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Common fixed point and invariant approximation in hyperbolic ordered metric spaces

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Abstract

We prove a common fixed point theorem for four mappings defined on an ordered metric space and apply it to find new common fixed point results. The existence of common fixed points is established for two or three noncommuting mappings where *T* is either ordered *S*-contraction or ordered asymptotically *S*-nonexpansive on a nonempty ordered starshaped subset of a hyperbolic ordered metric space. As applications, related invariant approximation results are derived. Our results unify, generalize, and complement various known comparable results from the current literature.
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1 Introduction

Metric fixed point theory has primary applications in functional analysis. The interplay between geometry of Banach spaces and fixed point theory has been very strong and fruitful. In particular, geometric conditions on underlying spaces play a crucial role for finding solution of metric fixed point problems. Although, it has purely metric flavor, it is still a major branch of nonlinear functional analysis with close ties to Banach space geometry, see for example [1-4] and references mentioned therein. Several results regarding existence and approximation of a fixed point of a mapping rely on convexity hypotheses and geometric properties of the Banach spaces. Recently, Khamsi and Khan [5] studied some inequalities in hyperbolic metric spaces, which lay foundation for a new mathematical field: the application of geometric theory of Banach spaces to fixed point theory. Meinardus [6] was the first to employ fixed point theorem to prove the existence of invariant approximation in Banach spaces. Subsequently, several interesting and valuable results have appeared about invariant approximations [7-9].

Existence of fixed points in ordered metric spaces was first investigated in 2004 by Ran and Reurings [10], and then by Nieto and Lopez [11].

In 2009, Dorić [12] proved some fixed point theorems for generalized (ψ, ϕ)-weakly contractive mappings in ordered metric spaces. Recently, Radenović and Kadelburg [13] presented a result for generalized weak contractive mappings in ordered metric spaces (see also, [14,15] and references mentioned theirin). Several authors studied the problem of existence and uniqueness of a fixed point for mappings satisfying different contractive conditions (e.g., [16-18,13,19]). The aim of this article is to study common fixed points of (i) four mappings on an ordered metric space (ii) ordered C_q-commuting mappings in the frame work of hyperbolic ordered metric spaces. Some results on invariant approximation for these mappings are also established which in turn extend and strengthen various known results.

2 Preliminaries

Let (X, d) be a metric space. A path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a metric segment joining x and y. When it is unique the metric segment is denoted by $[x, y]$. We shall denote by $(1 - \lambda)x \oplus \lambda y$ the unique point z of $[x, y]$ which satisfies

$$\mathrm{d}(x, z) = \lambda \mathrm{d}(x, y), \quad \text{and} \quad \mathrm{d}(z, y) = (1 - \lambda)\mathrm{d}(x, y).$$

Such metric spaces are usually called convex metric spaces (see Takahashi [20] and Khan at el. [21]). Moreover, if we have for all p, x, y in X

$$\mathrm{d}\left(\frac{1}{2}p \oplus \frac{1}{2}x, \frac{1}{2}p \oplus \frac{1}{2}y\right) \le \frac{1}{2}\mathrm{d}(x, y),$$

then X is called a hyperbolic metric space. It is easy to check that in this case for all x, y, z, w in X and $\lambda \in [0, 1]$

$$\mathrm{d}((1 - \lambda)x \oplus \lambda y, (1 - \lambda)z \oplus \lambda w) \le (1 - \lambda)\mathrm{d}(x, z) + \lambda \mathrm{d}(y, w).$$

Obviously, normed linear spaces are hyperbolic spaces [5]. As nonlinear examples one can consider Hadamard manifolds [2], the Hilbert open unit ball equipped with the hyperbolic metric [3] and CAT(0) spaces [4].

Let X be a hyperbolic ordered metric space. Throughout this article, we assume that $(1 - \lambda)x \oplus \lambda y \le (1 - \lambda)z \oplus \lambda w$ for all x, y, z, w in X with $x \le z$ and $y \le w$. A subset Y of X is said to be ordered convex if Y includes every metric segment joining any two of its comparable points. The set Y is said to be an ordered *q-starshaped* if there exists q in Y such that Y includes every metric segment joining any of its point comparable with q.

Let Y be an ordered q-starshaped subset of X and $f, g : Y \to Y$. Put,

$$Y_q^f = \{y_\lambda : y_\lambda = (1 - \lambda)q \oplus \lambda fx \quad \text{and} \quad \lambda \in [0, 1], \ q \le x \quad \text{or} \quad x \le q\}.$$

Set, for each x in X comparable with q in Y, $d(gx, Y_q^f) = \inf_{\lambda \in [0,1]} d(gx, y_\lambda)$.

Definition 2.1. A selfmap f on an ordered convex subset Y of a hyperbolic ordered metric space X is said to be affine if

$$f((1 - \lambda)x \oplus \lambda y) = (1 - \lambda)fx \oplus \lambda fy$$

for all comparable elements $x, y \in Y$, and $\lambda \in [0, 1]$.

Let f and g be two selfmaps on X. A point $x \in X$ is called (1) a fixed point of f if $f(x) = x$; (2) *coincidence point of a pair* (f, g) if $fx = gx$; (3) *common fixed point of a pair* (f, g) if $x = fx = gx$. If $w = fx = gx$ for some x in X, then w is called a point of coincidence of f and g. A pair (f, g) is said to be weakly compatible if f and g commute at their coincidence points.

We denote the set of fixed points of f by $Fix(f)$.

Definition 2.2. Let $(X, \leq)$ be an ordered set. A pair (f, g) on X is said:

(i) weakly increasing if for all $x \in X$, we have $fx \leq gfx$ and $gx \leq fgx$, ([22])

(ii) partially weakly increasing if $fx \leq gfx$, for all $x \in X$.

