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Fixed point results under c-distance in tvs-cone metric spaces

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Abstract

Fixed point and common fixed point results for mappings in tvs-cone metric spaces (with the underlying cone which is not normal) under contractive conditions expressed in the terms of *c*-distance are obtained. Respective results concerning mappings without periodic points are also deduced. Examples are given to distinguish these results from the known ones.

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1 Introduction

Cone metric spaces were considered by Huang and Zhang in [1], who reintroduced the concept which has been known since the middle of 20th century (see, e.g., [2-4]). Topological vector space-valued version of these spaces was treated in [5-13]; see also [14] for a survey of fixed point results in these spaces.

Fixed point theorems in metric spaces with the so-called *w*-distance were obtained for the first time by Kada et al. in [15] where nonconvex minimization problems were treated. Further results were given, e.g., in [16-18]. Cone metric version of this notion (usually called a *c*-distance) was used, e.g., in [19,20].

In this paper, we consider fixed point and common fixed point results for mappings in tvs-cone metric spaces (with the underlying cone which is not normal) under contractive conditions expressed in the terms of *c*-distance. Respective results concerning mappings without periodic points are also deduced. Examples are given to distinguish these results from the known ones.

2 Preliminaries

Let E be a real Hausdorff topological vector space (*tvs* for short) with the zero vector θ. A proper nonempty and closed subset P of E is called a *cone* if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \{\theta\}$. We shall always assume that the cone P has a nonempty interior int P (such cones are called *solid*).

Each cone P induces a partial order $\preccurlyeq$ on E by $x \preccurlyeq y \Leftrightarrow y - x \in P$. $x \pi y$ will stand for ($x \preccurlyeq y$ and $x \neq y$), while $x \ll y$ will stand for $y - x \in \operatorname{int} P$. The pair (E, P) is an *ordered topological vector space*.

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For a pair of elements x, y in E such that $x \preccurlyeq y$, put $[x, y] = \{z \in E : x \preccurlyeq z \preccurlyeq y\}$. A subset A of E is said to be *order-convex* if $[x, y] \subset A$, whenever $x, y \in A$ and $x \preccurlyeq y$.

Ordered topological vector space (E, P) is *order-convex* if it has a base of neighborhoods of θ consisting of order-convex subsets. In this case, the cone P is said to be *normal*. If E is a normed space, this condition means that the unit ball is order-convex, which is equivalent to the condition that there is a number k such that $x, y \in E$ and $0 \preccurlyeq x \preccurlyeq y$ implies that $||x|| \leq k||y||$. A proof of the following assertion can be found, e.g., in [2].

Theorem 1 *If the underlying cone of an ordered tvs is solid and normal, then such tvs must be an ordered normed space.*

Note that completions of cone metric spaces in the case of nonnormal underlying cone were treated in [21].

From [1,5-7], we give the following

Definition 1 Let X be a nonempty set and (E, P) an ordered *tvs*. A function $d : X \times X \to E$ is called a *tvs-cone metric* and (X, d) is called a *tvs-cone metric space* if the following conditions hold:

(c1) $\theta \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(c2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c3) $d(x, z) \preccurlyeq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Taking into account Theorem 1, proper generalizations when passing from norm-valued cone metric spaces of [1] to tvs-cone metric spaces can be obtained only in the case of nonnormal cones. We shall make use of the following properties:

(p_1) If $u, v, w \in E$, $u \preccurlyeq v$ and $v \ll w$ then $u \ll w$.

(p_2) If $u \in E$ and $\theta \preccurlyeq u \ll c$ for each $c \in \operatorname{int} P$ then $u = \theta$.

(p_3) If $u_n, v_n, u, v \in E$, $\theta \preccurlyeq u_n \preccurlyeq v_n$ for each $n \in \mathbb{N}$, and $u_n \to u$, $v_n \to v$ $(n \to \infty)$, then $\theta \preccurlyeq u \preccurlyeq v$.

(p_4) If $x_n, x \in X$, $u_n \in E$, $d(x_n, x) \preccurlyeq u_n$ and $u_n \to \theta$ $(n \to \infty)$, then $x_n \to x$ $(n \to \infty)$.

(p_5) If $u \preccurlyeq \lambda u$, where $u \in P$ and $0 \leq \lambda < 1$, then $u = \theta$.

(p_6) If $c \gg \theta$ and $u_n \in E$, $u_n \to \theta$ $(n \to \infty)$, then there exists n_0 such that $u_n \ll c$ for all $n \geq n_0$.

