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On the eleventh question of Allen Shields

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Abstract

In this article, we will give sufficient conditions for the boundedness of the analytic projection on the set of multipliers of the formal Laurent series spaces. This answers a question that has been raised by A. L. Shields. Also, we will characterize the fixed points of some weighted composition operators acting on weighted Hardy spaces.

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1 Introduction

Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers satisfying $\beta(0) = 1$. If $1 < p < \infty$, then the space $L^p(\beta)$ consists of all formal Laurent series $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$ such that the norm $||f||^p = ||f||_\beta^p = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p$ is finite. When n just runs over $\mathbb{N} \cup \{0\}$, the space $L^p(\beta)$ only contains formal power series $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, and it is usually denoted by $H^p(\beta)$. These spaces are also called as weighted Hardy spaces. If $p = 2$, such spaces were introduced by Allen L. Shields to study weighted shift operators in his article [1] which is one of basic studies in this area, and is a pretty large study that contains a number of interesting results, and indeed it is mainly of auxiliary nature. Actually, Shields showed a close relation between injective weighted shifts and the multiplication operator M_z acting on $L^2(\beta)$ or $H^2(\beta)$ (see [[1], Proposition 7]). These are reflexive Banach spaces with the norm $||\cdot||_\beta$. Let $\hat{f}_k(n) = \delta_k(n)$. Hence, $f_k(z) = z^k$ and $\{f_k\}_{k \in \mathbb{Z}}$ is a basis for $L^p(\beta)$ such that $||f_k|| = \beta(k)$. Clearly M_z, the operator of multiplication by z on $L^p(\beta)$, shifts the basis $\{f_k\}_k$. The operator M_z is bounded, if and only if $\{\beta(k+1)/\beta(k)\}_k$ is bounded, and in this case $||M_z^n|| = \sup_k[\beta(k+n)/\beta(k)]$ for all $n \in \mathbb{N} \cup \{0\}$.

We say that a complex number λ is a bounded point evaluation on $L^p(\beta)$ if the functional $e(\lambda) : L^p(\beta) \to \mathbb{C}$ defined by $e(\lambda)(f) = f(\lambda)$ is bounded.

Let X be a Banach space. It is convenient and helpful to introduce the notation $< x, x^* >$to stand for $x^*(x)$, for $x \in X$ and $x^* \in X^*$. Also, the set of bounded linear operators on X is denoted by $B(X)$. If $A \in B(X)$, then by $\sigma(A)$ we mean the spectrum of A and by $r(A)$ we mean the spectral radius of A.

By the same method used in [2], we can see that $L^p(\beta)^* = L^q(\beta^{\frac{p}{q}})$ where $\frac{1}{p} + \frac{1}{q} = 1$. Also, if $f(z) = \sum_n \hat{f}(n)z^n \in L^p(\beta)$ and $g(z) = \sum_n \hat{g}(n)z^n \in L^q(\beta^{\frac{p}{q}})$, then clearly $< f, g >= \sum_n \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p$. For some sources on these topics, we refer to [1-14].

If Ω is a domain in the complex plane $\mathbb{C}$, then by $H(\Omega)$ and $H^\infty(\Omega)$ we mean the set of analytic functions and the set of bounded analytic functions on Ω, respectively. By $||\cdot||_\Omega$, we denote the supremum norm on Ω.

We denote the set of multipliers $\{\phi \in L^p(\beta) : \phi L^p(\beta) \subseteq L^p(\beta)\}$ by $L^p_\infty(\beta)$ and the linear transformation of multiplication by ϕ on $L^p(\beta)$ by M_ϕ. The space $L^p_\infty(\beta)$ is a commutative Banach algebra with the norm $||\phi||_\infty = ||M_\phi||$.

By an analytic projection we mean the map which send each two-sided sequence $\{a_n\}_{n=-\infty}^{\infty}$ into the corresponding one-sided sequence $\{a_n\}_{n=0}^{\infty}$[1].

