx_n - (u - r_nBu) - (u_n - u)||^2\} \\
&\le \frac{1}{2}\{||u_n-u||^2 + ||x_n-u||^2 - ||x_n - r_nBx_n - (u - r_nBu) - (u_n - u)||^2\} \\
&= \frac{1}{2}\{||u_n-u||^2 + ||x_n-u||^2 - ||x_n-u_n||^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle - r_n^2||Bx_n-Bu||^2\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
||u_n-u||^2 &\le ||x_n-u||^2 - ||x_n-u_n||^2 + 2r_n\langle Bx_n-Bu, x_n-u_n\rangle - r_n^2||Bx_n-Bu||^2 \\
&\le ||x_n-u||^2 - ||x_n-u_n||^2 + 2r_n\langle Bx_n-Bu, x_n-u_n\rangle.
\end{aligned}$$

Then, by (3.5), we have

$$\begin{aligned}
||z_n-u||^2 &\le (1-\alpha_n-\beta_n)||x_n-u||^2 + \alpha_n[||u_n-u||^2 + b^2||Au_n||^2] + \beta_n[||u_n-u||^2 + 3(1-\gamma_n)b^2||Au_n||^2] \\
&\le (1-\alpha_n-\beta_n)||x_n-u||^2 + \alpha_n[(||x_n-u||^2 - ||x_n-u_n||^2 + 2r_n\langle Bx_n-Bu, x_n-u_n\rangle) + b^2||Au_n||^2] \\
&\quad + \beta_n[(||x_n-u||^2 - ||x_n-u_n||^2 + 2r_n\langle Bx_n-Bu, x_n-u_n\rangle) + 3(1-\gamma_n)b^2||Au_n||^2] \\
&\le ||x_n-u||^2 + (-\alpha_n-\beta_n)||x_n-u_n||^2 + 2r_n(\alpha_n+\beta_n)||Bx_n-Bu||\,||x_n-u_n|| + (3\beta_n - 3\beta_n\gamma_n + \alpha_n)b^2||Au_n||^2
\end{aligned}$$

Hence,

$$\begin{aligned}
(\alpha_n+\beta_n)||x_n-u_n||^2 &\le ||x_n-u||^2 - ||z_n-u||^2 + 2r_n(\alpha_n+\beta_n)||Bx_n-Bu||\,||x_n-u_n|| + (3\beta_n - 3\beta_n\gamma_n + \alpha_n)b^2||Au_n||^2 \\
&\le (||x_n-u|| + ||z_n-u||)||x_n-z_n|| + 2r_n(\alpha_n+\beta_n)||Bx_n-Bu||\,||x_n-u_n|| + (3\beta_n - 3\beta_n\gamma_n + \alpha_n)b^2||Au_n||^2.
\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \beta_n > 0$, $\lim_{n\to\infty} \gamma_n = 1$, $||x_n - z_n|| \to 0$, $||Bx_n - Bu|| \to 0$ and the sequences $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ are bounded, we obtain $||x_n - u_n|| \to 0$. From $||z_n - t_n|| \le ||z_n - x_n|| + ||x_n - u_n|| + ||u_n - t_n||$, we have $||z_n - t_n|| \to 0$.

From $||t_n - x_n|| \le ||t_n - u_n|| + ||x_n - u_n||$, we also have $||t_n - x_n|| \to 0$.

Since $z_n = (1 - \alpha_n - \beta_n)x_n + \alpha_n y_n + \beta_n S_n t_n$, we have $\beta_n(S_n t_n - t_n) = (1 - \alpha_n - \beta_n)(t_n - x_n) + \alpha_n(t_n - y_n) + (z_n - t_n)$. Then

$$\beta_n||S_n t_n - t_n|| \le (1 - \alpha_n - \beta_n)||t_n - x_n|| + \alpha_n||t_n - y_n|| + ||z_n - t_n||$$

and hence $||S_n t_n - t_n|| \to 0$. At the same time, observe that for all $i \in \{1, 2, ...\}$,

$$\begin{aligned} ||S_i t_n - t_n|| &\le ||S_i t_n - S_i(S_n t_n)|| + ||S_i(S_n t_n) - S_n t_n|| \, || + ||S_n t_n - t_n||. \\ &\le 2||S_n t_n - t_n|| + \sup_{x\in K} ||S_i(S_n x) - S_n x||. \end{aligned}$$

It follows from (3.8) and the condition (*) that for all $i \in \{1, 2, ...\}$,

$$\lim_{n\to\infty} ||S_i t_n - t_n|| = 0. \tag{3.9}$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{ni} \rightharpoonup w$. From $||x_n - u_n|| \to 0$, we obtain that $u_{ni} \rightharpoonup w$. From $||u_n - t_n|| \to 0$, we also obtain that $t_{ni} \rightharpoonup w$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $w \in C$.

