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A strong convergence theorem on solving common solutions for generalized equilibrium problems and fixed-point problems in Banach space

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Abstract

In this paper, the common solution problem (P1) of generalized equilibrium problems for a system of inverse-strongly monotone mappings $\{A_k\}_{k=1}^N$ and a system of bifunctions $\{f_k\}_{k=1}^N$ satisfying certain conditions, and the common fixed-point problem (P2) for a family of uniformly quasi-φ-asymptotically nonexpansive and locally uniformly Lipschitz continuous or uniformly Hölder continuous mappings $\{S_i\}_{i=1}^\infty$ are proposed. A new iterative sequence is constructed by using the generalized projection and hybrid method, and a strong convergence theorem is proved on approximating a common solution of (P1) and (P2) in Banach space.

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1. Introduction

Recently, common solution problems (i.e., to find a common element of the set of solutions of equilibrium problems and/or the set of fixed points of mappings and/or the set of solutions of variational inequalities) with their applications have been discussed. Some authors such as in references [1-7] presented various iterative schemes and showed some strong or weak convergence theorems on common solution problems in Hilbert spaces. In 2008-2009, Takahashi and Zembayashi [8,9] introduced several iterative sequences on finding a common solution of an equilibrium problem and a fixed-point problem for a relatively nonexpansive mapping, and established some strong or weak convergence theorems. In 2010, Chang et al. [10] discussed the common solution of a generalized equilibrium problem and a common fixed-point problem for two relatively nonexpansive mappings, and established a strong convergence theorem on the common solution problem. The frameworks of spaces in [8-10] are the uniformly smooth and uniformly convex Banach spaces. Chang et al. [11] established a strong convergence theorem on solving the common fixed-point problem for a family of uniformly quasi-φ-asymptotically nonexpansive and uniformly Lipschitz continuous mappings in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property. Some other problems such as optimization problems (e.g. see [1,4,6]) and common zero-point problems (e.g. see [10]) are closely related to common solution problems.

Throughout this paper, unless other stated, $\mathbb{R}$ and $\mathbb{J}$ are denoted by the set of the real numbers and the set $\{1, 2,..., N\}$, respectively, where N is any given positive integer. Let E be a real Banach space with the norm $||\cdot||$, E^* be the dual of E, and $\langle\cdot,\cdot\rangle$ be the pairing between E and E^*. Suppose that C is a nonempty closed convex subset of E.

Let $\{A_k\}_{k=1}^N : C \to E^*$ be N mappings and $\{f_k\}_{k=1}^N : C \times C \to \mathbb{R}$ be N bifunctions. For each $k \in \mathbb{J}$, the generalized equilibrium problem for f_k and A_k is to seek $\bar{u} \in C$ such that

$$f_k(\bar{u}, y) + \langle y - \bar{u}, A_k\bar{u}\} \geq 0, \quad \forall y \in C. \tag{1.1}$$

The *common solution problem* (P1) of generalized equilibrium problems for $\{A_k\}_{k=1}^N$ and $\{f_k\}_{k=1}^N$ is to seek an element in $\mathbb{G}$, where $\mathbb{G} = \bigcap_{k=1}^N G(k)$ and $G(k)$ is the set of solutions of (1.1). We write G instead of $\mathbb{G}$ in the case of $N = 1$.

Let $\{S_i\}_{i=1}^\infty : C \to C$ be a family of mappings. The *common fixed-point problem* (P2) for $\{S_i\}_{i=1}^\infty$ is to seek an element in $\mathbb{F}$, where $\mathbb{F} = \bigcap_{i=1}^\infty F(S_i)$ and $F(S_i)$ is the set of fixed points of S_i.

Motivated by the works in [8-11], in this paper we will produce a new iterative sequence approximating a common solution of (P1) and (P2) (i.e., some point belonging to $\mathbb{F} \cap \mathbb{G}$), and show a strong convergence theorem in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property, where $\{S_i\}_{i=1}^\infty$ in (P2) is a family of uniformly quasi-φ-asymptotically nonexpansive mappings and for each $i \geq 1$, S_i is locally uniformly Lipschitz continuous or uniformly Hölder continuous with order Θ_i .

2. Preliminaries

Let E be a real Banach space, and $\{x_n\}$ be a sequence in E. We denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong convergence and weak convergence of $\{x_n\}$, respectively. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = ||x||^2 = ||f||^2\}, \quad \forall x \in E.$$

By the Hahn-Banach theorem, $Jx \neq \emptyset$ for each $x \in E$.

A Banach space E is said to be *strictly convex* if $\frac{||x+y||}{2} < 1$ for all $x, y \in U = \{u \in E : ||u|| = 1\}$ with $x \neq y$; to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\gamma > 0$ such that $\frac{||x+y||}{2} < 1 - \gamma$ for all $x, y \in U$ with $||x - y|| \geq \varepsilon$; to be *smooth* if the limit

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t} \tag{2.1}$$

exists for every $x, y \in U$; to be *uniformly smooth* if the limit (2.1) exists uniformly for all $x, y \in U$.

Remark 2.1. The basic properties below hold (see [12]).

(i) If E is a real uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.

(ii) If E is a strictly convex reflexive Banach space, then J^{-1} is hemicontinuous, that is, J^{-1} is norm-to-weak*-continuous.

(iii) If E is a smooth and strictly convex reflexive Banach space, then J is single-valued, one-to-one and onto.

(iv) Each uniformly convex Banach space E has the *Kadec-Klee property*, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $||x_n|| \to ||x||$, then $x_n \to x$.

(v) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.

(vi) A Banach space E is strictly convex if and only if J is strictly monotone, that is,

$$\langle x - y, \ x^* - y^* \rangle > 0 \quad \text{whenever } x, \ y \in E, \quad x \neq y \text{ and } x^* \in Jx, \ y^* \in Jy.$$

(vii) Both uniformly smooth Banach spaces and uniformly convex Banach spaces are reflexive.