It is known that the analytic projection for the unweighted shift is unbounded ([[15], Prob. 9, Chapter 14]). In this paper, we want to investigate conditions under which the analytic projection on $L^p_\infty(\beta)$ to be bounded. Also, we investigate the fixed points of some weighted composition operators acting on weighted Hardy spaces.

2 Main results

First we note that the multiplication operator M_z on $L^p(\beta)$ is unitarily equivalent to an injective bilateral weighted shift and conversely, every injective bilateral weighted shift is unitarily equivalent to M_z acting on $L^p(\beta)$ for a suitable choice of β. Throughout this article, M_z is a bounded operator on $L^p(\beta)$.

Let $\varphi(z) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)z^k = \varphi_1(z) + \varphi_2(z)$ be in $L^p_\infty(\beta)$ where

$$\varphi_1(z) = \sum_{k=0}^{\infty} \hat{\varphi}(k)z^k, \quad \varphi_2(z) = \sum_{k=1}^{\infty} \hat{\varphi}(-k)z^{-k}.$$

Define the analytic projection $J : L^p_\infty(\beta) \to L^p(\beta)$ by $J(\phi) = \phi_1$. Then, the projection J is clearly bounded since $||J(\phi)||_p = ||\phi_1||_p \le ||\phi||_p \le ||\phi||_\infty$. The problem of boundedness raised when we consider the projection J from $L^p_\infty(\beta)$ into $L^p_\infty(\beta)$. It is certainly bounded if M_z is not invertible on $L^p(\beta)$, since in this case we will see that for $\varphi(z) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)z^k$ in $L^p_\infty(\beta)$, it should be $\hat{\varphi}(n) = 0$ for all $n < 0$. However, this projection is not necessarily a bounded operator on $L^p_\infty(\beta)$. For example, as stated in [1], it is not bounded when $\beta(n) = 1$ for all n in $\mathbb{Z}$ (see also [[16], Chapter VII, equations (2.2) and (2.3)]). We want to investigate those weighted Hardy spaces that admit the analytic projection as a bounded linear operator on $L^p_\infty(\beta)$. This answers the following question that has been considered by Shields in [[1], p. 91, Question 11]:

Question. For which bilateral shifts is the analytic projection a bounded operator on $L^2_\infty(\beta)$?

We will use the following notations:

$$
\begin{array}{lll}
r_{01} = \overline{\lim}\beta(-n)^{\frac{-1}{n}} & , \ \Omega_{01} = \{z \in \mathbb{C} : |z| > r_{01}\} \\
r_{11} = \underline{\lim}\beta(n)^{\frac{1}{n}} & , \ \Omega_{11} = \{z \in \mathbb{C} : |z| < r_{11}\} \\
r_{12} = r(M_z^{-1})^{-1} & , \ \Omega_{12} = \{z \in \mathbb{C} : |z| > r_{12}\} \\
r_{22} = r(M_z) & , \ \Omega_{22} = \{z \in \mathbb{C} : |z| < r_{22}\} \\
r_{23} = ||M_z^{-1}||^{-1} & , \ \Omega_{23} = \{z \in \mathbb{C} : |z| > r_{23}\} \\
r_{33} = ||M_z|| & , \ \Omega_{33} = \{z \in \mathbb{C} : |z| < r_{33}\} \\
\Omega_1 = \Omega_{01} \cap \Omega_{11} & = \{z \in \mathbb{C} : r_{01} < |z| < r_{11}\} \\
\Omega_2 = \Omega_{12} \cap \Omega_{22} & = \{z \in \mathbb{C} : r_{12} < |z| < r_{22}\} \\
\Omega_3 = \Omega_{23} \cap \Omega_{33} & = \{z \in \mathbb{C} : r_{23} < |z| < r_{33}\}.
\end{array}
$$

If $r_{01} < r_{11}$, then by the same method used for the formal power series in [2], we can see that each point of Ω_1 is a bounded point evaluation on $L^p(\beta)$.