First, we show $w \in GEP(F, B)$. By $u_n = T_{r_n}(x_n - r_n B x_n) \in C$, we know that

$$F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r^n}\langle y - u_n, u_n - x_n\rangle \ge 0, \ \forall y \in C.$$

It follows from (A2) that

$$\langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n), \forall y \in C.$$

Hence,

$$\langle Bx_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \ge F(y, u_{n_i}), \forall y \in C. \tag{3.10}$$

For t with $0 < t \le 1$ and $y \in C$, let $y_t = t_y + (1 - t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$. So, from (3.10) we have

$$\begin{aligned} \langle y_t - u_{n_i}, By_t\rangle &\ge \langle y_t - u_{n_i}, By_t\rangle - \langle y_t - u_{n_i}, Bx_{n_i}\rangle \\ &\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i}) \\ &= \langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle + \langle y_t - u_{n_i}, Bu_{n_i} - Bx_{n_i}\rangle \\ &\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i}). \end{aligned}$$

Since $||u_{n_i} - x_{n_i}|| \to 0$, we have $||Bu_{n_i} - Bx_{n_i}|| \to 0$. Further, from the inverse-strongly monotonicity of B, we have $\langle y_t - u_{n_i}, By_t - Bu_{n_i}\rangle \ge 0$. Hence, from (A4), $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, we have

$$\langle y_t - w, By_t\rangle \ge F(y_t, w), \tag{3.11}$$

as $i \to \infty$. From (A1), (A4) and (3.11), we also have

$$\begin{aligned} 0 = F(y_t, y_t) &\le tF(y_t, y) + (1 - t)F(y_t, w) \\ &\le tF(y_t, y) + (1 - t)\langle y_t - w, By_t\rangle \\ &= tF(y_t, y) + (1 - t)t\langle y - w, By_t\rangle. \end{aligned}$$

and hence

$$0 \leq F(y_t, y) + (1 - t)\langle y - w, By_t \rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(w, y) + \langle y - w, Bw \rangle \geq 0.$$

This implies that $w \in GEP(F, B)$.

We next show that $w \in \cap_{i=1}^{\infty} \mathrm{Fix}(S_i)$. Assume $w \notin \cap_{i=1}^{\infty} \mathrm{Fix}(S_i)$. Since $t_{n_i} \rightharpoonup w$ and $w \neq S_{i_0} w$ for some $i_0 \in \{1, 2,...\}$ from the Opial condition, we have

$$\begin{aligned}
\liminf_{i\to\infty} \|t_{n_i} - w\| &< \liminf_{i\to\infty} \|t_{n_i} - S_{i_0} w\| \\
&\leq \liminf_{i\to\infty} \{\|t_{n_i} - S_{i_0} t_{n_i}\| + \|S_{i_0} t_{n_i} - S_{i_0} w\|\} \\
&\leq \liminf_{i\to\infty} \|t_{n_i} - w\|.
\end{aligned}$$

This is a contradiction. Hence, we get $w \in \cap_{i=1}^{\infty} \mathrm{Fix}(S_i)$.

Finally we show $w \in VI(C, A)$. Let

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \in C. \end{cases}$$

where $N_C v$ is the normal cone to C at $v \in C$. We have already mentioned that in this case the mapping T is maximal monotone, and $0 \in Tv$ if and only if $v \in VI(C, A)$. Let $(v, g) \in G(T)$. Then $Tv = Av + N_C v$ and hence $g - Av \in N_C v$.