Now let E be a smooth and strictly convex reflexive Banach space. As Alber [13] and Kamimura and Takahashi [14] did, the *Lyapunov functional* $\varphi : E \times E \to \mathbb{R}^+$ is defined by

$$\phi(x, \ y) = ||x||^2 - 2\langle x, \ Jy\rangle + ||y||^2, \quad \forall x, \ y \in E.$$

It follows from [15] that $\varphi(x, y) = 0$ if and only if $x = y$, and that

$$(||x|| - ||y||)^2 \leq \phi(x, \ y) \leq (||x|| + ||y||)^2. \tag{2.2}$$

Further suppose that C is a nonempty closed convex subset of E. The *generalized projection* (see [13]) Π_C: $E \to C$ is defined by for each $x \in E$,

$$\Pi_C(x) = \arg\min_{y \in C} \phi(y, \ x).$$

A mapping $A : C \to E^*$ is said to be *δ-inverse-strongly monotone*, if there exists a constant $\delta > 0$ such that

$$\langle x - y, \ Ax - Ay \rangle \geq \delta || \ Ax - Ay \ ||^2, \quad \forall x, \ y \in C.$$

A mapping $S : C \to C$ is said to be *closed* if for each $\{x_n\} \subset C$, $x_n \to x$ and $Sx_n \to y$ imply $Sx = y$; to be *quasi-φ-asymptotically nonexpansive* (see [16]) if $F(S) \neq \emptyset$, and there exists a sequence $\{l_n\} \subset [1, \infty)$ with $l_n \to 1$ such that

$$\phi(u, \ S^n x) \leq l_n \phi(u, \ x), \quad \forall x \in C, \ u \in F(S), \quad \forall n \geq 1.$$

It is easy to see that if $A : C \to E^*$ is δ-inverse-strongly monotone, then A is $\frac{1}{\delta}$-Lipschitz continuous. The class of quasi-φ-asymptotically nonexpansive mappings contains properly the class of relatively nonexpansive mappings (see [17]) as a subclass.

Definition 2.1 (see [11]). Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be a sequence of mappings. $\{S_i\}_{i=1}^{\infty}$ is said to be *a family of uniformly quasi-φ-asymptotically nonexpansive mappings*, if $\mathbb{F} \neq \emptyset$ and there exists a sequence $\{l_n\} \subset [1, \infty)$ with $l_n \to 1$ such that for each $i \geq 1$,

$$\phi(u, \ S_i^n x) \leq l_n \phi(u, \ x), \quad \forall u \in \mathbb{F}, \ x \in C, \quad \forall n \geq 1.$$

Now we introduce the following concepts.

Definition 2.2. A mapping $S : C \to C$ is said

(1) to be *locally uniformly Lipschitz continuous* if for any bounded subset D in C, there exists a constant $L_D > 0$ such that

$$||S^n x - S^n y|| \leq L_D ||x - y||, \quad \forall x, \ y \in D, \quad \forall n \geq 1;$$

(2) to be *uniformly Hölder continuous with order* Θ ($\Theta > 0$) if there exists a constant $L > 0$ such that

$$||S^n x - S^n y|| \le L||x - y||^{\Theta}, \quad \forall x,\ y \in C, \quad \forall n \ge 1.$$

Remark 2.2. It is easy to see that any uniformly Lipschitz continuous mapping (see [11]) is locally uniformly Lipschitz continuous, and is also uniformly Hölder continuous with order $\Theta = 1$. However, the converse is not true.

Example 2.1. Suppose that $S : \mathbb{R} \to \mathbb{R}$ is defined by

$$S(x) = \begin{cases} x^2, \text{ if } x < 0, \\ 0, \text{ if } x \ge 0. \end{cases}$$

Then S is locally uniformly Lipschitz continuous. In fact, for any bounded subset D in $\mathbb{R}$, setting $M = 1 + \sup\{|x| : x \in D\}$, we have $|S^n x - S^n y| \le 2M\ |x - y|$, $x, y \in D$, $\forall n \ge 1$. But S fails to be uniformly Lipschitz continuous.

Example 2.2. Suppose that $S : \mathbb{R} - \mathbb{R}$ is defined by

$$S(x) = \begin{cases} \sqrt{-x}, \text{ if } x < 0, \\ 0, \quad \text{ if } x \ge 0. \end{cases}$$

S is uniformly Hölder continuous with order $\Theta = \frac{1}{2}$, since $|S^n x - S^n y| \le 2|x - y|^{\frac{1}{2}}$, $\forall x$, $y \in \mathbb{R}$, $\forall n \ge 1$. But S fails to be uniformly Lipschitz continuous.

Lemma 2.1 (see [13,14]). *If C is a nonempty closed convex subset of a smooth and strictly convex reflexive Banach space E, then*

(1) $\varphi(x, \Pi_C(y)) + \varphi(\Pi_C(y), y) \ge \varphi(x, y)$, $\forall x \in C$, $y \in E$;

(2) *for* $\times \in E$ and $u \in C$, *one has*

$$u = \Pi_C(x) \Leftrightarrow \langle u - y,\ Jx - Ju\rangle \ge 0, \quad \forall y \in C.$$

□

Lemma 2.2. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property, $\{x_n\}$ and$\{y_n\}$ be two sequences of E, and $\bar{u} \in E$. If $x_n \to \bar{u}$* and *$\varphi(x_n, y_n) \to 0$, then $y_n \to \bar{u}$.*

Proof. We complete this proof by two steps.

Step 1. Show that there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to \bar{u}$.