In the sequel, E will always denote a topological vector space, with the zero vector θ and with order relation $\preccurlyeq$, generated by a solid cone P. For notions such as convergent and Cauchy sequences, completeness, continuity etc. in (tvs)-cone metric spaces, we refer to [1,7,14] and references therein.

Kada et al. [15] introduced the notion of w-distance in metric spaces and proved some fixed point results using this notion (see also [16-18]). Cho et al. [19] transferred it to the setting of cone metric spaces (see also [20]).

Definition 2 [19] Let (X, d) be a tvs-cone metric space. A function $q : X \times X \to E$ is called a *c-distance* in X if:

(q1) $\theta \preccurlyeq q(x, y)$ for all $x, y \in X$;

(q2) $q(x, z) \preccurlyeq q(x, y) + q(y, z)$ for all $x, y, z \in X$;

(q3) If a sequence $\{y_n\}$ in X converges to a point $y \in X$, and for some $x \in X$ and $u = u_x \in P$, $q(x, y_n) \preccurlyeq u$ holds for each $n \in \mathbb{N}$, then $q(x, y) \preccurlyeq u$;

(q4) For each $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \preccurlyeq e$, such that $q(z, x) \ll e$ and $q(z, y) \ll e$ implies $d(x, y) \ll c$.

Each w-distance q in a metric space (X, d) (in the sense of [15]) is a c-distance in the tvs-cone metric space (X, d) (with $E = \mathbb{R}$ and $P = [0, +\infty)$). Indeed, only property (q3) has to be checked. Let $y_n \in X$, $y_n \to y$ in the cone metric d ($n \to \infty$), and let $q(x, y_n) \leq u_x \in [0, +\infty)$. Since q is (as a w-distance) lower semi-continuous, we have that $q(x, y) \leq \liminf_{n \to \infty} q(x, y_n) \leq \liminf_{n \to \infty} u_x = u_x$, i.e., $q(x, y) \leq u_x$ holds true.

The first two of the following examples are variations of [[19], Examples 2.7, 2.8], adjusted to the case of a tvs-cone metric.

Example 1 Let (X, d) be a tvs-cone metric space such that the metric $d(\cdot,\cdot)$ is a continuous function in second variable. Then, $q(x, y) = d(x, y)$ is a c-distance. Indeed, only property (q3) is nontrivial and it follows from $q(x, y_n) = d(x, y_n) \preccurlyeq u$, passing to the limit when $n \to \infty$ and using continuity of d.

Example 2 Let (X, d) be a tvs-cone metric space, and let $u \in X$ be fixed. Then, $q(x, y) = d(u, y)$ defines a c-distance on X. Indeed, (q1) and (q3) are clear. (q2) follows from $q(x, z) = d(u, z) \preccurlyeq d(u, y) + d(u, z) = q(x, y) + q(y, z)$. Finally, (q4) is obtained by taking $e = c/2$.

Example 3 Consider the Banach space $E = C[0, 1]$ of real-valued continuous functions with the max-norm and ordered by the cone $P = \{f \in E : f(t) \geq 0 \text{ for } t \in [0, 1]\}$. This cone is normal in the Banach-space topology on E. Let τ^* be the strongest locally convex topology on the vector space E. Then, the cone P is solid, but it is not normal in the topology τ^*. Indeed, if this were the case, Theorem 1 would imply that the topology τ^* is normed, which is impossible since an infinite dimensional space with the strongest locally convex topology cannot be metrizable (see, e.g., [14]).

Let now $X = [0, +\infty)$ and $d : X \times X \to (E, \tau^*)$ be defined by $d(x, y)(t) = |x - y|\phi(t)$ for a fixed element $\phi \in P$. Then, (X, d) is a tvs-cone metric space which is not a cone metric space in the sense of [1]. We can introduce two c-distances on this space:

$$q_1(x, y)(t) = x \cdot \varphi(t), \quad \text{and} \quad q_2(x, y)(t) = y \cdot \varphi(t).$$

They are the examples of c-distances in tvs-cone metric spaces which are not c-distances in cone metric spaces of [19,20].

These examples show, among other things, that for a c-distance q:

1. $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$;
2. $q(x, y) = \theta$ is not necessarily equivalent to $x = y$.