Note that for the algebra $B(X)$ of all the bounded operators on a Banach space X, the weak operator topology is the one such that $A_\alpha \to A$ in the weak operator topology if and only if $A_\alpha x \to Ax$ weakly, $x \in X$.

For the proof of the main theorems, we need the following lemmas.

Lemma 1. If there exists a constant $c > 0$ such that $||M_{J(p)}|| \leq c||M_p||$ for all Laurent polynomials p, then $J \in B(L^p_\infty(\beta))$.

Proof. Let $\varphi(z) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)z^k$ be an arbitrary element in $L^p_\infty(\beta)$ and define

$$P_n(\varphi) = \sum_{k=-n}^{n} (1 - \frac{|k|}{n+1})\hat{\varphi}(k)z^k, \quad n \in \mathbb{N} \cup \{0\}.$$

Indeed, the functions $P_n(\phi)$ are the averages of the partial sums of ϕ. Now, clearly $J(P_n(\phi)) = P_n(J(\phi))$ and we can also see that $||M_{P_n(\varphi)}|| \leq ||M_\varphi||$. But $P_n(\phi)$ is a Laurent polynomial; thus by the assumption, we have

$$||M_{P_n(J(\varphi))}|| \leq c||M_{P_n(\varphi)}|| \leq c||M_\varphi||,$$

for all $n \in \mathbb{N} \cup \{0\}$. Hence, $\{M_{P_n(J(\varphi))}\}_{n=0}^{\infty}$ is a norm-bounded sequence in $B(L^p(\beta))$. Now, since the unit ball of $B(L^p(\beta))$ is compact in the weak operator topology, we may assume, by passing to a subsequence if necessary, that $M_{P_n(J(\varphi))} \to A$ for some operator A in the weak operator topology. Put $Q_n = P_n(J(\phi))$. Then, clearly we have $\lim_n \widehat{Q_n}(k) = \widehat{J(\varphi)}(k)$. Thus, we get

$$\lim_n < M_{P_n(J(\varphi))}f_0, \frac{f_k}{\beta(k)} >= \lim_n \widehat{Q_n}(k) = \widehat{J(\varphi)}(k) =< J(\varphi)f_0, \frac{f_k}{\beta(k)} >,$$

where f_0 and f_k are considered, respectively, as elements of $L^p(\beta)$ and $L^q(\beta^{\frac{p}{q}})$. Hence, we obtain

$$\lim_n < M_{P_n(J(\varphi))}f_j, f_k >=< J(\varphi)f_j, f_k >$$

for all j and k (here $f_j \in L^p(\beta)$ and $f_k \in L^q(\beta^{\frac{p}{q}})$). However, $\{M_{P_n(J(\varphi))}\}_n$ is bounded, and $M_{P_n(J(\varphi))} \to A$ in the weak operator topology, and so $A = M_{J(\phi)}$ and indeed

$J(\varphi) \in L_\infty^p(\beta)$. Now, by applying the closed graph theorem and the continuity of the coefficient functionals on $L_\infty^p(\beta)$, we conclude that $J \in B(L_\infty^p(\beta))$. □

Lemma 2. Let $J_1 = I - J$ and $||M_{J_1(p)}|| \leq c||M_p||$ for all Laurent polynomials p, then $J \in B(L_\infty^p(\beta))$.

Proof. By the same method used in the proof of Lemma 1, we can see that $J_1 \in B(L_\infty^p(\beta))$ and this implies that J is also a bounded operator from $L_\infty^p(\beta)$ into $L_\infty^p(\beta)$. Hence, the proof is complete. □

Theorem 3. Let M_z be invertible on $L^p(\beta)$, $r_{22} < r_{33}$ and let for some $d > 0$, $||p||_{\Omega_3} \leq d||M_p||$ for all Laurent polynomials p. Then, $J \in B(L_\infty^p(\beta))$.