In fact, since $\varphi(x_n, y_n) \to 0$, by (2.2) we have $||x_n|| - ||y_n|| \to 0$. It follows from $x_n \to \bar{u}$ that

$$||y_n|| \ \to \ ||\bar{u}|| \ \text{ (as } n \to \infty), \tag{2.3}$$

and so

$$||Jy_n|| \ \to \ ||J\bar{u}|| \ \text{ (as } n \ \to \infty). \tag{2.4}$$

Then $\{Jy_n\}$ is bounded in E^*. It follows from Remark 2.1(v) and (vii) that E^* is reflexive. Hence there exist a point $f_0 \in E^*$ and a subsequence $\{Jy_{n_k}\}$ of $\{Jy_n\}$ such that

$$Jy_{n_k} \rightharpoonup f_0 (\text{as } k \to \infty). \tag{2.5}$$

It follows from Remark 2.1(vii) and (iii) that there exists a point $x \in E$ such that $Jx = f_0$. By the definition of φ, we obtain

$$\begin{aligned}\phi(x_{n_k}, y_{n_k}) &= ||x_{n_k}||^2 - 2\langle x_{n_k}, Jy_{n_k}\rangle + ||y_{n_k}||^2\\ &= ||x_{n_k}||^2 - 2\langle x_{n_k}, Jy_{n_k}\rangle + ||Jy_{n_k}||^2.\end{aligned}$$

By weak lower semicontinuity of norm $||\cdot||$, we have

$$\begin{aligned}0 &= \liminf_{k\to\infty}\phi(x_{n_k}, y_{n_k})\\ &\geq ||\bar{u}||^2 - 2\langle\bar{u}, f_0\rangle + ||f_0||^2\\ &= ||\bar{u}||^2 - 2\langle\bar{u}, Jx\rangle + ||Jx||^2\\ &= ||\bar{u}||^2 - 2\langle\bar{u}, Jx\rangle + ||x||^2 = \phi(\bar{u}, x),\end{aligned}$$

which implies that $\bar{u} = x$ and $f_0 = J\bar{u}$. It follows from Remark 2.1(iv) and (v) that E^* has the Kadec-Klee property, and so $Jy_{n_k} \to J\bar{u}$ by (2.4) and (2.5). By Remark 2.1(vii) and (ii), we have $y_{n_k} \rightharpoonup \bar{u}$, which implies that $y_{n_k} \to \bar{u}$ by (2.3) and the Kadec-Klee property of E.

Step 2. Show that $y_n \to \bar{u}$.

In fact, suppose that $y_n \not\to \bar{u}$. For some given number $\varepsilon_0 > 0$, there exists a positive integer sequence $\{n_k\}$ with $n_1 < n_2 < \cdots < n_k < \cdots$, such that

$$||y_{n_k} - \bar{u}|| \geq \varepsilon_0. \tag{2.6}$$

Replacing $\{y_n\}$ by $\{y_{n_k}\}$ in Step 1, there exists a subsequence $\{y_{n_{k_i}}\}$ of $\{y_{n_k}\}$ such that $y_{n_{k_i}} \to \bar{u}$, which contradicts (2.6). □

Lemma 2.3. *Let C be a nonempty closed convex subset of a smooth and strictly convex reflexive Banach space E, and let $A : C \to E^*$ be a δ-inverse-strongly monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions*

(B_1) $f(z, z) = 0$, $\forall z \in C$;

(B_2) $\limsup_{t\downarrow 0} f(z + t(x - z), y) \leq f(z, y), \quad \forall x, y, \ z \in C$;

(B_3) *for any $z \in C$, the function $y \,\alpha\, f(z, y)$ is convex and lower semicontinuous*;

(B_4) *for some $\beta \geq 0$ with $\beta \leq \delta$,*

$$f(z, y) + f(y, z) \leq \beta||Az - Ay||^2, \quad \forall z, y \in C.$$

Then the following conclusions hold:

(1) *For any $r > 0$ and $u \in E$, there exists a unique point $z \in C$ such that*

$$f(z, y) + \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Ju\rangle \geq 0, \quad \forall y \in C. \tag{2.7}$$

(2) *For any given $r > 0$, define a mapping $K_r : E \to C$ as follows*: $\forall u \in E$,

$$K_r u = z \text{ such that } f(z, y) + \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Ju\rangle \geq 0, \quad \forall y \in C.$$

We have (i) $F(K_r) = G$ *and G is closed convex in C, where*

$$G = \{z \in C : f(z, y) + \langle y - z, Az\rangle \geq 0, \quad \forall y \in C\};$$

(ii) $\varphi(z, K_r u) + \varphi(K_r u, u) \leq \varphi(z, u)$, $\forall z \in F(K_r)$.

(3) *For each $n \geq 1$, $r_n > a > 0$ and $u_n \in C$ with* $\lim_{n\to\infty} u_n = \lim_{n\to\infty} K_{r_n} u_n = \bar{u}$, *we have*

$$f(\bar{u}, y) + \langle y - \bar{u}, A\bar{u}\rangle \geq 0, \quad \forall y \in C.$$

Proof. (1) We consider the bifunction $\tilde{f} : (z, y) \mapsto f(z, y) + \langle y - z, Az\rangle$ instead of f. It follows from the proof of Lemma 2.5 in [10] that $\tilde{f}$ satisfies (B_1)-(B_3). Since A is δ-inverse-strongly monotone, by (B_4), we have

$$\begin{aligned}&(f(z, y) + \langle y - z, Az\rangle) + (f(y, z) + \langle z - y, Ay\rangle)\\ &= f(z, y) + f(y, z) - \langle z - y, Az - Ay\rangle\\ &\leq (\beta - \delta)||Az - Ay||^2 \leq 0, \forall y, z \in C,\end{aligned} \tag{2.8}$$

which implies $\tilde{f}$ is monotone. By Blum amd Oettli [18], for any $r > 0$ and $u \in E$, there exists $z \in C$ such that (2.7) holds. Next we show that (2.7) has a unique solution. If for any given $r > 0$ and $u \in E$, z_1 and z_2 are two solutions of (2.7), then

$$f(z_1, z_2) + \langle z_2 - z_1, Az_1\rangle + \frac{1}{r}\langle z_2 - z_1, Jz_1 - Ju\rangle \geq 0,$$

and

$$f(z_2, z_1) + \langle z_1 - z_2, Az_2\rangle + \frac{1}{r}\langle z_1 - z_2, Jz_2 - Ju\rangle \geq 0.$$