Remark 2.3. A pair (f, g) is weakly increasing if and only if ordered pair (f, g) and (g, f) are partially weakly increasing.

Example 2.4. Let $X = [0, 1]$ be endowed with usual ordering. Let $f, g : X \to X$ be defined by $fx = x^2$ and $gx = \sqrt{x}$. Then $fx = x^2 \leq x = gfx$ for all $x \in X$. Thus (f, g) is partially weakly increasing. But $gx = \sqrt{x} \nleq x = fgx$ for $x \in (0, 1)$. So (g, f) is not partially weakly increasing.

Definition 2.5. Let $(X, \leq)$ be an ordered set. A mapping f is called weak annihilator of g if $fgx \leq x$ for all $x \in X$.

Example 2.6. Let $X = [0, 1]$ be endowed with usual ordering. Define $f, g : X \to X$ by $fx = x^2$ and $gx = x^3$. Then $fgx = x^6 \leq x$ for all $x \in X$. Thus f is a weak annihilator of g.

Definition 2.7. Let $(X, \leq)$ be an ordered set. A selfmap f on X is called dominating map if $x \leq fx$ for each x in X.

Example 2.8. Let $X = [0, 1]$ be endowed with usual ordering. Let $f : X \to X$ be defined by $fx = x^{\frac{1}{3}}$. Then $x \leq x^{\frac{1}{3}} = fx$ for all $x \in X$. Thus f is a dominating map.

Example 2.9. Let $X = [0, \infty)$ be endowed with usual ordering. Define $f : X \to X$ by

$$fx = \begin{cases} \sqrt[n]{x} & \text{for } x \in [0, 1), \\ x^n & \text{for } x \in [1, \infty), \end{cases}$$

$n \in N$. Then for all $x \in X$, $x \leq fx$ so that f is a dominating map.

Definition 2.10. Let $(X, \leq)$ be a ordered set and f and g be selfmaps on X. Then the pair (f, g) is said to be order limit preserving if

$$gx_0 \leq fx_0,$$

for all sequences $\{x_n\}$ in X with $gx_n \leq fx_n$ and $x_n \to x_0$.

Definition 2.11. Let X be a hyperbolic ordered metric space, Y an ordered q-starshaped subset of X, f and g be selfmaps on X and $q \in Fix(g)$. Then f is said to be:

(1) *ordered g-contraction* if there exists $k \in (0, 1)$ such that

$$\mathrm{d}(fx, fy) \leq k\mathrm{d}(gx, gy);$$

for $x, y \in Y$ with $x \leq y$.

(2) ordered *asymptotically S-nonexpansive* if there exists a sequence $\{k_n\}$, $k_n \geq 1$, with $\lim_{n\to\infty} k_n = 1$ such that

$$\mathrm{d}(f^n(x), f^n(y)) \leq k_n \mathrm{d}(gx, gy)$$

for each x, y in Y with $x \leq y$ and each $n \in N$. If $k_n = 1$, for all $n \in N$, then f is known as ordered g-nonexpansive mapping. If $g = I$ (identity map), then f is ordered asymptotically nonexpansive mapping;

(3) *R-weakly commuting* if there exists a real number $R > 0$ such that

$$d(fgx,\ gfx) \leq Rd(fx,\ gx);$$

for all x in Y.

(4) ordered *R-subweakly commuting* [23] if there exists a real number $R > 0$ such that

$$d(fgx,\ gfx) \leq Rd(gx,\ Y_q^f)$$

for all $x \in Y$.

(5) ordered *uniformly R-subweakly commuting* [23] if there exists a real number $R > 0$ such that

$$d(f^n gx,\ gf^n x) \leq Rd(gx,\ Y_q^{f^n})$$

for all $x \in Y$.

(6) ordered *C_q-commuting* [24], if $gfx = fgx$ for all $x \in C_q(f, g)$, where $C_q(f, g) = U\{C(g, f_k) : 0 \leq k \leq 1\}$ and $f_k x = (1 - k)q \oplus kfx$.

(7) ordered *uniformly C_q-commuting*, if $gf^n x = f^n gx$ for all $x \in C_q(g, f^n)$ and $n \in N$.

(8) uniformly asymptotically regular on Y if, for each $\eta > 0$, there exists $N(\eta) = N$ such that $d(f^n x, f^{n+1} x) < \eta$ for all $\eta \geq N$ and all $x \in Y$.

For other related notions of noncommuting maps, we refer to [7]; in particular, here Example 2.2 and Remark 3.10(2) provide two maps which are not C_q-commuting. Also, uniformly C_q-commuting maps on X are C_q-commuting and uniformly R-subweakly commuting maps are uniformly C_q-commuting but the converse statements do not hold, in general [23,25]. Fixed point theorems in a hyperconvex metric space (an example of a convex metric space) have been established by Khamsi [26] and Park [27].

Let Y be a closed subset of an ordered metric space X. Let $x \in X$. Define $d(x, Y) = \inf\{d(x, y) : y \in Y, y \leq x \text{ or } x \leq y\}$. If there exists an element y_0 in Y comparable with x such that $d(x, y_0) = d(x, Y)$, then y_0 is called an ordered *best approximation to X out of Y*. We denote by $P_Y(x)$, the set of all ordered best approximation to x out of Y. The reader interested in the interplay of fixed points and approximation theory in normed spaces is referred to the pioneer work of Park [28] and Singh [9].

3 Common fixed point in ordered metric spaces

We begin with a common fixed point theorem for two pairs of partially weakly increasing functions on an ordered metric space. It may regarded as the main result of this article.