Proof. First, note that since M_z is invertible, then $\sigma(M_z) = \bar{\Omega}_2$, and hence, the inequality $r_{22} < r_{33}$ implies that $||M_z|| \notin \sigma(M_z)$. Thus, by the Cauchy integral formula we have

$$M_p = p(M_z) = \frac{1}{2\pi i}\int_{|w|=||M_z||} p(w)(w - M_z)^{-1}dw$$

for all polynomials p. Therefore,

$$||M_p|| < \frac{1}{2\pi}||p||_{\Omega_{33}}\int_{|w|=||M_z||} ||(w - M_z)^{-1}||d|w|.$$

However, $t \to ||(e^{it}||M_z|| - M_z)^{-1}||$ is continuous on $[0, 2\pi]$ and so it should be constant, say c_1. Thus, for all polynomials p we get

$$||M_p|| \leq c||p||_{\Omega_{33}} \quad (*)$$

where $c = c_1||M_z||$. Now, let q be a Laurent polynomial. Then,

$$q(z) = \sum_{k=-n}^{n} \hat{q}(k)z^k = q_1(z) + q_2(z)$$

where

$$q_1(z) = \sum_{k=0}^{n} \hat{q}(k)z^k, \quad q_2(z) = \sum_{k=1}^{n} \hat{q}(-k)z^{-k}.$$

We have $||M_{q_1}|| \leq c||q_1||_{\Omega_{33}}$. On the other hand, since $q \in H^\infty(\Omega_3)$, by using the Lemma in [[1], page 81], we get $||q_1||_{\Omega_{33}} \leq c_0||q||_{\Omega_3}$ where $c_0 = 1 + r_{23}(r_{33}^2 - r_{23}^2)^{\frac{1}{2}}$. By the assumption $||q||_{\Omega_3} \leq d||M_q||$, thus $||q_1||_{\Omega_{33}} \leq c_0 d||M_q||$. Therefore, $||M_{J(q)}|| \leq cdc_0||M_q||$ for all Laurent polynomials q and so by Lemma 1 the proof is complete. □

Theorem 4. Let M_z be invertible on $L^p(\beta)$, $r_{23} < r_{12}$, and let for some $d > 0$, the relation $||q||_{\Omega_3} \leq d||M_q||$ holds for all Laurent polynomials q. Then, $J \in B(L_\infty^p(\beta))$.

Proof. It is sufficient to prove that $J_1 = I - J$ is a bounded operator on $L_\infty^p(\beta)$. Note that since $r_{23} < r_{12}$, thus $||M_z^{-1}|| \notin \sigma(M_z^{-1})$. Now by using the relation (*) with replacing z by $\frac{1}{z}$ and M_z by M_z^{-1} we get $||M_q|| \leq c||q||_{\Omega_{23}}$ for all polynomials q in z^{-1}. Now, let p be a Laurent polynomial. Then,

$$p(z) = \sum_{k=-n}^{n} \hat{p}(k)z^k = p_1(z) + p_2(z)$$

where

$$p_1(z) = \sum_{k=0}^{n} \hat{p}(k)z^k, \quad p_2(z) = \sum_{k=1}^{n} \hat{p}(-k)z^{-k}.$$

Hence, $\|M_{p_2}\| \le c\|p_2\|_{\Omega_{23}}$. Since $p \in H^\infty(\Omega_3)$, by using the Lemma in [[1], p. 81], we get $\|p_2\|_{\Omega_{23}} \le c_0\|p\|_{\Omega_3}$ where $c_0 = 1 + r_{23}(r_{33}^2 - r_{23}^2)^{\frac{1}{2}}$. By the assumption we obtain

$$\|M_{J_1(p)}\| = \|M_{p_2}\| \le cdc_0\|M_p\|$$

for all Laurent polynomials p. Now by Lemma 2, $J \in B(L_\infty^p(\beta))$ and so the proof is complete. □

Proposition 5. If M_z is not invertible on $L^p(\beta)$, then $J \in B(L_\infty^p(\beta))$.