Adding these two inequalities, we have

$$f(z_1, z_2) + f(z_2, z_1) - \langle z_2 - z_1, Az_2 - Az_1\rangle - \frac{1}{r}\langle z_2 - z_1, Jz_2 - Jz_1\rangle \geq 0.$$

It follows from (2.8) that

$$\langle z_2 - z_1, Jz_2 - Jz_1\rangle \leq 0,$$

which implies that $z_1 = z_2$ by Remark 2.1(vi).

(2) Since $\tilde{f}$ satisfies (B_1)-(B_3) and is monotone, the conclusion (2) follows from Lemmas 2.8 and 2.9 in [9].

(3) Since

$$f(K_{r_n}u_n, y) + \langle y - K_{r_n}u_n, AK_{r_n}u_n\rangle + \frac{1}{r_n}\langle y - K_{r_n}u_n, JK_{r_n}u_n - Ju_n\rangle \geq 0, \quad \forall y \in C,$$

we have

$$\begin{aligned}\frac{1}{r_n}\langle y - K_{r_n}u_n, JK_{r_n}u_n - Ju_n\rangle &\geq -(f(K_{r_n}u_n, y) + \langle y - K_{r_n}u_n, AK_{r_n}u_n\rangle)\\ &\geq f(y, K_{r_n}u_n) + \langle K_{r_n}u_n - y, Ay\rangle, \quad \forall y \in C,\end{aligned} \tag{2.9}$$

by the monotonicity of $\tilde{f}$. It follows from $\lim_{n\to\infty} u_n = \lim_{n\to\infty} K_{r_n}u_n = \bar{u}$. $r_n > a > 0$ and Remark 2.1(i) that

$$\lim_{n\to\infty} \frac{||Ju_n - JK_{r_n}u_n||}{r_n} = 0.$$

Since $y \mapsto \tilde{f}(z, y)$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n \to \infty$ in (2.9), we have $f(y, \bar{u}) + \langle \bar{u} - y, Ay\rangle \leq 0$, $\forall y \in C$. For any $t \in (0, 1]$ and $y \in C$, setting $y_t = ty + (1 - t)\bar{u}$, we have $y_t \in C$ and $f(y_t, \bar{u}) + \langle \bar{u} - y_t, Ay_t\rangle \leq 0$, which together with ($B_1$) implies that

$$\begin{aligned}0 &= f(y_t,\ y_t) + \langle y_t - y_t,\ Ay_t\rangle \\ &= f(y_t,\ ty + (1-t)\bar{u}) + \langle ty + (1-t)\bar{u} - y_t,\ Ay_t\rangle \\ &\le t[f(y_t,\ y) + \langle y - y_t,\ Ay_t\rangle] + (1-t)[f(y_t, \bar{u}) + \langle \bar{u} - y_t,\ Ay_t\rangle] \\ &\le t[f(y_t,\ y) + \langle y - y_t,\ Ay_t\rangle].\end{aligned}$$

Thus $f(y_t, y) + \langle y - y_t, Ay_t\rangle \geq 0$, $\forall y \in C$, $\forall t \in (0, 1]$. Letting $t \downarrow 0$, since $z \alpha f(z, y) + \langle y - z, Az\rangle$ satisfies (B_2), we have $f(\bar{u},\ y) + \langle y - \bar{u},\ A\bar{u}\rangle \geq 0$, $\forall y \in C$.

Remark 2.3. If $\beta = 0$ in (B_4), that is, f is monotone, then the conclusions (1) and (2) in Lemma 2.3 reduce to the relating results of Lemmas 2.5 and 2.6 in [10], respectively.

Next we give an example to show that there exist the mapping A and the bifunction f satisfying the conditions of Lemma 2.3. However, f is not monotone.

Example 2.3. Define $A : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $Ax = 2x + \sqrt{1 + x^2} \in \ \forall x \in \mathbb{R}$ and $f(x,\ y) = \frac{(x-y)^2}{10}$, $\forall (x, y) \in \mathbb{R} \times \mathbb{R}$, respectively. It is easy to see that A is $\frac{1}{3}$-inverse-strongly monotone, f satisfies (B_1)-(B_3), and $f(x,\ y) + f(y,\ x) \leq \frac{1}{5}|Ax - Ay|^2$, $\forall (x, y) : \mathbb{R} \times \mathbb{R}$ with $\frac{1}{5} \leq \frac{1}{3}$.

Lemma 2.4 (see [12]). *Let C be a nonempty closed convex subset of a real uniformly smooth and strictly convex Banach space E with the Kadec-Klee property, $S : C \to C$ be a closed and quasi-φ-asymptotically nonexpansive mapping with a sequence $\{l_n\} \subset [1, \infty)$, $l_n \to 1$. Then $F(S)$ is closed convex in C.*

Lemma 2.5 (see [11]). *Let E be a uniformly convex Banach space, $\eta > 0$ be a positive number and $B_\eta(0)$ be a closed ball of E. Then, for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_\eta(0)$ and for any given $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous, strictly increasing and convex function $g : [0, 2\eta) \to [0, \infty)$ with $g(0) = 0$ such that for any positive integers i, j with $i < j$,*

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

□

3. Strong convergence theorem

In this section, let C be a nonempty closed convex subset of a real uniformly smooth and strictly convex Banach space E with the Kadec-Klee property.