Theorem 3.1. *Let $(X, \leq, d)$ be an ordered metric space. Let f, g, S, and T be selfmaps on X, (T, f) and (S, g) be partially weakly increasing with $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$, and dominating maps f and g be weak annihilator of T and S, respectively. Also, for every two comparable elements $x, y \in X$,*

$$d(fx,\ gy) \leq hM(x,\ y),$$

where

$$M(x, y) = \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{d(Sx, gy) + d(fx, Ty)}{2}\} \tag{3.1}$$

for $h \in [0, 1)$ *is satisfied. If one of* $f(X)$, $g(X)$, $S(X)$, *or* $T(X)$ *is complete subspace of* X, *then* $\{f, S\}$ *and* $\{g, T\}$ *have unique point of coincidence in* X *provided that for a nondecreasing sequence* $\{x_n\}$ *with* $x_n \le y_n$ *for all* n *and* $y_n \to u$ *implies* $x_n \le u$. *Moreover, if* $\{f, S\}$ *and* $\{g, T\}$ *are weakly compatible, then* f, g, S, *and* T *have a common fixed point.*

Proof. For any arbitrary point x_0 in X, construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n-1} = fx_{2n-2} = Tx_{2n-1} \le fTx_{2n-1}, \quad \text{and} \quad y_{2n} = gx_{2n-1} = Sx_{2n} \le gSx_{2n}.$$

Since dominating maps f and g are weak annihilator of T and S, respectively so for all $n \ge 1$,

$$x_{2n-2} \le fx_{2n-2} = Tx_{2n-1} \le fTx_{2n-1} \le x_{2n-1},$$

and

$$x_{2n-1} \le gx_{2n-1} = Sx_{2n} \le gSx_{2n} \le x_{2n}.$$

Thus, we have $x_n \le x_{n+1}$ for all $n \ge 1$. Now (3.1) gives that.

$$\mathrm{d}(y_{2n+1}, y_{2n+2}) = \mathrm{d}(fx_{2n}, gx_{2n+1}) \le hM(x_{2n}, x_{2n+1})$$

for $n = 1, 2, 3,...$, where

$$\begin{aligned}
&M(x_{2n}, x_{2n+1}) \\
&= \max\{\mathrm{d}(Sx_{2n}, Tx_{2n+1}), \mathrm{d}(fx_{2n}, Sx_{2n}), \mathrm{d}(gx_{2n+1}, Tx_{2n+1}), \\
&\quad \frac{\mathrm{d}(fx_{2n}, Tx_{2n+1}) + \mathrm{d}(gx_{2n+1}, Sx_{2n})}{2}\Big\} \\
&= \max\{\mathrm{d}(y_{2n}, y_{2n+1}), \mathrm{d}(y_{2n+1}, y_{2n}), \mathrm{d}(y_{2n+2}, y_{2n+1}), \frac{\mathrm{d}(y_{2n+1}, y_{2n+1}) + \mathrm{d}(y_{2n+2}, y_{2n})}{2}\} \\
&= \max\{\mathrm{d}(y_{2n}, y_{2n+1}), \mathrm{d}(y_{2n+1}, y_{2n+2}), \frac{\mathrm{d}(y_{2n}, y_{2n+1}) + \mathrm{d}(y_{2n+1}, y_{2n+2})}{2}\} \\
&= \max\{\mathrm{d}(y_{2n}, y_{2n+1}), \mathrm{d}(y_{2n+1}, y_{2n+2})\}.
\end{aligned}$$

Now if $M(x_{2n}, x_{2n+1}) = \mathrm{d}(y_{2n}, y_{2n+1})$, then $\mathrm{d}(y_{2n+1}, y_{2n+2}) \le h\mathrm{d}(y_{2n}, y_{2n+1})$.
And if $M(x_{2n}, x_{2n+1}) = \mathrm{d}(y_{2n+1}, y_{2n+2})$, then $\mathrm{d}(y_{2n+1}, y_{2n+2}) \le h\mathrm{d}(y_{2n+1}, y_{2n+2})$ which implies that $\mathrm{d}(y_{2n+1}, y_{2n+2}) = 0$, and $y_{2n+1} = y_{2n+2}$. Hence

$$\mathrm{d}(y_n, y_{n+1}) \le h\mathrm{d}(y_{n-1}, y_n) \quad \text{for} \quad n = 3, 4, \ldots$$

Therefore

$$\begin{aligned}
\mathrm{d}(y_n, y_{n+1}) &\le h\mathrm{d}(y_{n-1}, x_n) \\
&\le h^2\mathrm{d}(y_{n-2}, y_{n-1}) \le \cdots \le h^n d(y_0, y_1)
\end{aligned}$$

for all $n \in \mathbb{N}$. Then, for $m > n$,

$$\begin{aligned}
\mathrm{d}(y_n, y_m) &\le \mathrm{d}(y_n, y_{n+1}) + \mathrm{d}(y_{n+1}, y_{n+2}) + \cdots + \mathrm{d}(y_{m-1}, y_m) \\
&\le [h^n + h^{n+1} + \cdots + h^m]\mathrm{d}(y_0, y_1) \\
&\le \frac{h^n}{1-h}\mathrm{d}(y_0, y_1),
\end{aligned}$$

and so $d(y_n, y_m) \to 0$ as $n, m \to \infty$. Hence $\{y_n\}$ is a Cauchy sequence. Suppose that $S(X)$ is complete. Then there exists u in $S(X)$, such that $Sx_{2n} = y_{2n} \to u$ as $n \to \infty$. Consequently, we can find v in X such that $Sv = u$. Now we claim that $fv = u$. Since, $x_{2n-2} \le x_{2n-1} \le gx_{2n-1} = Sx_{2-n}$ and $Sx_{2n} \to Sv$. So that $x_{2n-1} \le Sv$ and since, $Sv \le gSv$ and $gSv \le v$, implies $x_{2n-1} \le v$. Consider

$$\begin{aligned} d(fv, u) &\le d(fv, gx_{2n-1}) + d(gx_{2n-1}, u) \\ &\le hM(v, x_{2n-1}) + d(gx_{2n-1}, u), \end{aligned}$$

where

$$M(v, x_{2n-1}) = \max\{d(Sv, Tx_{2n-1}), d(fv, Sv), d(gx_{2n-1}, Tx_{2n-1}), \frac{d(fv, Tx_{2n-1}) + d(gx_{2n-1}, Sv)}{2}\}$$

for all $n \in \mathbb{N}$. Now we have four cases:

If $M(v, x_{2n-1}) = d(Sv, Tx_{2n-1})$, then $d(fv, u) \le hd(Sv, Tx_{2n-1}) + d(gx_{2n-1}, u) \to 0$ as $n \to \infty$ implies that $fv = u$.