3 Results

3.1 Fixed point and common fixed point results under c-distance

We will call a sequence $\{u_n\}$ in P a *c-sequence* if for each $c \gg \theta$ there exists $n_0 \in \mathrm{N}$ such that $u_n \ll c$ for $n \geq n_0$. It is easy to show that if $\{u_n\}$ and $\{v_n\}$ are c-sequences in E and $\alpha, \beta > 0$, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence.

Note that in the case that the cone P is normal, a sequence in E is a c-sequence iff it is a θ -sequence (see property (p_6)). However, when the cone is not normal, a c-sequence need not be a θ -sequence (see [7,14]). Also, from [7], we know that the cone metric d need not be a continuous function.

The following lemma is a tvs-cone metric version of lemmas from [15,19].

Lemma 1 *Let (X, d) be a tvs-cone metric space and let q be a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\times$ and $x, y, z \in X$. Suppose that $\{u_n\}$ and $\{v_n\}$ are c-sequences in P. Then the following hold:*

(1) If $q(x_n, y) \preccurlyeq u_n$ and $q(x_n, z) \preccurlyeq v_n$ for $n \in \mathbb{N}$, then $y = z$. In particular, if $q(x, y) = \theta$ and $q(x, z) = \theta$, then $y = z$.

(2) If $q(x_n, y_n) \preccurlyeq u_n$ and $q(x_n, z) \preccurlyeq v_n$ for $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.

(3) If $q(x_n, x_m) \preccurlyeq u_n$ for $m > n > n_0$, then $\{x_n\}$ is a Cauchy sequence in X.

(4) If $q(y, x_n) \preccurlyeq u_n$ for $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.

Proof We will prove assertions (1) and (2). Proofs of the other two are similar.

(1) In order to prove that $y = z$, according to (p_2), it is enough to show that $d(y, z) \ll c$ for each $c \gg \theta$. For the given c choose $e \gg \theta$ such that property (q4) is satisfied. Choose then $n_0 \in \mathbb{N}$ such that $u_n \ll e$ and $v_n \ll e$ for $n \geq n_0$. Then, by property (p_1), we get that $q(x_n, y) \ll e$ and $q(x_n, z) \ll e$ and (q4) imply that $d(y, z) \ll c$.

(2) Let again $c \ll \theta$ be arbitrary and choose a corresponding $e \gg \theta$ satisfying property (q4). If $n_0 \in$ N is such that $u_n \ll e$ and $v_n \ll e$ for $n \geq n_0$, then (p_1) implies that $q(x_n, y_n) \ll e$ and $q(x_n, z) \ll e$ for $n \geq n_0$. Then, by (q4), $d(y_n, z) \ll c$ and $y_n \to z$ $(n \to \infty)$. ■

Our first result is the following theorem of Hardy-Rogers type.

Theorem 2 *Let (X, d) be a complete tvs-cone metric space and let q be a c-distance on X. Suppose that a continuous self-map $f : X \to X$ satisfies the following two conditions:*

$$q(fx, fy) \preccurlyeq Aq(x, y) + Bq(x, fx) + Cq(y, fy) + Dq(x, fy) + Eq(y, fx), \tag{3.1}$$

$$q(fy, fx) \preccurlyeq Aq(y, x) + Bq(fx, x) + Cq(fy, y) + Dq(fy, x) + Eq(fx, y) \tag{3.2}$$

for all $x, y \in X$, where A, B, C, D, E are nonnegative constants such that $A + B + C + 2D + 2E < 1$. Then f has a fixed point in X. If $fu = u$, then $q(u, u) = \theta$.

Proof Let $x_0 \in X$ be arbitrary and form the sequence $\{x_n\}$ with $x_n = f^n x_0$. In order to prove that it is a Cauchy sequence, put $x = x_n$ and $y = x_{n-1}$ in (3.1) to get

$$\begin{aligned} q(x_{n+1}, x_n) &\preccurlyeq Aq(x_n, x_{n-1}) + Bq(x_n, x_{n+1}) + Cq(x_{n-1}, x_n) \\ &\quad + Dq(x_n, x_n) + Eq(x_{n-1}, x_{n+1}) \\ &\preccurlyeq Aq(x_n, x_{n-1}) + (B + D + E)q(x_n, x_{n+1}) \\ &\quad + (C + E)q(x_{n-1}, x_n) + Dq(x_{n+1}, x_n). \end{aligned} \tag{3.3}$$