Proof. Let $\varphi \in L_\infty^p(\beta)$. Since M_z is not invertible, $\inf_m \frac{\beta(m+1)}{\beta(m)} = 0$. Now by the relation (**) in the proof of Theorem 6, we have

$$|\hat{\varphi}(k)| \le \|M_\varphi\|\,\|M_z^k\|^{-1} = \|M_\varphi\| \inf_m \frac{\beta(m)}{\beta(m+k)}, \quad k \in \mathbb{Z}.$$

Hence, for $k = -1$ we get $\hat{\varphi}(-1) = 0$ for all $\varphi \in L_\infty^p(\beta)$. Multiplying by z we have $\hat{\varphi}(-2) = z\hat{\varphi}(-1) = 0$, etc. Thus, $\hat{\varphi}(n) = 0$ for all $n \le -1$ and so J is identity, i.e., $J(\phi)(z) = \phi(z)$ for all $\varphi \in L_\infty^p(\beta)$ and so the proof is complete. □

Definition 6. Let X be a Banach space. A compact subset K of the plane is called a spectral set of $A \in B(X)$, if it contains the spectrum of A, $\sigma(A)$, and $\|f(A)\| \le \max\{|f(z)| : z \in K\}$ for all rational functions f with poles off K.

From now on suppose that M_z is invertible on $L^p(\beta)$ and Ω_2 is nonempty.

Theorem 7. If $\sigma(M_z)$ is a spectral set for M_z, then the analytic projection is a bounded operator on $L_\infty^p(\beta)$.

Proof. Suppose that $J : L_\infty^p(\beta) \to L^p(\beta)$ denotes the analytic projection. Note that $\sigma(M_z) = \bar{\Omega}_2$. Since $\sigma(M_z)$ is a spectral set, we have $\|M_p\| \le c\|p\|_{\Omega_2}$ for all polynomials p. Also, since Ω_2 is nonempty, we have $r_{12} < r_{22}$. Now, let $q(z) = q_1(z) + q_2(z)$ be a Laurent polynomial such that $q_1(z) = \sum_{k=0}^{n} \hat{q}(k)z^k$ and $q_2(z) = \sum_{k=1}^{n} \hat{q}(-k)z^{-k}$. Thus, we have $\|J(q)\|_{\Omega_{22}} = \|q_1\|_{\Omega_{22}} \le c_1\|q\|_{\Omega_2}$ where $c_1 = 1 + r_{12}(r_{22}^2 - r_{12}^2)^{\frac{1}{2}}$ (see the Lemma in [[1], p. 81]). Therefore,

$$\|M_{q_1}\| \le c\|q_1\|_{\Omega_2} \le c\|q_1\|_{\Omega_{22}} \le cc_1\|q\|_{\Omega_2}.$$

But $L_\infty^p(\beta) \subset H(\Omega_2)$ and each point of Ω_2 is a bounded point evaluation on $L_\infty^p(\beta)$. Indeed, if $\varphi = \sum_n \hat{\varphi}(n)z^n \in L_\infty^p(\beta)$, then the relation $< M_\varphi f_m, f_n >= \hat{\varphi}(n-m)\beta(n)^p$ implies that

$$\begin{aligned}|\hat{\varphi}(n-m)|\beta(n)^p &\le \|M_\varphi\|\,\|f_m\|_{L^p(\beta)}.\|f_n\|_{L^q(\beta^{\frac{p}{q}})} \\ &= \|M_\varphi\|\beta(m)\beta(n)^{\frac{p}{q}}.\end{aligned}$$