If $M(v, x_{2n-1}) = d(fv, Sv)$, then $d(fv, u) \le hd(fv, Sv) + d(gx_{2n-1}, u)$. Taking limit as $n \to \infty$ we get $d(fv, u) \le hd(fv, u)$. Since $h < 1$, so that $fv = u$.

If $M(v, x_{2n-1}) = d(gx_{2n-1}, Tx_{2n-1})$, then $d(fv, u) \le hd(gx_{2n-1}, Tx_{2n-1}) + d(gx_{2n-1}, u) \to 0$ as $n \to \infty$ implies that $fv = u$.

If $M(v, x_{2n-1}) = \frac{d(fv, Tx_{2n-1}) + d(gx_{2n-1}, Sv)}{2}$, then

$$d(fv, u) \le h\frac{[d(fv, Tx_{2n-1}) + d(gx_{2n-1}, Sv)]}{2} + d(gx_{2n-1}, u).$$

Taking limit as $n \to \infty$ we get $d(fv, u) \le \frac{h}{2}d(fv, u)$. Since $h < 1$, so that $fv = u$. Therefore, in all the cases $fv = Sv = u$.

Since $u \in f(X) \subset T(X)$, there exists $w \in X$ such that $Tw = u$. Now we shall show that $gw = u$. As, $x_{2n-1} \le x_{2n} \le fx_{2n} = Tx_{2n+1}$ and $Tx_{2n+1} \to Tw$ and so $x_{2n} \le Tw$. Hence, $Tw \le fTw$ and $fTw \le w$, imply $x_{2n} \le w$. Consider

$$\begin{aligned} d(gw, u) &\le d(gw, fx_{2n}) + d(fx_{2n}, u) \\ &= d(fx_{2n}, gw) + d(fx_{2n}, u) \\ &\le hM(x_{2n}, w) + d(fx_{2n}, u), \end{aligned}$$

where

$M(x_{2n}, w) = \max\left\{d(Sx_{2n}, Tw), d(fx_{2n}, Sx_{2n}), d(gw, Tw), \frac{d(fx_{2n}, Tw) + d(gw, Sx_{2n})}{2}\right\}$ for all $n \in \mathbb{N}$.

Again we have four cases:

If $M(x_{2n},w) = d(Sx_{2n}, Tw)$, then $d(gw, u) \le h\, d(Sx_{2n}, Tw) + d(fx_{2n}, u) \to 0$ as $n \to \infty$.

If $M(x_{2n},w) = d(fx_{2n}, Sx_{2n})$, then $d(gw, u) \le h\, d(fx_{2n}, Sx_{2n}) + d(fx_{2n}, u) \to 0$ as $n \to \infty$.

If $M(x_{2n},w) = d(gw, Tw)$, then $d(gw, u) \le hd(gw, Tw) + d(fx_{2n}, u) = hd(gw, u) + d(fx_{2n}, u)$. Taking limit as $n \to \infty$ we get $d(gw, u) \le hd(gw, u)$ which implies that $gw = u$. If $M(x_{2n}, w) = \frac{d(fx_{2n}, Tw) + d(gw, Sx_{2n})}{2}$, then

$$\begin{aligned} d(gw, u) &\le h\frac{d(fx_{2n}, Tw) + d(gw, Sx_{2n})}{2} + d(fx_{2n}, u) \\ &\le \frac{h}{2}[d(fx_{2n}, u) + d(gw, Sx_{2n})] + d(fx_{2n}, u). \end{aligned}$$

Taking limit as $n \to \infty$ we get $d(gw, u) \le \frac{h}{2}d(gw, u)$ which implies that $gw = u$. Following the arguments similar to those given above, we obtain $gw = Tw = u$. Thus $\{f, S\}$ and $\{g, T\}$ have a unique point of coincidence in X. Now, if $\{f, S\}$ and $\{g, T\}$ are weakly compatible, then $fu = fSv = Sfv = Su = w_1$ (say) and $gu = gTw = Tgw = Tu = w2$ (say). Now

$$d(w_1, w_2) = d(fu, gu) \le hM(u, u),$$

where

$$\begin{aligned} M(u, u) &= \max\{d(Su, Tu), d(fu, Su), d(gu, Tu), \frac{d(fu, Tu) + d(gu, Su)}{2}\} \\ &= d(w_1, w_2). \end{aligned}$$

Therefore $d(w_1, w_2) \le hd(w_1, w_2)$ gives $w_1 = w_2$. Hence

$$fu = gu = Su = Tu.$$

That is, u is a coincidence point of f, g, S,, and T. Now we shall show that $u = gu$. Since, $v \le fv = u$,

$$\begin{aligned} d(u, gu) &= d(fv, gu) \\ &\le hM(v, u) \end{aligned}$$

where

$$\begin{aligned} M(v, u) &= \max\left\{d(Sv, Tu), d(fv, Sv), d(gw, Tu), \frac{d(fv, Tu) + d(gu, Sv)}{2}\right\} \\ &= d(u, gu). \end{aligned}$$

Thus, $d(u, gu) \le hd(u, gu)$ implies that $gu = u$. In similar way, we obtain $fu = u$. Hence, u is a common fixed point of f, g, S, and T.