Hence, we have $\langle v - t, g - Av \rangle \geq 0$ for all $t \in C$. On the other hand, from $t_n = P_C(u_n - \lambda_n A y_n)$ and $v \in C$, we have

$$\langle u_n - \lambda_n A y_n - t_n, t_n - v \rangle \geq 0$$

and hence

$$\langle v - t_n, \frac{t_n - u_n}{\lambda_n} + Ay_n \rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
\langle v - t_{n_i}, g \rangle &\geq \langle v - t_{n_i}, Av \rangle \\
&\geq \langle v - t_{n_i}, Av \rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \rangle \\
&= \langle v - t_{n_i}, Av - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\
&= \langle v - t_{n_i}, Av - At_{n_i} + At_{n_i} - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\
&= \langle v - t_{n_i}, Av - At_{n_i} \rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle \\
&\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle - \langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle
\end{aligned}$$

Hence, we obtain $\langle v - w, g \rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C, A)$. This implies that $w \in \Omega$.

From $l_0 = P_\Omega x$, $w \in \Omega$ and (3.6), we have

$$||l_0 - x|| \le ||w - x|| \le \liminf_{i\to\infty} ||x_{n_i} - x|| \le \limsup_{i\to\infty} ||x_{n_i} - x|| \le ||l_0 - x||.$$

Hence, we obtain

$$\lim_{i\to\infty} ||x_{n_i} - x|| = ||w - x||.$$

From $x_{n_i} - x \rightharpoonup w - x$, we have $x_{n_i} - x \to w - x$, and hence $x_{n_i} \to w$. Since $x_n = P_{Q_n} x$ and $l_0 \in \Omega \subset C_n \cap Q_n \subset Q_n$, we have

$$-||l_0 - x_{n_i}||^2 \le \langle l_0 - x_{n_i}, x_{n_i} - x\rangle + \langle l_0 - x_{n_i}, x - l_0\rangle \ge \langle l_0 - x_{n_i}, x - l_0\rangle.$$

As $i \to \infty$, we obtain $- ||l_0 - w||^2 \ge \langle l_0 - w, x - l_0\rangle \ge 0$ by $l_0 = P_\Omega x$ and $w \in \Omega$. Hence, we have $w = l_0$. This implies that $x_n \to l_0$. It is easy to see $u_n \to l_0$, $y_n \to l_0$ and $z_n \to l_0$. The proof is now complete.

By combining the arguments in the proof of Theorem 3.1 and those in the proof of Theorem 3.1 in [3], we can easily obtain the following weak convergence theorem for an iterative algorithm based on the extragradient method which solves the problem of finding a common element of the set of solutions of a generalized equilibrium problem, the set of fixed points of a family of infinitely nonexpansive mappings and the set of solutions of the variational inequality for a monotone, Lipschitz-continuous mapping in a Hilbert space.

Theorem 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A4). Let A be a monotone, and k-Lipschitz-continuous mapping of C into H and B be an α-inverse-strongly monotone mapping of C into H. Let S_1, S_2,... be a family of infinitely nonexpansive mappings of C into itself such that $\Omega = \cap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap VI(C, A) \cap \mathrm{GEP}(F, B) \neq \emptyset$. Assume that for all $i \in \{1, 2,...\}$ and for any bounded subset K of C, thenthere holds

$$\lim_{n\to\infty} \sup_{x\in K} ||S_n x - S_i(S_n x)|| = 0. \qquad (\star)$$

Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be the sequences generated by

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S_n P_C(u_n - \lambda_n A y_n) \end{cases} \tag{3.12}$$

for every $n = 1, 2,...$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\beta_n\} \subset [\delta, \varepsilon]$ for some δ, $\varepsilon \in (0, 1)$ and $\{r_n\} \subset [d, e]$ for some d, $e \in (0, 2\alpha)$. Then, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge weakly to $w \in \Omega$, where $w = \lim_{n\to\infty} P_\Omega x_n$.

4. Applications

By Theorems 3.1 and 3.2, we can obtain many new and interesting convergence theorems in a real Hilbert space. We give some examples as follows:

Let $A = 0$, by Theorems 3.1 and 3.2, respectively, we obtain the following results.