Theorem 3.1. *Suppose that*

(C_1) *for each $k \in \mathbb{J}$, the mapping $A_k : C \to E^*$ is δ_k-inverse-strongly monotone, the bifunction $f_k : C \times C \to \mathbb{R}$ satisfies (B_1)-(B_3), and for some $\beta_k \geq 0$ with $\beta_k \leq \delta_k$,*

$$f_k(z,\ y) + f_k(y,\ z) \leq \beta_k ||A_k z - A_k y||, \quad \forall z,\ y \in C;$$

(C_2) $\{S_i\}_{i=1}^{\infty} : C \to C$ *is a family of closed and uniformly quasi-φ-asymptotically nonexpansive mappings with a sequence $\{l_n\} \subset [1, \infty)$, $l_n \to 1$;*

(C_3) *for each $i \geq 1$, S_i is either locally uniformly Lipschitz continuous or uniformly Hölder continuous with order Θ_i ($\Theta_i > 0$), and $\mathbb{F}$ is bounded in C.*

(C_4) *$\mathbb{F} \cap \mathbb{G} \neq \emptyset$. Take the sequence $\{x_n\}_{n=1}^{\infty}$generated by*

$$\begin{cases} x_0 \in C,\ H_0 = W_0 = C, \\ u_{0,n} = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}J\, S_i^n x_n), \\ u_{1,n} \in C \text{ such that} \\ f_1(u_{1,n},\ y) + \langle y - u_{1,n},\ A_1 u_{1,n}\rangle + \frac{1}{r_{1,n}}\langle y - u_{1,n},\ Ju_{1,n} - Ju_{0,n}\rangle \geq 0, \quad \forall y \in C, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ u_{N,n} \in C \text{ such that} \\ f_N(u_{N,n},\ y) + \langle y - u_{N,n},\ A_N u_{N,n}\rangle + \frac{1}{r_{N,n}}\langle y - u_{N,n},\ Ju_{N,n} - Ju_{N-1,n}\rangle \geq 0, \quad \forall y \in C, \\ H_{n+1} = \{v\ \in H_n : \phi(v,\ u_{N,n}) \leq \phi(v,\ x_n) + \xi_n\}, \\ W_{n+1} = \{z \in W_n : \langle x_n - z,\ Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_{n+1}\cap W_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where for each $k \in \mathbb{J}$, $\{r_{k,n}\}_{n=0}^{\infty} \subset [a,\ \infty)$with some $a > 0$, $\{\alpha_{n,i}\}_{n=0,i=0}^{\infty} \subset [0, 1]$, and $\xi_n = \sup_{u\in\mathbb{F}}(l_n - 1)\phi(u,\ x_n)$. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$, $\forall n \geq 0$ and $\liminf_{n\to\infty} \alpha_{n,0}\ \alpha_{n,\ i} > 0$, $\forall i \geq 1$, then $x_n \to \Pi_{\mathbb{F}\cap\mathbb{G}} x_0$.

Proof. We shall complete this proof by seven steps below.

Step 1. Show that $\mathbb{F}$, $\mathbb{G}$, H_n and W_n for all $n \geq 0$ are closed convex.

In fact, $\mathbb{F} = \bigcap_{i=1}^{\infty} F(S_i)$ is closed convex since for each $i \geq 1$, $F(S_i)$ is closed convex by (C_2) and Lemma 2.4. $\mathbb{G}$ is closed convex since for each $k \in \mathbb{J}$, $G(k)$ is closed convex by (C_1) and Lemma 2.3(2)(i). $H_0 = C$ is closed convex. Since $\varphi(v,u_{N,n}) \leq \varphi(v,x_n) + \xi_n$ is equivalent to

$$2\langle v,\ Jx_n - Ju_{N,n}\rangle \leq\ ||x_n||^2 - ||u_{N,n}||^2 + \xi_n,$$

we know that $H_n(n \geq 0)$ are closed convex. Finally, W_n is closed convex by its definition. Thus $\Pi_{\mathbb{F}\cap\mathbb{G}} x_0$ and $\Pi_{H_n\cap W_n} x_0$ are well defined.

Step 2. Show that $\{x_n\}$ and $\{S_i^n x_n\}_{i,n=1}^{\infty}$ are bounded.

From $x_n = \Pi_{H_n\cap W_n} x_0$, $\forall n \geq 0$ and Lemma 2.1(1), we have

$$\phi(x_n,\ x_0) \leq \phi(u,\ x_0) - \phi(u,\ x_n) \leq \phi(u,\ x_0), \quad \forall u \in C, \quad \forall n \geq 0, \tag{3.1}$$

which implies that $\{\varphi(x_n, x_0)\}$ is bounded, and so is $\{x_n\}$ by (2.2). It follows from (C_2) that for all $u \in \mathbb{F}$, $i \geq 1$, $n \geq 1$,

$$\phi(u,\ S_i^n x_n) \leq l_n \phi(u,\ x_n) \leq l_n(||u|| + ||x_n||)^2 \leq \sup_{u\in\mathbb{F}} l_n(||u|| + ||x_n||)^2.$$

Hence for all $i \geq 1$, $\{\phi(u,\ S_i^n x_n)\}_{n=1}^{\infty}$ is uniformly bounded, and so is $\{S_i^n x_n\}_{n=1}^{\infty}$ by (2.2). Obviously,

$$\xi_n = \sup_{u\in\mathbb{F}}(l_n - 1)\phi(u,\ x_n) \leq \sup_{u\in\mathbb{F}}(l_n - 1)(||u|| + ||x_n||)^2 \to 0 \text{ (as } n\ \to \infty). \tag{3.2}$$

Step 3. Show that $\mathbb{F} \cap \mathbb{G} \subset H_n \cap W_n$, $\forall n \geq 0$.