Similarly, putting $y = x_{n-1}$ and $x = x_n$ in (3.2), one obtains

$$\begin{aligned} q(x_n, x_{n+1}) &\preccurlyeq Aq(x_{n-1}, x_n) + Bq(x_{n+1}, x_n) + Cq(x_n, x_{n-1}) \\ &\quad + Dq(x_n, x_n) + Eq(x_{n+1}, x_{n-1}) \\ &\preccurlyeq Aq(x_{n-1}, x_n) + (B + D + E)q(x_{n+1}, x_n) \\ &\quad + (C + E)q(x_n, x_{n-1}) + Dq(x_n, x_{n+1}). \end{aligned} \tag{3.4}$$

Denote $u_n = q(x_{n+1}, x_n) + q(x_n, x_{n+1})$. Adding up (3.3) and (3.4), we get that

$$u_n \preccurlyeq (A + C + E)u_{n-1} + (B + 2D + E)u_n,$$

i.e. $u_n \preccurlyeq hu_{n-1}$ with

$$0 \leq h = \frac{A + C + E}{1 - B - 2D - E} < 1,$$

since $A + B + C + 2D + 2E < 1$ and, e.g., $A + C + E > 0$.

By induction, $u_n \preccurlyeq h^n u_0$ and $q(x_n, x_{n+1}) \preccurlyeq u_n \preccurlyeq h^n(q(x_1, x_0) + q(x_0, x_1))$. In the usual way, it follows that

$$q(x_n, x_m) \preccurlyeq \frac{h^n}{1-h}(q(x_1, x_0) + q(x_0, x_1)) = v_n$$

for $m > n$, where $\{v_n\}$ is a c-sequence. Lemma 1.(3) implies that $\{x_n\}$ is a Cauchy sequence in X and, since X is complete, $x_n \to x^* \in X$ $(n \to \infty)$. Continuity of f implies that $x_{n+1} = fx_n \to fx^*$, and since the limit of a sequence in tvs-cone metric space in unique, we get that $fx^* = x^*$.

Suppose that $fu = u$. Then, (3.1) implies that

$$\begin{aligned} q(u, u) = q(fu, fu) &\preccurlyeq Aq(u, u) + Bq(u, u) + Cq(u, u) + Dq(u, u) + Eq(u, u) \\ &= (A + B + C + D + E)q(u, u), \end{aligned}$$

which is, by property (p_5) and $A + B + C + D + E < A + B + C + 2D + 2E < 1$, possible only if $q(u, u) = \theta$. ■

Some special cases of the previous theorem, for example Banach-type and Kannan-type fixed point results, need only one condition:

$$q(fx, fy) \preccurlyeq \lambda q(x, y), \qquad \lambda \in [0, 1),$$

and

$$q(fx, fy) \preccurlyeq \lambda(q(x, fx) + q(y, fy)), \qquad \lambda \in [0, 1/2),$$

respectively.

Remark 1 If the underlying cone P of the given tvs-cone metric space (X, d) is normal (and, hence, this space is a cone metric space in the sense of [1], see Theorem 2.1), then continuity of f in Theorem 2 can be replaced by the condition

$$\inf\{||q(x, y)|| + ||q(x, fx)|| \ : x \in X\} > 0 \text{ for all } y \in X \text{ with } y \neq fy.$$

It may be of interest to note that in this case, property (q3) of c-distance has to be used in the course of the proof (see, e.g., the respective procedure in ordered cone metric spaces in [19]), while in our case (when f is continuous), this property is not needed.

The next is a result including two mappings and the existence of their common fixed point.

Theorem 3 *Let (X, d) be a complete tvs-cone metric space and let q be a c-distance on X. Suppose that continuous self-maps $f, g : X \to X$ satisfy the following two conditions:*

$$q(fx, gy) \preccurlyeq Aq(x, y) + B[q(x, fx) + q(y, gy)] + D[q(x, gy) + q(y, fx)], \tag{3.5}$$

$$q(gy, fx) \preccurlyeq Aq(y, x) + B[q(fx, x) + q(gy, y)] + D[q(gy, x) + q(fx, y)] \tag{3.6}$$

for all $x, y \in X$, where A, B, D are nonnegative constants, such that $A + 2B + 4D < 1$. Then f and g have a common fixed point in X. If $fu = gu = u$, then $q(u, u) = \theta$.