By taking $k = n - m$, we get $|\hat{\varphi}(k)| \le ||M_\varphi|| \frac{\beta(m)}{\beta(m+k)}$ for all m. Therefore,

$$|\hat{\varphi}(k)| \le ||M_\varphi||\, ||M_z^k||^{-1} \quad (**)$$

and so $|\varphi(z)| \le ||M_\varphi|| \sum_n \frac{|z|^n}{||M_z^n||}$ where by the root test the series $\sum_n \frac{|z|^n}{||M_z^n||}$ converges on Ω_2. This implies that $L_\infty^p(\beta) \subset H(\Omega_2)$ and each points of Ω_2 is a bounded point evaluation on $L_\infty^p(\beta)$. Also, Ω_2 is the largest open annulus such that $L_\infty^p(\beta) \subset H(\Omega_2)$ (see [2] for the case of formal power series). Since $L_\infty^p(\beta)$ is a commutative Banach algebra and e_λ is multiplicative, it should be $||e_\lambda|| = 1$ for all $\lambda \in \Omega_2$, and this implies that $||\psi||_{\Omega_2} \le ||M_\psi||$ for all ψ in $L_\infty^p(\beta)$. Therefore, $||M_{J(q)}|| \le cc_1||M_q||$ for all Laurent polynomials q. So the proof is complete. □

Corollary 8. If $||M_p|| \le c||p||_{\Omega_2}$ for all polynomials p, then the analytic projection is a bounded operator on $L_\infty^p(\beta)$.

Proof. In the proof of Theorem 7, we only used the inequality in the definition of the spectral set for polynomials instead of rational functions. Hence, Theorem 7 is also consistent if we substitute the statement "$||M_p|| \le c||p||_{\Omega_2}$ for all polynomials p in z", instead of the statement "$\sigma(M_z)$ be a spectral set for M_z". This completes the proof. □

Corollary 9. If $||M_q|| \le c||q||_{\Omega_2}$ for all polynomials q in z^{-1}, then the analytic projection is a bounded operator on $L_\infty^p(\beta)$.

Proof. Put $J_1 = I - J$ where $J : L_\infty^p(\beta) \to L^p(\beta)$ denotes the analytic projection. So for a Laurent polynomial

$$p(z) = \sum_{k=-n}^{n} \hat{p}(k) z^k = p_1(z) + p_2(z),$$

where $p_1(z) = \sum_{k=0}^{n} \hat{p}(k) z^k$ and $p_2(z) = \sum_{k=1}^{n} \hat{p}(-k) z^{-k}$, we have $J_1(p) = p_2$. Now by the Lemma in [[1], p. 81], we have $||p_2||_{\Omega_{12}} = ||J_1(p)||_{\Omega_{12}} \le c_1||p||_{\Omega_2} \le c_1||M_p||$ where $c_1 = 1 + r_{12}(r_{22}^2 - r_{12}^2)^{\frac{1}{2}}$. Thus, by this hypothesis, we get

$$||M_{J_1(p)}|| = ||M_{p2}|| \le c||p_2||_{\Omega_2} \le c||p_2||_{\Omega_{12}} \le cc_1||M_p||.$$

Now, the result follows from Lemma 2. □

Now we consider the special case when $p = 2$.

Theorem 10. Let M_z be invertible on $L^2(\beta)$. If $0 < r_{12} = r_{23} < r_{22} = r_{33}$, then $J \in B(L_\infty^2(\beta))$.

Proof. Note that for the Hilbert space $L^2(\beta)$, the Von Neumann's inequality holds and since $r_{23} < r_{33}$, there exists a number $c > 0$ such that $||M_p|| \le c||p||_{\Omega_3}$ for all Laurent polynomials p (see Proposition 23 in [[1], p. 82]). Also, note that $\Omega_2 = \Omega_3$.