In the following result, we establish existence of a common fixed point for a pair of partially weakly increasing functions on an ordered metric space by using a control function $r : R^+ \to R^+$.

Theorem 3.2. *Let* $(X, \le, d)$ *be an ordered metric space. Let f and g be R-weakly commuting selfmaps on X,* (g, f) *be partially weakly increasing with* $f(X) \subseteq g(X)$*, dominating map f is weak annihilator of g. Suppose that for every two comparable elements* $x, y \in X$,

$$d(fx, fy) \le r(d(gx, gy)),$$

where $r : R^+ \to R^+$ *is a continuous function such that* $r(t) < t$ *for each* $t > 0$*. If either f or g is continuous and one of* $f(X)$ *or* $g(X)$ *is complete subspace of X, then f and g have a common fixed point provided that for a nondecreasing sequence* $\{x_n\}$ *with* $x_n \le y_n$ *for all n and* $y_n \to u$ *implies* $x_n \le u$.

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that

$$fx_n = gx_{n+1} \preceq fgx_{n+1}.$$

Since dominating map f is weak annihilator of g, so that for all $n \geq 1$,

$$x_n \preceq fx_n = gx_{n+1} \preceq fgx_{n+1} \preceq x_{n+1}.$$

Thus, we have $x_n \leq x_n$+1 for all $n \geq 1$. Now

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq r(d(gx_n, gx_{n+1})) \\ &= r(d(fx_{n-1}, fx_n)) \\ &< d(fx_{n-1}, fx_n). \end{aligned}$$

Thus $\{d(fx_n, fx_{n+1})\}$ is a decreasing sequence of positive real numbers and, therefore, tends to a limit L. We claim that $L = 0$. For if $L > 0$, the inequality

$$d(fx_n, fx_{n+1}) \leq r(d(fx_{n-1}, fx_n))$$

on taking limit as $n \to \infty$ and in the view of continuity of r yields $L \leq r(L) < L$, a contradiction. Hence, $L = 0$.

For a given $\varepsilon > 0$, since $r(\varepsilon) < \varepsilon$, there is an integer k_0 such that

$$d(fx_n, fx_{n+1}) < \varepsilon - r(\varepsilon) \quad \forall n \geq k_0. \tag{3.2}$$

For $m, n \in N$ with $m > n$, we claim that

$$d(fx_n, fx_m) < \varepsilon \quad \forall n \geq k_0. \tag{3.3}$$

We prove inequality (3.3) by induction on m. Inequality (3.3) holds for $m = n + 1$, using inequality (3.2) and the fact that $\varepsilon - r(\varepsilon) < \varepsilon$. Assume inequality (3.3) holds for $m = k$. For $m = k + 1$, we have

$$\begin{aligned} d(fx_n, fx_m) &\leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_m) \\ &< \varepsilon - r(\varepsilon) + r(d(gx_{n+1}, gx_m)) \\ &= \varepsilon - r(\varepsilon) + r(d(fx_n, fx_{m-1})) \\ &= \varepsilon - r(\varepsilon) + r(d(fx_n, fx_k)) \\ &< \varepsilon - r(\varepsilon) + r(\varepsilon) = \varepsilon. \end{aligned}$$

By induction on m, we conclude that inequality (3.3) holds for all $m \geq n \geq k_0$.

So $\{fx_n\}$ is a Cauchy sequence. Suppose that $g(X)$ is a complete metric space. Hence $\{fx_n\}$ has a limit z in $g(X)$. Also $gx_n \to z$ as $n \to \infty$.

Let us suppose that the mapping f is continuous. Then $ffx_n \to fz$ and $fgx_n \to fz$. Further, since f and g are R - weakly commuting, we have

$$d(fgx_n, gfx_n) \leq Rd(fx_n, gx_n).$$

Taking limit as $n \to \infty$, the above inequality yields $gffx_n \to fz$. We now assert that $z = fz$. Otherwise, since $x_n \leq fx_n$, so we have the inequality

$$d(fx_n, ffx_n) \leq r(d(gx_n, gfx_n)).$$

Taking limit as $n \to \infty$ gives $d(z, fz) \leq r(d(z, fz)) < d(z, fz)$, a contradiction.

Hence, $z = fz$. As $f(X) \subseteq g(X)$, there exists z_1 in X such that $z = fz = gz_1$.

Now, since $fx_n \leq ffx_n$ and $ffxn \to fz = gz_1$ and $gz_1 \leq fgz_1 \leq z_1$ imply $fx_n \leq z_1$. Consider,

$$d(ffx_n, fz_1) \leq r(d(gfx_n, gz_1)) < d(gfx_n, gz_1).$$

Taking limit as $n \to \infty$ implies that $fz = fz_1$. This in turn implies that

$$d(fz, gz) = d(fgz_1, gfz_1) \leq Rd(fz_1, gz_1) = 0,$$

i.e., $z = fz = gz$. Thus z is a common fixed point of f and g. The same conclusion is found when g is assumed to be continuous since continuity of g implies continuity of f.

4 Results in hyperbolic ordered metric spaces

In this section, existence of common fixed points of ordered C_q-commuting and ordered uniformly C_q-commuting mappings is established in hyperbolic ordered metric spaces by utilizing the notions of ordered S-contractions and ordered asymptotically S-nonexpansive mappings.