Proof Let $x_0 \in X$ be arbitrary and form the sequence $\{x_n\}$ such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for $n \geq 0$. Denote $u_n = q(x_{2n}, x_{2n+1})+q(x_{2n+1}, x_{2n})$ and $v_n = q(x_{2n+1}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})$.

Putting $x = x_{2n+2}$, $y = x_{2n+1}$ in (3.5) we obtain that

$$\begin{aligned} q(x_{2n+3}, x_{2n+2}) &\preccurlyeq Aq(x_{2n+2}, x_{2n+1}) + B[q(x_{2n+2}, x_{2n+3}) + q(x_{2n+1}, x_{2n+2})] \\ &\quad + D[q(x_{2n+2}, x_{2n+2}) + q(x_{2n+1}, x_{2n+3})] \\ &\preccurlyeq Aq(x_{2n+2}, x_{2n+1}) + (B+2D)q(x_{2n+2}, x_{2n+3}) \\ &\quad + (B+D)q(x_{2n+1}, x_{2n+2}) + Dq(x_{2n+3}, x_{2n+2}). \end{aligned} \tag{3.7}$$

Similarly, putting the same values for x, y in (3.6), we get

$$\begin{aligned} q(x_{2n+2}, x_{2n+3}) &\preccurlyeq Aq(x_{2n+1}, x_{2n+2}) + B[q(x_{2n+3}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})] \\ &\quad + D[q(x_{2n+2}, x_{2n+2}) + q(x_{2n+3}, x_{2n+1})] \\ &\preccurlyeq Aq(x_{2n+1}, x_{2n+2}) + (B+2D)q(x_{2n+3}, x_{2n+2}) \\ &\quad + (B+D)q(x_{2n+2}, x_{2n+1}) + Dq(x_{2n+2}, x_{2n+3}). \end{aligned} \tag{3.8}$$

It follows by adding up (3.7) and (3.8) that

$$u_{n+1} \preccurlyeq (A+B+D)v_n + (B+3D)u_{n+1},$$

i.e.,

$$u_{n+1} \preccurlyeq hv_n, \quad n \in \mathbb{N},$$

where $0 < h = \dfrac{A+B+D}{1-B-3D} < 1$, since $A + B + D > 0$ and $A + 2B + 4D < 1$.

By a similar procedure, starting with $x = x_{2n}$ and $y = x_{2n+1}$, one can get

$$v_n \preccurlyeq hu_n, \quad n \in \mathbb{N}.$$

Combining the last two inequalities, it follows that

$$u_{n+1} \preccurlyeq h^2 u_n \quad \text{and} \quad v_n \preccurlyeq h^2 v_{n-1},$$

and we get that $\{u_n\}$ and $\{v_n\}$ are c-sequences. We have that $q(x_{2n}, x_{2n+1}) \preccurlyeq u_n$, $q(x_{2n+1}, x_{2n+2}) \preccurlyeq v_n$ and it follows that $q(x_n, x_{n+1}) \preccurlyeq u_n + v_n$, where $u_n + v_n$ is a c-sequence. Using Lemma 1.(3), we obtain that $\{x_n\}$ is a Cauchy sequence in X. Hence, $x_n \to x^* \in X$ $(n \to \infty)$. Since f and g are continuous, it easily follows from the definition of $\{x_n\}$ that $fx^* = gx^* = x^*$.

Thus, mappings f and g have a common fixed point. Suppose that $u \in X$ is any point satisfying $fu = gu = u$. Then, (3.5) implies that

$$\begin{aligned} q(u,u) = q(fu, gu) &\preccurlyeq Aq(u,u) + B[q(u,u) + q(u,u)] + D[q(u,u) + q(u,u)] \\ &= (A + 2B + 2D)q(u,u) \end{aligned}$$

and, since $0 < A + 2B + 2D < A + 2B + 4D < 1$, property ($p_5$) implies that $q(u, u) = \theta$. ■

As corollaries, we obtain, for example, common fixed point result for self-maps f and g satisfying

$$q(fx, gy) \preccurlyeq Aq(x,y), \quad q(gy, fx) \preccurlyeq Aq(y,x), \quad 0 < A < 1, \tag{3.9}$$

or for a self-map f satisfying

$$\begin{aligned} &q(f^n x, f^m y) \preccurlyeq Aq(x,y) + B[q(x, f^n x) + q(y, f^m y)] + D[q(x, f^m y) + q(y, f^n x)], \\ &q(f^m y, f^n x) \preccurlyeq Aq(y,x) + B[q(f^n x, x) + q(f^m y, y)] + D[q(f^m y, x) + q(f^n x, y)], \end{aligned}$$

where $m, n \in \mathbb{N}$, $A + 2B + 4D < 1$.