Let

$$p(z) = \sum_{k=-n}^{n} \hat{p}(k) z^k = p_1(z) + p_2(z)$$

where

$$p_1(z) = \sum_{k=0}^{n} \hat{p}(k)z^k, \quad p_2(z) = \sum_{k=1}^{n} \hat{p}(-k)z^{-k}.$$

Thus, $||M_{p_1}|| \le c||p_1||_{\Omega_3}$. Since $p \in H^\infty(\Omega_3)$, we have $||p_1||_{\Omega_{33}} \le c_0||p||_{\Omega_3}$ where $c_0 = 1 + r_{23}(r_{33}^2 - r_{23}^2)^{\frac{1}{2}}$. On the other hand, since $r_{12} < r_{22}$, as we saw earlier, $L^2_\infty(\beta) \subset H^\infty(\Omega_2)$ and $||\varphi||_{\Omega_2} \le ||M_\varphi||$ for all ϕ in $L^p_\infty(\beta)$. Therefore, we get

$$\begin{aligned} ||M_{J(p)}|| &= ||M_{p_1}|| \le c||p_1||_{\Omega_3} \le c||p_1||_{\Omega_{33}} \\ &\le cc_0||p||_{\Omega_3} = cc_0||p||_{\Omega_2} \le cc_0||M_p||. \end{aligned}$$

Now by Lemma 1, the proof is complete. □

Studying the fixed points of weighted composition operators entails a study of the iterate behavior of holomorphic self-maps. The holomorphic self maps of the open unit disc U are divided into classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in U. The maps that are not elliptic are called of non-elliptic type. The iterate of a non-elliptic map can be characterized by the Denjoy-Wolff Iteration Theorem [3]. By ψ_n we denote the nth iterate of ψ and by $\psi'(w)$ we denote the angular derivative of ψ at $w \in \partial U$. Note that if $w \in U$, then $\psi'(w)$ has the natural meaning of derivative.

Denjoy-Wolff Iteration Theorem. Suppose ψ is a holomorphic self-map of U that is not an elliptic automorphism.

(*i*) If ψ has a fixed point $w \in U$, then $\psi_n \to w$ uniformly on compact subsets of U, and $|\psi'(w)| < 1$.

(*ii*) If ψ has no fixed point in U, then there is a point $w \in \partial U$ such that $\psi_n \to w$ uniformly on compact subsets of U, and the angular derivative of ψ exists at w, with $0 < \psi'(w) \le 1$.

We call the unique attracting point w, the Denjoy-Wolff point of ψ.

Suppose that ψ is a holomorphic self-map of U such that the composition operator C_ψ acts boundedly on a Banach space of formal power series $H^p(\beta)$, and ϕ belongs to $H^p_\infty(\beta)$, the set of multipliers of $H^p(\beta)$. Then, the weighted composition operator $C_{\psi,\phi}$ acting on $H^p(\beta)$ is defined by $C_{\phi,\psi} = M_\phi C_\psi$. Note that if $\underline{\lim}\beta(n)^{\frac{1}{n}} = 1$, then $H^p(\beta) \subset H(U)$, and each point of U is a bounded point evaluation on $H^p(\beta)$.

Theorem 11. Suppose that ψ is a holomorphic self-map of U such that $||\psi||_U < 1$, and the composition operator C_ψ acts boundedly on a Banach space of formal power series $H^p(\beta)$ with $\underline{\lim}\beta(n)^{\frac{1}{n}} = 1$. Also suppose that ϕ is a multiplier of $H^p(\beta)$, and w is the Denjoy-Wolff point of ψ. If $H^p_\infty(\beta) = H^\infty(U)$ and $\phi(w) \ne 0$, then the operator $C_{\Phi,\psi}$ has a nonzero fixed point on $H^p(\beta)$ where $\Phi = \frac{\varphi}{\varphi(w)}$.