Theorem 4.1. *Let Y be a nonempty closed ordered subset of a hyperbolic ordered metric space X. Let T and S be ordered R- subweakly commuting selfmaps on Y such that $T(Y) \subset S(Y)$, $cl(T(Y))$ is compact, $q \in Fix(S)$ and $S(Y)$ is complete and q-starshaped where each x in X is comparable with q. Let (T, S) be partially weakly increasing, order limit preserving and weakly compatible pair such that dominating map T is weak annihilator of S. If T is continuous, S-ordered nonexpansive and S is affine, then $Fix(T) \cap Fix(S)$ is nonempty provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \to u$ implies that $x_n \leq u$.*

Proof. Define $T_n : Y \to Y$ by

$$T_n(x) = (1 - \lambda_n)q \oplus \lambda_n Tx,$$

for each $n \geq 1$, where $\lambda_n \in (0, 1)$ with $\lim_{n\to\infty} \lambda_n = 1$. Then T_n is a selfmap on Y for each $n \geq 1$. Since S is ordered affine and $T(Y) \subset S(Y)$, therefor we obtain $T_n(Y) \subset S(Y)$. Note that,

$$\begin{aligned} d(T_nS_x, ST_nx) &= d((1-\lambda_n)q \oplus \lambda_n TSx, (1-\lambda_n)q \oplus \lambda_n STx) \\ &\leq (1-\lambda_n)d(q, q) + \lambda_n d(TSx, STx) \\ &= \lambda_n d(TSx, STx) \\ &\leq \lambda_n Rd(Sx, (1-\lambda_n)q \oplus \lambda_n Tx) \\ &= \lambda_n Rd(Sx, T_nx). \end{aligned}$$

This implies that the pair $\{T_n, S\}$ is ordered $\lambda_n R$-weakly commuting for each n. Also for any two comparable elements x and y in X, we get

$$\begin{aligned} d(T_nx, T_ny) &= d((1-\lambda_n)q \oplus \lambda_n Tx, (1-\lambda_n)q \oplus \lambda_n Ty) \\ &\leq \lambda_n d(Tx, Ty) \leq \lambda_n d(Sx, Sy). \end{aligned}$$

Now following lines of the proof of Theorem 3.2, there exists x_n in Y such that x_n is a common fixed point of S and T_n for each $n \geq 1$. Note that

$$\begin{aligned} d(x_n, Tx_n) &= d(T_nx_n, Tx_n) = d((1-\lambda_n)q \oplus \lambda_n Tx_n, Tx_n) \\ &= (1-\lambda_n)d(q, Tx_n). \end{aligned}$$

Since $cl(T(Y))$ is compact, there exists a positive integer M such that

$$d(x_n, Tx_n) \leq (1-\lambda_n)M.$$

The compactness of $cl(T_n(Y))$ implies that there exists a subsequence $\{x_k\}$ of $\{x_n\}$ such that $x_k \to x_0 \in Y$ as $k \to \infty$. Now,

$$d(x_0, Tx_0) \leq d(Tx_0, Tx_k) + d(Tx_k, x_k) + d(x_k, x_0)$$

and continuity of T give that $x_0 \in Fix(T)$. Since, T is dominating map, therefore $Sx_k \leq TSx_k$. As T is weak annihilator of S and T is dominating, so $TSx_k \leq x_k \leq Tx_k$. Thus $Sx_k \leq Tx_k$ and order limit preserving property of (T, S) implies that $Sx_0 \leq Tx_0 = x_0$. Also $x_0 \leq Sx_0$. Consequently, $Sx_0 = Tx_0 = x_0$. Hence the result follows.

Theorem 4.2. *Let Y be a nonempty closed subset of a complete hyperbolic ordered metric space X and let T and S be mappings on Y such that $T(Y - \{u\}) \subset S(Y - \{u\})$, where $u \in Fix(S)$. Suppose that T is an S-contraction and continuous. Let (T, S) be partially weakly increasing, dominating maps T is weak annihilator of S. If T is continuous, and S and T are R-weakly commuting mappings on $Y - \{u\}$, then $Fix(T) \cap Fix(S)$ is nonempty provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies $x_n \leq u$.*

Proof. Similar to the proof of Theorem 3.2.

Theorem 3.1 yields a common fixed point result for a pair of maps on an ordered startshaped subset Y of a hyperbolic ordered metric space as follows.

Theorem 4.3. *Let Y be a nonempty closed q- starshaped subset of a complete hyperbolic ordered metric space X and let T and S be uniformly C_q- commuting selfmapps on $Y - \{q\}$ such that $S(Y) = Y$ and $T(Y - \{q\}) \subset S(Y - \{q\})$,* where $q \in Fix(S)$. *Let (T, S) be partially weakly increasing, order limit preserving and weakly compatible pair, dominating map T is weak annihilator of S, T is continuous and asymptotically S- nonexpansive with sequence $\{k_n\}$, as in Definition 2.11 (2), and S is an affine mapping. For each $n \geq 1$, define a mapping T_n on Y by $T_n x = (1 - \alpha_n)q \oplus \alpha_n T^n x$, where $\alpha_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\lim_{n\to\infty} \lambda_n = 1$. Then for each $n \in N$, $F(T_n) \cap Fix(S)$ is nonempty provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies $x_n \leq u$.*

Proof. For all $x, y \in Y$, we have

$$\begin{aligned} & d(T_n(x), T_n(y)) \\ = & d((1-\alpha_n)q \oplus \alpha_n T^n x, (1-\alpha_n)q \oplus \alpha_n T^n y) \\ \leq & \alpha_n d(T^n(x), T^n(y)) \leq \lambda_n d(Sx, Sy). \end{aligned}$$

Moreover, since T and S are uniformly C_q-commuting and S is affine on Y with $Sq = q$, for each $x \in C_n(S, T) \subseteq C_q(S, T)$, we have

$$\begin{aligned} ST_n x &= S((1-\alpha_n)q \oplus \alpha_n T^n x) = (1-\alpha_n)q \oplus \alpha_n ST^n x \\ &= (1-\alpha_n)q \oplus \alpha_n T^n Sx = T_n Sx. \end{aligned}$$

Thus S and T_n are weakly compatible for all n. Now, the result follows from Theorem 3.1.