Remark 2 Similarly as in Remark 1, we note that if the cone P is normal, then continuity of mappings f and g in Theorem 3 can be replaced by conditions

$$\inf\{||q(x,y)|| + ||q(x,fx)|| \ : x \in X\} > 0 \text{ for all } y \in X \text{ with } y \neq fy,$$
$$\inf\{||q(x,y)|| + ||q(x,gx)|| \ : x \in X\} > 0 \text{ for all } y \in X \text{ with } y \neq gy.$$

Example 4 Let $E = \mathbb{R}$ and $P = [0, +\infty)$. Let $X = [0, +\infty)$, $d(x, y) = |x - y|$ and define $q(x, y) = x$. It is easy to check that q is a c-distance on a cone metric space (X, d).

Take functions f, $g : X \to X$ defined by $fx = \frac{x}{4}$ and $gx = \frac{x}{2}$. If $x = 5$, $y = \frac{15}{2}$, then $d(fx, gy) = d\left(\frac{5}{4}, \frac{15}{4}\right) = \frac{5}{2}$ and $d(x, y) = d\left(5, \frac{15}{2}\right) = \frac{5}{2}$. Hence, there is no $A \in (0, 1)$ (and hence no triple (A, B, D)) such that $d(fx, gy) \leq Ad(x, y)$ for each $x, y \in [0, +\infty)$, i.e., the existence of a common fixed point of f and g cannot be deduced from the well-known metric version of Theorem 3.

However, conditions of the c-distance version (Theorem 3) are satisfied. Indeed, take arbitrary A, $\frac{1}{2} \leq A < 1$ and $B = D = 0$. Then, for each $x, y \in [0, +\infty)$, $q(fx, gy) = fx = \frac{x}{4} \leq Ax = Aq(x, y)$ and $q(gy, fx) = gy = \frac{y}{2} \leq Ay = Aq(y, x)$ (see (3.9)). Note that f and g have a (trivial) common fixed point $u = 0$ and that $q(u, u) = q(0, 0) = 0$.

This example can be easily modified to the tvs-cone metric case. It is enough to define tvs-cone metric on X by $d(x, y)(t) = |x - y|\phi(t)$ with fixed $\phi \in P = \{f \in C[0, 1] : f(t) \geq 0 \text{ for } t \in [0, 1]\}$ and take c-distance $q_1(x, y)(t) = x \cdot \phi(t)$ (see Example 3).

3.2 Mappings without periodic points

The first part of the following result was given with an incorrect proof in [20] (using lim inf which may not be defined in the case of an arbitrary cone metric space).

Recall that a map $f : X \to X$ is said to have property (P) if it satisfies $F(f) = F(f^n)$ for each $n \in \mathbb{N}$, where $F(f)$ stands for the set of all fixed points of f [22].

Theorem 4 *Let (X, d) be a tvs-cone metric space and $q : X \times X \to E$ be a c-distance on X. Suppose that a continuous self-map $f : X \to X$ satisfies*

$$q(fx, f^2x) \preccurlyeq \lambda q(x, fx) \tag{3.10}$$

for some $\lambda \in (0, 1)$ and each $x \in X$. Then:

1. f has a fixed point and if $fu = u$, then $q(u, u) = \theta$;

2. f has property (P).

Proof (1) Let $x_0 \in X$ and $x_{n+1} = fx_n$, $n \geq 0$. If $x_{n_0} + 1 = x_{n_0}$ for some $n_0 \in \mathbb{N}_0$, then x_{n_0} is a fixed point of f. Otherwise, we get from (3.10) that

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, f^2x_{n-1}) \preccurlyeq \lambda q(x_{n-1}, fx_{n-1}) = \lambda q(fx_{n-2}, f^2x_{n-2}) \\ &\preccurlyeq \lambda^2 q(x_{n-2}, fx_{n-2}) \preccurlyeq \cdots \preccurlyeq \lambda^n q(x_0, x_1). \end{aligned}$$

Using Lemma 1.(3) again, one obtains that $\{x_n\}$ is a Cauchy sequence in X.

Hence, $x_n \to x^*$, and continuity of f implies that $x_{n+1} = fx_n \to fx^*$ and $fx^* = x^*$.