Theorem 4.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ to R satisfying (A1)-(A4). Let B be an α-inverse-strongly monotone mapping of C into H. Let S_1, S_2,... be a family of infinitely nonexpansive mappings of C into itself such that $\sum = \cap_{i=1}^{\infty}\mathrm{Fix}(S_i) \cap \mathrm{GEP}(F, B) \neq \emptyset$. Assume that for all $i \in \{1, 2,...\}$ and for any bounded subset K of C, thenthere holds

$$\lim_{n\to\infty} \sup_{x\in K} ||S_n x - S_i(S_n x)|| = 0. \qquad (\star)$$

Let $\{x_n\}$, $\{u_n\}$ $\{y_n\}$, and $\{z_n\}$ be the sequences generated by

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ z_n = (1 - \alpha_n - \beta_n)x_n + \alpha_n u_n + \beta_n S_n u_n, \\ C_n = \{z \in C : ||z_n - z||^2 \leq \ ||x_n - z||^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 1, 2,....$ where $\{r_n\} \subset [d, e]$ for some $d, e \in (0, 2\alpha)$, and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in [0, 1] satisfying the conditions:

(i) $\alpha_n + \beta_n \leq 1$ for all $n \in N$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\liminf_{n\to\infty} \beta_n > 0$ for all $n \in N$;

Then, $\{x_n\}$, $\{u_n\}$, and $\{z_n\}$ converge strongly to $w = P_{\Sigma}(x)$.

Theorem 4.2. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ to R satisfying (A1)-(A4). Let B be an α-inverse-strongly monotone mapping of C into H. Let S_1, S_2,... be a family of infinitely nonexpansive mappings of C into itself such that $\sum = \cap_{i=1}^{\infty}\mathrm{Fix}(S_i) \cap \mathrm{GEP}(F, B) \neq \emptyset$. Assume that for all $i \in \{1, 2,...\}$ and for any bounded subset K of C, thenthere holds

$$\lim_{n\to\infty} \sup_{x\in K} ||S_n x - S_i(S_n x)|| = 0. \qquad (\star)$$

Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S_n u_n \end{cases}$$

for every $n = 1, 2,....$ If $\{\beta_n\} \subset [\delta, \varepsilon]$ for some $\delta, \varepsilon \in (0, 1)$ and $\{r_n\} \subset [d, e]$ for some $d, e \in (0, 2\alpha)$. Then, $\{x_n\}$ and $\{u_n\}$ converge weakly to $w \in \Sigma$, where $w = \lim_{n\to\infty} P_{\Sigma} x_n$.

Theorem 4.3. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ to R satisfying (A1)-(A4). Let A be a monotone and k-Lipschitz-continuous mapping of C into H and B be an α-inverse-strongly monotone mapping of C into H. Let S_1, S_2,... be a family of infinitely nonexpansive mappings of C into itself such that $\Omega = \cap_{i=1}^{\infty}\mathrm{Fix}(S_i) \cap VI(C, A) \cap \mathrm{GEP}(F, B) \neq \emptyset$. Assume that for all $i \in \{1, 2,...\}$ and for any bounded subset K of C, thenthere holds

$$\lim_{n\to\infty} \sup_{x\in K} ||S_n x - S_i(S_n x)|| = 0. \qquad (\star)$$

Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \langle Bx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = (1-\beta_n)x_n + \beta_n S_n P_C(u_n - \lambda_n A y_n), \\ C_n = \{z \in C : ||z_n - z||^2 \leq \ ||x_n - z||^2, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 1, 2, \ldots$ where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{4k})$, $\{r_n\} \subset [d, e]$ for some $d, e \in (0, 2\alpha)$, and $\{\beta_n\}$ is a sequence in [0, 1] satisfying $\liminf_{n\to\infty} \beta_n > 0$. Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $w = P_\Omega(x)$.

Proof. Putting $\gamma_n = 1$ and $\alpha_n = 0$, by Theorem 3.1, we obtain the desired result.

Let $B = 0$, by Theorems 3.1, 3.2, and 4.3, we obtain the following results.