Since Banach space E is uniformly smooth, E^* is uniformly convex, by Remark 2.1(v). For any given $p \in \mathbb{F}$, any $n \geq 1$ and any positive integer j, by (C_2) and Lemma 2.5, we have

$$\begin{aligned}
\phi(p,\ u_{0,n}) &= \phi(p,\ J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n)) \\
&= \ ||p||^2 - \ 2\langle p,\ \alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n\rangle \ + \left\|\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n\right\|^2 \\
&\le \ ||p||^2 - 2\alpha_{n,0}\langle p,\ Jx_n\rangle - 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle p,\ JS_i^n x_n\rangle + \alpha_{n,0}||x_n||^2 \\
&\quad + \sum_{i=1}^{\infty}\alpha_{n,i}||S_i^n x_n||^2 - \ \alpha_{n,0}\alpha_{n,j}g(||Jx_n - JS_j^n x_n||) \quad \text{(By Lemma 2.5)} \\
&= \alpha_{n,0}\phi(p,\ x_n) + (1-\alpha_{n,0})||p||^2 - \ 2\sum_{i=1}^{\infty}\alpha_{n,i}\langle p,\ JS_i^n x_n\rangle \\
&\quad + \sum_{i=1}^{\infty}\alpha_{n,i}||S_i^n x_n||^2 - \ \alpha_{n,0}\alpha_{n,j}g(||Jx_n - JS_j^n x_n||) \\
&= \alpha_{n,0}\phi(p,\ x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}\phi(p,\ S_i^n x_n) - \alpha_{n,0}\alpha_{n,j}g(||Jx_n - JS_j^n x_n||) \\
&\le \alpha_{n,0}\phi(p,\ x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}l_n\phi(p,\ x_n) - \alpha_{n,0}\alpha_{n,j}g(||Jx_n - JS_j^n x_n||) \\
&\le l_n\phi(p,\ x_n) - \alpha_{n,0}\alpha_{n,j}g(||Jx_n - JS_j^n x_n||) \\
&\le \phi(p,\ x_n) + \sup_{p\in\mathbb{F}}(l_n - 1)\phi(p,\ x_n) - \alpha_{n,0}\alpha_{n,j}g(||Jx_n - JS_j^n x_n||) \\
&= \phi(p,\ x_n) + \xi_n - \alpha_{n,0}\alpha_{n,j}g(||Jx_n - JS_j^n x_n||).
\end{aligned} \tag{3.3}$$

Put $u_{k,n} = K_{r_{k,n}}u_{k-1,n}$, $k \in \mathbb{J}$, $\forall n \ge 0$. It follows from (3.3) and Lemma 2.3(2)(ii) that

$$\begin{aligned}
\phi(p,\ u_{k,n}) = \phi(p,\ K_{r_{k,n}}u_{k-1,n}) \le \phi(p,\ u_{k-1,n}) \le \phi(p,\ x_n) + \xi_n, \\
\forall p \in \mathbb{F}\cap\mathbb{G}, \quad \forall k \in \mathbb{J}, \quad \forall n \ge 0,
\end{aligned} \tag{3.4}$$

which implies that if $p \in \mathbb{F}\cap\mathbb{G}$, then $p \in H_n$, $\forall n \ge 0$. Hence, $\mathbb{F}\cap\mathbb{G} \subset H_n$, $\forall n \ge 0$. By induction, now we prove that $\mathbb{F}\cap\mathbb{G} \subset W_n$, $\forall n \ge 0$. In fact, it follows from $W_0 = C$ that $\mathbb{F}\cap\mathbb{G} \subset W_0$. Suppose that $\mathbb{F}\cap\mathbb{G} \subset W_m$ for some $m \ge 0$. By the definition of $x_m = \Pi_{H_m\cap W_m}x_0$ and Lemma 2.1(2), we have

$$\langle x_m - z,\ Jx_0 - Jx_m\rangle \ge 0, \quad \forall z \in H_m \cap W_m,$$

and so

$$\langle x_m - z,\ Jx_0 - Jx_m\rangle \ge 0, \quad \forall z \in \mathbb{F}\cap\mathbb{G},$$

which shows $z \in W_{m+1}$, so $\mathbb{F}\cap\mathbb{G} \subset W_{m+1}$.

Step 4. Show that there exists $\bar{u} \in C$ such that $x_n \to \bar{u}$.

Without loss of generalization, we can assume that $x_n \rightharpoonup \bar{u}$, since $\{x_n\}$ is bounded and E is reflexive. Moreover, it follows that $\bar{u} \in H_n \cap W_n$, $\forall n \ge 0$ from $H_{n+1} \cap W_{n+1} \subset H_n \cap W_n$ and the closeness and convexity of $H_n \cap W_n$. Noting that

$$\begin{aligned}
&\liminf_{n\to\infty}\phi(x_n,\ x_0) = \liminf_{n\to\infty}(||x_n||^2 - \ 2\langle x_n,\ Jx_0\rangle \ + ||x_0||^2) \\
&\ge \ ||\bar{u}||^2 - \ 2\langle\bar{u},\ Jx_0\rangle \ + ||x_0||^2 = \phi(\bar{u},\ x_0),
\end{aligned}$$

we have

$$\phi(\bar{u}, x_0) \leq \liminf_{n\to\infty} \phi(x_n, x_0) \leq \limsup_{n\to\infty} \phi(x_n, x_0) \leq \phi(\bar{u}, x_0).$$

by (3.1). It follows that

$$\lim_{n\to\infty} \phi(x_n, x_0) = \phi(\bar{u}, x_0), \tag{3.5}$$

and so $||x_n|| \to ||\bar{u}||$ by $x_n \rightharpoonup \bar{u}$. Hence,

$$x_n \to \bar{u} \text{ (as } n \to \infty) \tag{3.6}$$

by the Kadec-Klee property of E, and so

$$Jx_n \to J\bar{u} \text{ (as } n \to \infty) \tag{3.7}$$

by Remark 2.1(i).