The above theorem leads to the following result.

Theorem 4.4. *Let Y be a nonempty closed q- starshaped subset of a hyperbolic ordered metric space X and let T and S be selmaps on Y such that $S(Y) = Y$ and $T(Y - \{q\}) \subset S(Y - \{q\})$, $q \in Fix(S)$. Let (T, S) be partially weakly increasing, order limit preserving, T is continuous, uniformly asymptotically regular, asymptotically S-nonexpansive* *and S is an affine mapping. If cl(Y - {q}) is compact and S and T are uniformly C_q-commuting selfmaps on Y - {q}, then Fix(T) ∩ Fix(S) is nonempty provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \leq yn$ for all n and $y_n \to u$ implies $x_n \leq u$.*

Proof. By Theorem 4.3, for each $n \in N$, $F(T_n) \cap Fix(S)$ is singleton in Y. Thus,

$$Sx_n = x_n = (1-\alpha_n)q \oplus \alpha_n T^n x_n.$$

Also,

$$\begin{aligned} d(x_n, T^n x_n) &= d((1-\alpha_n)q \oplus \alpha_n T^n x_n, T^n x_n) \\ &= (1-\alpha_n)d(q, T^n x_n). \end{aligned}$$

Since $T(Y - \{q\})$ is bounded so $d(x_n, T^n x_n) \to 0$ as $n \to \infty$. Note that,

$$\begin{aligned} & d(x_n, Tx_n) \\ \leq\ & d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\ \leq\ & d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_l d(ST^n x_n, Sx_n) \\ \leq\ & d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_l d(ST^n x_n, S((1-\alpha_n)q \oplus \alpha_n T^n x_n)) \\ \leq\ & d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_l d(ST^n x_n, (1-\alpha_n)q \oplus \alpha_n ST^n x_n) \\ \leq\ & d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1(1-\alpha_n)d(ST^n x_n, Sq) \\ \leq\ & d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1(1-\alpha_n)d(ST^n x_n, Sq). \end{aligned}$$

Consequently, $d(x_n, Tx_n) \to 0$, when $n \to \infty$. Since $cl(Y - \{q\})$ is compact and Y is closed, therefore there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x_0 \in Y$ as $i \to \infty$. By the continuity of T , we have $T(x_0) = x_0$. Since, T is dominating map, therefore $Sx_k \leq TSx_k$. As T is weak annihilator of S and T is dominating, so $TSx_k \leq x_k \leq Tx_k$. Thus, $Sx_k \leq Tx_k$ and order limit preserving property of (T, S) implies that $Sx_0 \leq Tx_0 = x_0$. Also $x_0 \leq Sx_0$. Consequently, $Sx_0 = Tx_0 = x_0$. Hence, the result follows.

As another application of Theorem 3.1, we obtain yet an other result for two maps satisfying a very general contractive condition on the set Y.

Theorem 4.5. Let Y be a nonempty q-starshaped complete subset of a hyperbolic ordered metric space and T, f, and g be selfmaps on Y . Suppose that T is continuous, $cl(T(Y))$ is compact and f and g are affine and continuous and $T(Y) \subset f(Y) \cap g(Y)$. Let (T, f) and (T, g) be partially weakly increasing, and dominating maps f and g be weak annihilators of T. If the pairs $\{T, f\}$ and $\{T, g\}$ are C_q-commuting and satisfy for all $x, y \in Y$,

$$\begin{aligned} d(Tx, Ty) \leq \max\{& d(fx, gy),\ d(fx, Y_q^T),\ d(gy, Y_q^T), \\ & \frac{1}{2}[d(fx, Y_q^T) + d(gy, Y_q^T)]\}, \end{aligned} \tag{4.1}$$

then T, f, and g have a common fixed point provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies $x_n \leq u$.

Proof. Define $T_n : Y \to Y$ by

$$T_n(x) = (1-\lambda_n)q \oplus \lambda_n Tx,$$

where $\lambda_n \in (0, 1)$ with $\lim_{n\to\infty} \lambda_n = 1$. Then T_n is a selfmap on Y for each $n \geq 1$. Since f and g are affine and $T(Y) \subset f(Y) \cap g(Y)$, therefore we obtain $T_n(Y) \subset f(Y) \cap g(Y)$.

Now f and T are C_q-commuting and f is affine on Y with $fq = q$, for each $x \in C_n(f, T) \subseteq C_q(f, T)$, so we have

$$\begin{aligned} fT_n x &= f((1-\lambda_n)q \oplus \lambda_n Tx) = (1-\lambda_n)q \oplus \lambda_n fTx \\ &= (1-\lambda_n)q \oplus \lambda_n Tfx = T_n fx. \end{aligned}$$

Thus, f and T_n are weakly compatible for all n. Also since g and T are C_q-commuting and g is affine on Y with $gq = q$, therefore, g and T_n are weakly compatible for all n. Moreover using (4.1) we have

$$\begin{aligned} d(T_n x, T_n y) &\le \lambda_n d(Tx, Ty) \\ &\le \lambda_n \max\{d(fx, gy), d(fx, Y_q^{T(x)}), \\ &\quad d(gy, Y_q^{T(y)}), \frac{1}{2}[d(fx, Y_q^{T(y)}) + d(gy, Y_q^{T(x)})]\Big\} \\ &\le \lambda_n \max\{d(fx, gy), d(fx, T_n x), \\ &\quad d(gy, T_n y), \frac{1}{2}[d(fx, T_n y) + d(gy, T_n x)]\}. \end{aligned}$$

By Theorem 3.1, for each $n \ge 1$, there exists x_n in Y such that x_n is a common fixed point of f, g and T_n. The compactness of $cl(T(Y))$ implies that there exists a subsequence $\{Tx_k\}$ of $\{Tx_n\}$ such that $Tx_k \to y$ as $k \to \infty$. Now, the definition of $T_k x_k$ gives that $x_k \to y$ and the result follows using continuity of T, f, and g.