Theorem 4.4. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ to R satisfying (A1)-(A4). Let A be a monotone and k-Lipschitz-continuous mapping of C into H. Let S_1, S_2,... be a family of infinitely nonexpansive mappings of C into itself such that $\Lambda = \cap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap VI(C, A) \cap EP(F) \neq \emptyset$. Assume that for all $i \in \{1, 2, \ldots\}$, and for any bounded subset K of C, there holds

$$\lim_{n\to\infty} \sup_{x\in K} ||S_n x - S_i(S_n x)|| \ = 0. \qquad (\star)$$

Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = (1-\gamma_n)u_n + \gamma_n P_C(u_n - \lambda_n A u_n), \\ z_n = (1-\alpha_n-\beta_n)x_n + \alpha_n y_n + \beta_n S_n P_C(u_n - \lambda_n A y_n), \\ C_n = \{z \in C : ||z_n - z||^2 \leq \ ||x_n - z||^2 + \ (3 - 3\gamma_n + \alpha_n)b^2 ||Au_n||^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 1, 2, \ldots.$ where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{4k})$, $\{r_n\} \subset [d, +\infty)$ for some $d > 0$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in [0, 1] satisfying the following conditions:

(i) $\alpha_n + \beta_n \leq 1$ for all $n \in N$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\liminf_{n\to\infty} \beta_n > 0$;
(iv) $\lim_{n\to\infty} \gamma_n = 1$ and $\gamma_n > \frac{3}{4}$ for all $n \in N$;

Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $w = P_\Lambda(x)$.

Theorem 4.5. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ to R satisfying (A1)-(A4). Let A be a monotone and k-Lipschitz-continuous mapping of C into H. Let S_1, S_2,... be a family of infinitely nonexpansive mappings of C into itself such that $\Lambda = \cap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap VI(C, A) \cap EP(F) \neq \emptyset$. Assume that for all $i \in \{1, 2, \ldots\}$ and for any bounded subset K of C, thenthere holds

$$\lim_{n\to\infty}\sup_{x\in K}||S_n x - S_i(S_n x)|| = 0. \qquad (\star)$$

Let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be the sequences generated by

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S_n P_C(u_n - \lambda_n A y_n) \end{cases}$$

for every $n = 1, 2,...$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$,$\{\beta_n\} \subset [\delta, \varepsilon]$, for some δ, $\varepsilon \in (0, 1)$ and $\{r_n\} \subset [d, +\infty]$ for some $d > 0$, then $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge weakly to $w \in \Lambda$, where $w = \lim_{n\to\infty} P_\Lambda x_n$.

Theorem 4.6. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A4). Let A be a monotone and k-Lipschitz-continuous mapping of C into H. Let S_1, S_2,... be a family of infinitely nonexpansive mappings of C into itself such that $\Lambda = \cap_{i=1}^{\infty}\mathrm{Fix}(S_i) \cap VI(C, A) \cap EP(F) \neq \emptyset$. Assume that for all $i \in \{1, 2,...\}$ and for any bounded subset K of C, thenthere holds

$$\lim_{n\to\infty}\sup_{x\in K}||S_n x - S_i(S_n x)|| = 0. \qquad (\star)$$

Let $\{x_n\}$, $\{u_n\}$ $\{y_n\}$, and $\{z_n\}$ be the sequences generated by

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = (1-\beta_n)x_n + \beta_n S_n P_C(u_n - \lambda_n A y_n), \\ C_n = \{z \in C : ||z_n - z||^2 \leq ||x_n - z||^2, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 1, 2,...$. where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \dfrac{1}{4k})$, $\{r_n\} \subset [d, +\infty)$ and for some $d > 0$, and $\{\beta_n\}$ is a sequence in [0, 1] satisfying $\liminf_{n\to\infty}\beta_n > 0$. Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $w = P_\Lambda(x)$.

Let $B = 0$ and $F(x, y) = 0$ for $x, y \in C$, by Theorems 3.1 and 4.3, we obtain the following results.

Theorem 4.7. Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz-continuous mapping of C into H. Let S_1, S_2,... be a family of infinitely nonexpansive mappings of C into itself such that $\Gamma = \cap_{i=1}^{\infty}\mathrm{Fix}(S_i) \cap VI(C, A) \neq \emptyset$. Assume that for all $i \in \{1, 2,...\}$ and for any bounded subset K of C, thenthere holds

$$\lim_{n\to\infty}\sup_{x\in K}||S_n x - S_i(S_n x)|| = 0. \qquad (\star)$$

Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by

$$\begin{cases} x_1 = x \in C, \\ y_n = (1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A x_n), \\ z_n = (1-\alpha_n-\beta_n)x_n + \alpha_n y_n + \beta_n S_n P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : ||z_n - z||^2 \leq ||x_n - z||^2 + (3 - 3\gamma_n + \alpha_n)b^2||Ax_n||^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 1, 2, \ldots$ where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{4k})$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$are three sequences in [0, 1] satisfying the following conditions:

(i) $\alpha_n + \beta_n \leq 1$ for all $n \in N$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\liminf_{n\to\infty} \beta_n > 0$;
(iv) $\lim_{n\to\infty} \gamma_n = 1$ and $\gamma_n > \frac{3}{4}$ for all $n \in N$;

Then, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $w = P_\Gamma(x)$.