Proof. Note that $w \in U$ since $||\psi||_U < 1$. Assume that $\phi(w) \ne 0$. Choose δ with $|\psi'(w)| < \delta < 1$. Without loss of generality, suppose that $w = 0$. Hence, $|\psi(z)| < \delta|z|$ when z is sufficiency near to zero. If K is a compact subset of U, then by the Denjoy-Wolff Theorem, $\psi_n \to 0$ uniformly on K and $|\psi_{n+k}(z)| < \delta^k|\psi_n(z)|$ for sufficiently large n and every $k \in \mathbb{N}$, and $z \in K$. This implies that $\sum_{i=0}^{\infty} |\psi_i(z)|$ converges uniformly on compact subsets of U. Since ϕ is bounded, an application of Schwarz's lemma shows that there exists a constant $M > 0$ such that $|\phi(0) - \phi(z)| < M|z|$ for every $z \in U$. However, $\phi(0) \neq 0$, thus

$$|1 - \frac{1}{\varphi(0)}\varphi(z)| < \frac{M}{|\varphi(0)|}|z| \quad (z \in U).$$

By substituting $\psi_i(z)$ instead of z in the above inequality, we get

$$|1 - \frac{1}{\varphi(0)}\varphi(\psi_i(z))| < \frac{M}{|\varphi(0)|}|\psi_i(z)|.$$

This implies that $\sum_{i=0}^{\infty} |1 - \frac{1}{\varphi(0)}\varphi(\psi_i(z))|$ and consequently $\prod_{i=0}^{\infty} \frac{1}{\varphi(0)}\varphi(\psi_i(z))$ converges uniformly on compact subsets of U. Set $g(z) = \prod_{i=0}^{\infty} \frac{1}{\varphi(0)}\varphi(\psi_i(z))$. Then g is a nonzero holomorphic function on U. Also, note that $\phi \cdot g \circ \psi = \phi(0)g$. Thus, generally there is a function $g \in H(U)$ such that

$$(**)\ C_{\varphi,\psi} g = \varphi(w)g.$$

Hence

$$\prod_{j=0}^{n-1} \varphi(\psi_j) g \circ \psi_n = \varphi(w)^n g,$$

so that

$$g = \varphi(w)^{-n} \prod_{j=0}^{n-1} \varphi(\psi_j) g \circ \psi_n.$$

Because $||\psi_n||_U < 1$, the function $g \circ \psi_n \in H^\infty(U)$; moreover, each factor of $\prod_{j=0}^{n-1} \varphi(\psi_j)$ belongs to $H^\infty(U)$ since ϕ belongs to $H^\infty(U)$. Thus, $g \in H^\infty(U) \subseteq H^p(\beta)$, and thus (**) shows that the operator $C_{\Phi,\psi}$ has a nonzero fixed point on $H^p(\beta)$ as desired. □

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Authors' contributions
The study of weighted Hardy spaces lies at the interface of analytic function theory and operator theory. As a part of operator theory, research on multiplication operators acting on weighted Hardy spaces is of fairly recent origin, dating back to valuable work of Allen Shields [1] in the mid 1970's. The multiplication and composition operators have been the focus of attention for several decades and many of it's properties have been studied. One of another basic works in this area is "composition operators on the spaces of analytic functions" due to Cowen and MacCluer [3], which is a pretty large work that contains a number of interesting results and indeed it is mainly of auxiliary nature. Some properties of composition operators, such as hyponormality and compactness, have been studied by Zorboska on these spaces (see [17, 18]). The properties of unicellularity and strictly cyclicity of the multiplication operator M_z have been studied on these spaces in [68]. We gave some necessary conditions for the multiplication operator M_z, acting on weighted Hardy spaces, to be reflexive (see [12]). This answers the eighteenth question raised by Allen Shields in [1]. In this article, we have given conditions under which the analytic projection is bounded on . This answers the

eleventh question of Allen Shields raised in [1]. Also, we characterized the fixed points of some weighted composition operators acting on weighted Hardy spaces. At the end, we note that the weighted Hardy spaces (similar to (cone) metric spaces) will be mentioned as potential materials for further studies on stability of some iteration procedures, and we will continue the investigation on these topics such as [19-28] in our future studies.

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