5 Invariant approximation

In this section, we obtain results on best approximation as a fixed point of R-subweakly and uniformly R-subweakly commuting mappings in the setting of hyperbolic ordered metric spaces. In particular, as an application of Theorem 4.4 (respectively Theorem 4.5), we demonstrate the existence of common fixed point for one pair (respectively two pairs) of maps from the set of best approximation.

Theorem 5.1. Let M be a nonempty subset of a hyperbolic ordered metric space X, T, and S be continuous selfmaps on X such that $T(\partial M \cap M) \subset M$, ∂M stands for boundary of M, and $u \in Fix(S) \cap Fix(T)$ for some u in X, where u is comparable with all $x \in X$. Let (T, S) be partially weakly increasing, order limit preserving, T is uniformly asymptotically regular, asymptotically S-nonexpansive and S is affine on $P_M(u)$ with $S(P_M(u)) = P_M(u)$, $q \in Fix(S)$, and $P_M(u)$ is q-starshaped. If $cl(P_M(u))$ is compact, $P_M(u)$ is complete and S and T are uniformly C_q-commuting mappings on $P_M(u) \cup \{u\}$ satisfying $d(Tx, Tu) \le d(Sx, Su)$, then $P_M(u) \cap Fix(T) \cap Fix(S) \ne \varphi$ provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \le y_n$ for all n and $y_n \to u$ implies $x_n \le u$.

Proof. Let $x \in P_M(u)$. Then $d(x, u) = d(u, M)$. Note that for any $\lambda \in (0, 1)$,

$$\begin{aligned} d(y_\lambda, u) &= d((1-\lambda)u \oplus \lambda x, u) \\ &= \lambda d(x, u) < d(x, u) = d(u, M). \end{aligned}$$

This shows that $Y_\lambda^I = \{y_\lambda : y_\lambda = (1-\lambda)u \oplus \lambda x\} \cap M = \phi$. So $x \in \partial M \cap M$ which further implies that $Tx \in M$. Since $Sx \in P_M(u)$, u is a common fixed point of S and T, therefore by the given contractive condition, we obtain

$$\begin{aligned} d(Tx, u) &= d(Tx, Tu) \\ &\le d(Sx, Su) = d(Sx, u) = d(u, M). \end{aligned}$$

Thus, $P_M(u)$ is T -invariant. Hence,

$$T(P_M(u)) \subset P_M(u) = S(P_M(u)).$$

Now the result follows from Theorem 4.4.

Theorem 5.2. Let M be a nonempty subset of a hyperbolic ordered metric space X, T, f, and g be selfmaps on X such that u is common fixed point of f, g, and T and $T(\partial M \cap M) \subset M$. Suppose that f and g are continuous and affine on $P_M(u)$, $q \in Fix(f) \cap Fix(g)$, and $P_M(u)$ is q-starshaped with $f(P_M(u)) = P_M(u) = g(P_M(u))$. Let (T, f) and (T, g) be partially weakly increasing, and dominating maps f and g be weak annihilator of T. Assume that the pairs $\{T, f\}$ and $\{T, g\}$ are C_q-commuting and satisfy for all $x \in P_M(u) \cup \{u\}$

$$d(Tx, Ty) \le \begin{cases} d(fx, gu), & \text{if } y = u \\ \max\{d(fx, gy), d(fx, Y_q^T), d(gy, Y_q^T), \\ \frac{1}{2}[d(fx, Y_q^T) + d(gy, Y_q^T)]\}, & \text{if } y \in P_M(u). \end{cases}$$

If $cl(P_M(u))$ is compact and $P_M(u)$ is complete, then $P_M(u) \cap Fix(T) \cap Fix(f) \cap Fix(g) \neq \varphi$ provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \le y_n$ for all n and $y_n \to u$ implies $x_n \le u$.

Proof. Let $x \in P_M(u)$. Then $d(x, u) = d(u, M)$. Note that for any $\lambda \in (0, 1)$

$$\begin{aligned} d(y_\lambda, u) &= d((1-\lambda)u \oplus \lambda x, u) \\ &= \lambda d(x, u) < d(x, u) = d(u, M), \end{aligned}$$

which shows that M and $Y_\lambda^x = \{y_\lambda : y_\lambda = (1-\lambda)u \oplus \lambda x\}$ are disjoint. So $x \in \partial M \cap M$ which further implies that $Tx \in M$. Since $fx \in P_M(u)$, u is a common fixed point of f, g, and T, therefore by the given contractive condition, we obtain

$$\begin{aligned} d(Tx, u) &= d(Tx, Tu) \\ &\le d(fx, gu) = d(fx, u) = d(u, M). \end{aligned}$$

Thus $P_M(u)$ is T -invariant. Hence,

$$T(P_M(u)) \subset P_M(u) = f(P_M(u)) = g(P_M(u)).$$

The result follows from Theorem 4.5.

Remark 5.3.

(a) Theorem 3.2 extends and improves Theorem 2.2 of Al-Thagafi [8] and Theorem 2.2(i) of Hussain and Jungck [25] in the setup of hyperbolic ordered metric spaces.

(b) Theorems 4.4 and 4.5 extend the results in [23] to more general classes of mappings defined on a hyperbolic ordered metric space.

(c) Theorems 5.1 and 5.2 set analogues of Theorems 2.11(i) and 2.12(i) in [25], respectively.

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Authors' contributions

The authors have contributed in this work on an equal basis. All authors have read and approved the final manuscript.

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