Theorem 4.8. Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz-continuous mapping of C into H. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Gamma = \cap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap VI(C, A) \neq \emptyset$. Assume that for all $i \in \{1, 2, \ldots\}$ and for any bounded subset K of C, thenthere holds

$$\lim_{n\to\infty} \sup_{x\in K} ||S_n x - S_i(S_n x)|| = 0. \qquad (\star)$$

Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be the sequences generated by

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = (1 - \beta_n)x_n + \beta_n S_n P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : ||z_n - z||^2 \leq ||x_n - z||^2, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 1, 2, \ldots$ where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{4k})$, and $\{\beta_n\}$ is a sequence in [0, 1] satisfying $\liminf_{n\to\infty} \beta_n > 0$. Then, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $w = P_\Gamma(x)$.

Let $F(x, y) = 0$ for $x, y \in C$, then by Theorem 3.2 and the proof of Theorem 4.7 in [3], we obtain the following result.

Theorem 4.9. Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz-continuous mapping of C into H and B be an α-inverse-strongly monotone mapping of C into H. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Xi = \cap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Assume that for all $i \in \{1, 2, \ldots\}$ and for any bounded subset K of C, thenthere holds

$$\lim_{n\to\infty} \sup_{x\in K} ||S_n x - S_i(S_n x)|| = 0. \qquad (\star)$$

Let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be the sequences generated by

$$\begin{cases} x_1 = x \in C, \\ u_n = P_C(x_n - r_n B x_n), \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n P_C(u_n - \lambda_n A y_n) \end{cases}$$

for every $n = 1, 2, \ldots$ if $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\beta_n\} \subset [\delta, \varepsilon]$ for some δ, $\varepsilon \in (0, 1)$ and $\{r_n\} \subset [d, e]$ for some d, $e \in (0, 2\alpha)$. Then, $\{x_n\}$ and $\{u_n\}$ converge weakly to $w \in \Xi$, where $w = \lim_{n\to\infty} P_{\Xi}x_n$.

Remark 4.1.

(i) For all $n \geq 1$, let $S_n = S$ be a nonexpansive mapping, by Theorems 3.2, 4.2, 4.7, 4.8, and 4.9 we recover Theorem 3.1 in [5], Theorem 3.1 in [1], Theorem 5 in [26], Theorem 3.1 in [23], and Theorem 4.7 in [3]. In addition, let $A = 0$, by Theorems 4.6 and 4.5, respectively, we recover Theorems 3.1 and 4.1 in [11].

(ii) For all $n \geq 1$, let $S_n = S$ be a nonexpansive mapping, by Theorems 3.1, 4.3, and 4.4, respectively, we recover Theorems 4.3, 4.4, and 4.7 in [4] with some modified conditions on F.

(iii) Theorems 3.1, 3.2, 4.3-4.7 also improve the main results in [10,12,13] because the inverse strongly monotonicity of A has been replaced by the monotonicity and Lipschitz continuity of A.

The following result illustrates that there are the nonexpansive mappings S_1, S_2 ,... satisfying the condition (*).

Lemma 4.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive mapping of C into itself such that $\mathrm{Fix}(T) \neq \emptyset$. If we define $S_n(x) = \dfrac{1}{n}\sum_{j=0}^{n-1} T^j x$ for $n \in \{1, 2, \ldots\}$, and $x \in C$, then the following results hold:

(a) For any bounded subset K of C, there holds

$$\lim_{n\to\infty} \sup_{x\in K} ||S_n x - T(S_n x)|| = 0.$$

(b) $\cap_{i=1}^{\infty} \mathrm{Fix}(S_i) = \mathrm{Fix}(T)$.