Step 5. Show that $\bar{u} \in \mathbb{F}$.

Since $x_{n+1} \in C$, setting $u = x_{n+1}$ in (3.1), we have

$$\phi(x_{n+1}, x_n) \leq \phi(x_{n+1}, x_0) - \phi(x_n, x_0).$$

By (3.5),

$$\phi(x_{n+1}, x_n) \to 0 \text{ (as } n \to \infty). \tag{3.8}$$

By $x_{n+1} \in H_{n+1}$, (3.2) and (3.8), we have

$$\phi(x_{n+1}, u_{N,n}) \leq \phi(x_{n+1}, x_n) + \xi_n \to 0 \text{ (as } n \to \infty),$$

which together with (3.6) and Lemma 2.2 implies that

$$\lim_{n\to\infty} u_{N,n} = \bar{u}. \tag{3.9}$$

For any $j \geq 1$ and any given $p \in \mathbb{F} \cap \mathbb{G}$, it follows from (3.2)-(3.4) and (3.9) that

$$\begin{aligned} &\alpha_{n,0}\alpha_{n,j} g(||Jx_n - JS_j^n x_n||) \leq \phi(p, x_n) + \xi_n - \phi(p, u_{0,n}) \\ &\leq \phi(p, x_n) + \xi_n - \phi(p, u_{N,n}) \to 0 \text{ (as } n \to \infty), \end{aligned} \tag{3.10}$$

which implies that

$$g(||Jx_n - JS_j^n x_n||) \to 0 \text{ (as } n \to \infty),$$

since $\liminf_{n\to 0} \alpha_{n,0}\alpha_{n,i} > 0$, $\forall i \geq 1$. We obtain

$$||Jx_n - JS_j^n x_n|| \to 0 \text{ (as } n \to \infty), \tag{3.11}$$

since $g(0) = 0$ and g is strictly increasing and continuous. By (3.7) and (3.11), we have $JS_j^n x_n \to J\bar{u}$ and $||S_j^n x_n|| \to ||\bar{u}||$ for all $j \geq 1$. It follows from Remark 2.1(ii) that $S_j^n x_n \rightharpoonup \bar{u}$, which implies

$$S_j^n x_n \to \bar{u} \text{ (as } n \to \infty), \quad \forall j \geq 1, \tag{3.12}$$

by the uniform boundedness of $\{S_j^n x_n\}_{n=1}^{\infty}$ and the Kadec-Klee property of E. Thus

$$||S_j^{n+1} x_{n+1} - S_j^n x_n|| \to 0 \text{ (as } n \to \infty), \quad \forall j \geq 1.$$

By (C_3) and (3.6), we have

$$||S_j^{n+1}x_n - S_j^{n+1}x_{n+1}|| \to 0 \text{ (as } n \to \infty), \quad \forall j \geq 1.$$

Hence, for each $j \geq 1$,

$$\begin{aligned} &||S_j(S_j^n x_n) - S_j^n x_n|| = ||S_j^{n+1}x_n - S_j^n x_n|| \\ &\leq ||S_j^{n+1}x_n - S_j^{n+1}x_{n+1}|| + ||S_j^{n+1}x_{n+1} - S_j^n x_n|| \to 0 \text{ (as } n \to \infty). \end{aligned}$$

By (3.12) and the closeness of S_j, we have $S_j\bar{u} = \bar{u}$ for all $j \geq 1$ and so $\bar{u} \in \mathbb{F}$.

Step 6. Show that $\bar{u} \in \mathbb{G}$.

In fact, it is easy to see that for each $k \in \{0\} \cup \mathbb{J}$, and $p \in \mathbb{F} \cap \mathbb{G}$, the sequence $\{\varphi(p, u_{k,n})\}$ is bounded by (3.2), (3.4) and the boundedness of $\{x_n\}$ and $\mathbb{F}$, which implies that $\{u_{k,n}\}$ is bounded in C by (2.2). Since $\bar{u} \in \mathbb{F}$, by (3.2), (3.3), (3.5) and (3.10), we have

$$\begin{aligned} &\phi(\bar{u}, u_{0,n}) \leq \phi(\bar{u}, x_n) + \xi_n - \alpha_{n,0}\alpha_{n,j}g(||Jx_n - JS_j^n x_n)||) \\ &\leq \phi(\bar{u}, x_n) + \xi_n \to 0 \text{ (as } n \to \infty). \end{aligned}$$

It follows from Lemma 2.2 that

$$u_{0,n} \to \bar{u} \text{ (as } n \to \infty). \tag{3.13}$$

Furthermore, it follows from (3.4) and Lemma 2.3(2)(ii) that for any given $p \in \mathbb{F} \cap \mathbb{G}$,

$$\phi(p, u_{N,n}) + \phi(u_{1,n}, u_{0,n}) \leq \phi(p, u_{1,n}) + \phi(u_{1,n}, u_{0,n}) \leq \phi(p, u_{0,n}),$$

which implies

$$\begin{aligned} &\phi(u_{1,n}, u_{0,n}) \leq \phi(p, u_{0,n}) - \phi(p, u_{N,n}) \\ &= ||u_{0,n}||^2 - ||u_{N,n}||^2 - 2\langle p, Ju_{0,n} - Ju_{N,n}\rangle \to 0 \text{ (as } n \to \infty), \end{aligned}$$

by Remark 2.1(i), (3.9) and (3.13). Then $u_{1,n} \to \bar{u}$ by (3.13) and Lemma 2.2. Similarly, we also obtain $u_{k,n} \to \bar{u}$ $(k = 2, 3, \ldots, N - 1)$. Hence, together with (3.9) and (3.13), for each $k \in \{0\} \cup \mathbb{J}$,