(2) Obviously, $F(f) \subseteq F(f^n)$ for each $n \in \mathbb{N}$. Let $u \in F(f^n)$, i.e., $f^n u = u$. Then, (3.10) implies that

$$\begin{aligned} q(u, fu) &= q(ff^{n-1}u, f^2f^{n-1}u) \preccurlyeq \lambda q(f^{n-1}u, f^nu) = \lambda q(ff^{n-2}u, f^2f^{n-2}u) \\ &\preccurlyeq \lambda^2 q(f^{n-2}u, f^{n-1}u) \preccurlyeq \cdots \preccurlyeq \lambda^n q(u, fu). \end{aligned}$$

By property (p_5), it follows that $q(u, fu) = \theta$.

Now, for arbitrary $k \in \{1, 2,..., n\}$, we have that $q(f^k u, f^{k+1}u) \preceq \lambda^k q(u, fu)\ \theta$ and so $q(f^k u, f^{k+1}u) = \theta$. It follows that $q(u, f^2u) \preceq q(u, fu)+q(fu, f^2u) = \theta$, i.e., $q(u, f^2u) = \theta$ and, similarly,

$$q(u, fu) = q(u, f^2u) = q(u, f^3u) = \cdots = q(u, f^n u) = \theta.$$

From $q(u, fu) = \theta = q(u, f^n u) = q(u, u)$ and Lemma 1.(1), we conclude that $fu = u$, i.e., $u \in F(f)$. ■

Another way to obtain property (P) is the following.

Theorem 5 *Let* (X, d) *be a tvs-cone metric space and* $q : X \times X \to E$ *be a c-distance on* X. *Suppose that a continuous self-map* $f : X \to X$ *satisfies*

$$q(fx, f^2x) + q(f^2x, fx) \preccurlyeq \lambda[q(x, fx) + q(fx, x)] \tag{3.11}$$

for some $\lambda \in (0, 1)$ *and each* $x \in X$. *Then* f *has property* (P).

Proof Denote $z_1(x) = q(x, fx)+ q(fx, x)$ and $z_2(x) = q(fx, f^2x)+ q(f^2x, fx)$.

Then, the given condition is written as $z_2(x) \preceq \lambda z_1(x)$ for each $x \in X$. Suppose that $f^n u = u$. Then,

$$\begin{aligned} z_1(u) &= q(u, fu) + q(fu, u) = q(f^n u, ff^n u) + q(ff^n u, f^n u) = z_2(f^{n-1}u) \\ &\preccurlyeq \lambda z_1(f^{n-1}u) = \lambda z_2(f^{n-2}u) \preccurlyeq \lambda^2 z_1(f^{n-2}u) \preccurlyeq \cdots \preccurlyeq \lambda^n z_1(u). \end{aligned}$$

Since $0 < \lambda^n < 1$, property (p_5) implies that $z_1(u) = q(u, fu) + q(fu, u) = \theta$. Again, the triangle inequality (q2) implies that $q(u, u) = q(fu, fu) = \theta$, and by Lemma 1.(1), we get that $fu = u$. ■

Corollary 1 *Let* q *be a c-distance on a tvs-cone metric space* (X, d) *and let* $f : X \to X$ *be continuous and such that for some nonnegative* A, B, C, D, E *such that* $A + B + C + 2D + 2E < 1$, *inequalities* (3.1) *and* (3.2) *hold for all* $x, y \in X$. *Then* f *has property* (P).

Proof Putting $x = x$ and $y = fx$ in conditions (3.1) and (3.2) leads to the following inequalities:

$$\begin{aligned} q(fx, f^2x) &\preccurlyeq (A + B + D)q(x, fx) + (C + D + E)q(fx, f^2x) + Eq(f^2x, fx), \\ q(f^2x, fx) &\preccurlyeq (A + B + D)q(fx, x) + (C + D + E)q(f^2x, fx) + Eq(fx, f^2x). \end{aligned}$$

Adding up, one obtains inequality (3.11) with $0 < \lambda = \dfrac{A + B + D}{1 - C - D - 2E} < 1$, since $A + B + C + 2D + 2E < 1$. ■

Similar results concerning property (Q) of two self-mappings f and g (i.e., property that $F(f) \cap F(g) = F(f^n) \cap F(g^n)$ for each $n \in \mathbb{N}$) can be obtained.

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Competing interests
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Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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