(c) for all $i \in \{1, 2, \ldots\}$ and for any bounded subset K of C, there holds

$$\lim_{n\to\infty} \sup_{x\in K} ||S_n x - S_i(S_n x)|| = 0.$$

Proof.

(a) It is due to Bruck [27,28] (please also see Lemma 3.1 in [22]).

(b) It follows from (a) that $\cap_{i=1}^{\infty} \mathrm{Fix}(S_i) \subseteq \mathrm{Fix}(T)$.

Moreover, it is obvious that $\cap_{i=1}^{\infty} \mathrm{Fix}(S_i) \supseteq \mathrm{Fix}(T)$. Hence, $\cap_{i=1}^{\infty} \mathrm{Fix}(S_i) = \mathrm{Fix}(T)$.

(c) It can be proved by mathematical induction. In fact, it is clear that this conclusion holds for $i = 1$. Assume that the conclusion holds for $i = m$, that is, for any bounded subset K of C, there holds

$$\lim_{n\to\infty} \sup_{x\in K} ||S_n x - S_m(S_n x)|| = 0. \tag{4.1}$$

We now prove that the conclusion also holds for $i = m + 1$. In fact, we observe that

$$\begin{aligned} \lim_{n\to\infty} \sup_{x\in K} ||S_n x - S_{m+1}(S_n x)|| &\leq \lim_{n\to\infty} \sup_{x\in K} ||S_n x - S_m(S_n x)|| + \lim_{n\to\infty} \sup_{x\in K} ||S_m(S_n x) - S_{m+1}(S_n x)|| \\ &\leq \lim_{n\to\infty} \sup_{x\in K} ||S_n x - S_m(S_n x)|| + \lim_{n\to\infty} \sup_{x\in K} \left[\frac{1}{m+1} ||T^m(S_n x)|| + \frac{1}{m(m+1)} \sum_{j=0}^{m-1} ||T^j(S_n x)|| \right]. \end{aligned} \tag{4.2}$$

It is easy to verify that S_1, S_2,... are nonexpansive mappings. It follows from (4.1) and (4.2) that for any bounded subset K of C, there holds

$$\lim_{n\to\infty} \sup_{x\in K} ||S_n x - S_{m+1}(S_n x)|| = 0.$$

From Lemma 4.1, we know that by Theorems 3.1 and 3.2, respectively, we can obtain the following results.

Theorem 4.10. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A4). Let A be a monotone and k-Lipschitz-continuous mapping of C into H and B be an α-inverse-strongly monotone mapping of C into H. Let T be a nonexpansive mapping of C into itself such that $\Theta = Fix(T) \cap VI(C, A) \cap GEP(F, B) \neq \emptyset$. Let $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{4k})$, $\{r_n\} \subset [d, e]$ and for some $d, e \in (0, 2\alpha)$, and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in [0, 1] satisfying the following conditions:

(i) $\alpha_n + \beta_n \leq 1$ for all $n \in N$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\liminf_{n\to\infty} \beta_n > 0$;
(iv) $\lim_{n\to\infty} \gamma_n = 1$ and $\gamma_n > \frac{3}{4}$ for all $n \in N$; If we define $S_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} T^j x$ for $n \in \{1, 2,...\}$, and $x \in C$, then the sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, and $\{z_n\}$ generated by algorithm (3.1) converge strongly to $w = P_\Theta(x)$.

Theorem 4.11. Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A4). Let A be a monotone and k-Lipschitz-continuous mapping of C into H and B be an α-inverse-strongly monotone mapping of C into H, and T be a nonexpansive mapping of C into itself such that $\Theta = Fix(T) \cap VI(C, A) \cap GEP(F, B) \neq \emptyset$. Assume that $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$ $\{\beta_n\} \subset [\delta, \varepsilon]$ for some $\delta, \varepsilon \in (0, 1)$, and $\{r_n\} \subset [d, e]$ some $d, e \in (0, 2\alpha)$. If we define $S_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} T^j x$ for $n \in \{1, 2,...\}$ and $x \in C$, then the sequences $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ generated by algorithm (3.12) converge weakly to $w \in \Theta$, where $w = \lim_{n\to\infty} P_\Theta x_n$.

5. Competing interests

The authors declare that they have no competing interests.

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