$$u_{k,n} \to \bar{u} \text{ (as } n \to \infty). \tag{3.14}$$

For each $k \in \mathbb{J}$, since $u_{k,n} = K_{r_{k,n}}u_{k-1,n}$, we have

$$f_k(u_{k,n}, y) + \langle y - u_{k,n}, A_k u_{k,n}\rangle + \frac{1}{r_{k,n}}\langle y - u_{k,n}, Ju_{k,n} - Ju_{k-1,n}\rangle \geq 0, \quad \forall y \in C,$$

which together with (3.14) and Lemma 2.3(3) implies that $f_k(\bar{u}, y) + \langle y - \bar{u}, A_k\bar{u}\rangle \geq 0$, $\forall y \in C$. Therefore $\bar{u} \in \mathbb{G}$ and so $\bar{u} \in \mathbb{F} \cap \mathbb{G}$.

Step 7. Show that $\bar{u} = \Pi_{\mathbb{F}\cap\mathbb{G}}x_0$.

In fact, letting $w = \Pi_{\mathbb{F}\cap\mathbb{G}}x_0$, by $w \in \mathbb{F} \cap \mathbb{G} \subset H_n \cap W_n$ and $x_n = \Pi_{H_n\cap W_n}x_0$, we have

$$\phi(x_n, x_0) \leq \phi(w, x_0), \quad \forall n \geq 0.$$

It follows from (3.6) that

$$\begin{aligned} &\phi(\bar{u}, x_0) = ||\bar{u}||^2 - 2\langle \bar{u}, Jx_0\} + ||x_0||^2 \\ &= \lim_{n\to\infty}\{||x_n||^2 - 2\langle x_n, Jx_0\} + ||x_0||^2\} \\ &= \lim_{n\to\infty}\phi(x_n, x_0) \leq \phi(w, x_0). \end{aligned}$$

Hence, $\bar{u} = w$, and so $x_n \to \bar{u} = \Pi_{\mathbb{F}\cap\mathbb{G}}x_0$. □

Setting $N = 1$, $u_{0,n} = y_n$ and $u_{N,n} = u_n$ in Theorem 3.1, we can obtain the following result.

Corollary 3.1 *Suppose that*

(D_1) *the mapping* $A : C \to E^*$ *is a mapping with* δ *-inverse-strongly monotone, the bifunction* $f : C \times C \to \mathbb{R}$ *satisfies* (B_1)-(B_3) *and for some* $\beta > 0$ *with* $\beta \le \delta$,

$$f(z, y) + f(y, z) \le \beta ||Az - Ay||^2, \quad \forall z,\ y \in C;$$

(D_2) *both* (C_2) *and* (C_3) *hold, and* $\mathbb{F} \cap \mathbb{G} \neq \emptyset$*Take the sequence* $\{x_n\}_{n=1}^{\infty}$*generated by*

$$\begin{cases} x_0 \in C,\ H_0 = W_0 = C, \\ y_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n), \\ u_n \in C \text{ such that} \\ \qquad f(u_n, y) + \langle y - u_n, Au_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\} \ge 0, \quad \forall y \in C, \\ H_{n+1} = \{v \in H_n : \phi(v, u_n) \le \phi(v, x_n) + \xi_n\}, \\ W_{n+1} = \{z \in W_n : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_{n+1}\cap W_{n+1}}x_0, \quad \forall n \ge 0, \end{cases}$$

where $\{\alpha_{n,i}\}_{n=0,i=0}^{\infty} \subset [0, 1]$, $\{r_n\}_{n=0}^{\infty} \in [a, \infty)$*for some* $a > 0$ *and* $\xi_n = \sup_{u\in\mathbb{F}}(l_n - 1)\phi(u, x_n)$. *If* $\sum_{i=0}^{\infty}\alpha_{n,i} = 1$, $\forall_n \ge 0$ *and* $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,i} > 0$, $\forall i \ge 1$, *then* $x_n \to \Pi_{\mathbb{F}\cap\mathbb{G}}x_0$. □

Furthermore, if $S_i = S$, $i \ge 1$ in Corollary 3.1, the following corollary can be obtained immediately.

Corollary 3.2. *Suppose that, besides* (D1),

(E_1) $S : C \to C$ *is closed and quasi-φ-asymptotically nonexpansive with* $\{l_n\} \subset [1, \infty)$, $l_n \to 1$;

(E_2) *S is either locally uniformly Lipschitz continuous or uniformly Hölder continuous with order* Θ ($\Theta > 0$), $F(S)$ *is bounded in C and* $F(S) \cap G \neq \emptyset$. *Take the sequence* $\{x_n\}_{n=1}^{\infty}$*generated by*

$$\begin{cases} x_0 \in C,\ H_0 = W_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JS^n x_n), \\ u_n \in C \text{ such that} \\ \qquad f(u_n,\ y) + \langle y - u_n, Au_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\} \ge 0, \quad \forall y \in C\ , \\ H_{n+1} = \{v \in H_n : \phi(v, u_n) \le \phi(v,\ x_n) + \xi_n\}, \\ W_{n+1} = \{z \in W_n : \langle x_n - z,\ Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_{n+1}\cap W_{n+1}}x_0, \quad \forall n \ge 0, \end{cases}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset (0, 1)$, $\{r_n\}_{n=0}^{\infty} \in [a, \infty)$*for some* $a > 0$ *and* $\xi = \sup_{u\in F(S)}(l_n - 1)\varphi(u, x_n)$. *If* $\liminf_{n\to\infty}\alpha_n(1 - \alpha_n) > 0$, *then* $x_n \to \Pi_{F(S)\cap G}x_0$. □

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Authors' contributions

All the authors read and approved the final manuscript.

Competing interests

The authors declare that they have no completing interests.

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