Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Generalization of fixed point theorems in ordered metric spaces concerning generalized distance

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Abstract

In this article, we consider ordered metric spaces concerning generalized distance and prove some fixed point theorems in these spaces. Our results generalize, improve, and simplify the proof of the previous results given by some authors.

Mathematics Subject Classification (2000)
47H10, 54H25

Keywords: Ordered metric space, Fixed point, Generalized distance

1. Introduction and Preliminary

Recently, Nieto and Rodriguez-Lopez [1,2], Ran and Reurins [3], Petrusel and Rus [4] presented some new results in partially ordered metric spaces. Their main idea was to combine the ideas of iterative technique in the contractive mapping with these in monotone technique.

Recently, Kada et al. [5,6] in 1996 introduced the concept of w-distance in a metric space and prove some fixed point theorems. For the study of fixed point theorem concerning generalized distance followed in other articles, see [5,7-15].

The aim of this article is to use the concept of w-distance to generalize the fixed point theorems in partially ordered metric spaces. Our results not only generalize some fixed point theorems, but also improve and simplify the previous results.

In the sequel, we state some definitions and a lemma which we will use in our main results.

Definition 1.1. ([5,8,10]) Let (X, d) be a metric space. Then, a function $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following conditions are satisfied:

(a) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;
(b) for any $x \in X$, $p(x, .) : X \to [0, \infty)$ is lower semi-continuous;
(c) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(x, z) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

We know that a real-valued function f defined in a metric space X is said to be lower semi-continuous at a point $x_0 \in X$ if either $\liminf_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf_{x_n \to x_0} f(x_n)$, whenever $x_n \in X$ for each $n \in \mathbf{N}$ and $x_n \to x_0$.

Lemma 1.2. ([5,7]) *Let (X, d) be a metric space and p be a w-distance on X. Let $\{x_n\}$, $\{y_n\}$ be sequences in X, $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to zero and let x, y,*

$z \in X$. Then, the following conditions hold:

(1) *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbf{N}$, then $y = z$. In particular, if $p(x. y) = 0$ and $p(x, z) = 0$, then $y = z$;*
(2) *If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbf{N}$, then $d(y_n, z) \to 0$;*
(3) *If $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbf{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
(4) *If $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbf{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Let $f : X \to X$ be an operator:

(1) $I(f)$ is the set of all nonempty invariant subsets of f, i.e., $I(f) = \{Y \subset X : f(Y) \subset Y\}$ and $F_f = \{x \in X : x = f(x)\}$.
(2) The operator f is called Picard operator (briefly, PO) if there exists $x^* \in X$ such that $F_f = \{x^*\}$ and, for all $x \in X$, $\{f^n(x)\}$ converges to x^*.
(3) The operator f is called orbitally U-continuous for any $U \subset X \times X$ if the following condition holds:

For any $x \in X$, $f^{n_i}(x) \to a \in X$ as $i \to \infty$ and $(f^{n_i}(x), a) \in U$ for any $i \in \mathbf{N}$ imply that $f^{n_i+1}(x) \to f(a)$ as $i \to \infty$.

(4) Let $(X, \leq)$ be a partially ordered set. Then,

$$X_{\leq} = \{(x, y) \in X \times X : x \leq y \text{ or } y \leq x\}$$

and $[x, y]_{\leq} = \{z \in X : x \leq z \leq y\}$, where $x, y \in X$ and $x \leq y$.

(5) If $g : Y \to Y$ is an operator, then the Cartesian product of f and g is the mapping $f \times g : X \times Y \to X \times Y$ defined by $(f \times g)(x, y) = (f(x), g(y))$ for all $(x, y) \in X \times Y$.
(6) $\phi : R_+ \to R_+$ is said to be a comparison function if it is increasing and $\phi^n(t) \to 0$ as $n \to \infty$. As a consequence, we also have $\phi(t) < t$ for any $t > 0$, $\phi(0) = 0$, and ϕ is right continuous at 0.

2. Main Results

Now, we give the main results of this article.

Theorem 2.1. *Let $(X, d, \leq)$ be an ordered metric space and $f : X \to X$ be an operator. Let p be a w-distance on (X, d) and suppose that*

(a) *$X_{\leq} \in I(f \times f)$;*
(b) *there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in X_{\leq}$;*
(c) *(c_1) f is orbitally continuous or*
(c_2) f is orbitally $X_{\leq}$-continuous and there exists a subsequence $\{f^{n_k}(x_0)\}$ of $\{f^n(x_0)\}$ such that $(f^{n_k}(x_0), x^*) \in X_{\leq}$ for any $k \in \mathbf{N}$;
(d) *there exists a comparison function $\phi : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ such that*

$$p(f(x), f(y)) \leq \varphi(M_{xy})$$

for all $(x, y) \in X_{\leq}$, *where*

$$M_{xy} = \max\{p(x,y), p(x,f(x)), p(y,f(y)), \frac{1}{2}(p(x,f(y)) + p(y,f(x)))\};$$

(e) *the metric d is complete.*

Then $F_f \neq \emptyset$.

Proof. If $f(x_0) = x_0$, then the proof is completed. Let $x_0 \in X$ be such that $(x_0, f(x_0)) \in X_{\leq}$. By (a), since $(f \times f)(X_{\leq}) \subset X_{\leq}$, we have $(f \times f)(x_0, f(x_o)) \in X_{\leq}$ and so $(f(x_0), f^2(x_o)) \in X_{\leq}$.

Continuing this process, we obtain

$$(f^n(x_0), f^{n+1}(x_0)) \in X_{\leq}$$

for any $n \in \mathbf{N}$.

Now, we show that

$$p(f^n(x_0), f^{n+1}(x_0)) \leq \varphi(p(f^{n-1}(x_0), f^n(x_0))) \tag{3.1}$$

for any $n \in \mathbf{N}$. Let $p_0 = p(x_0, f(x_0))$ and $p_n = p(f^n(x_0), f^{n+1}(x_0))$ for any $n \in \mathbf{N}$. Then we have

$$\begin{aligned} p_n &\leq \varphi(\max\{p_{n-1}, p_n, p_{n-1}, \frac{1}{2}(p(f^{n-1}(x_0), f^n(x_0)) + p(f^n(x_0), f^{n+1}(x_0)))\}) \\ &\leq \varphi(\max\{p_{n-1}, p_n, \frac{1}{2}(p_{n-1} + p_n)\}) \\ &\leq \varphi(\max\{p_{n-1}, p_n\}) \end{aligned} \tag{3.2}$$

for any $n \in \mathbf{N}$. If $\max\{p_{n-1}, p_n\} = p_{n-1}$, then (3.1) follows. Otherwise, $\max\{p_{n-1}, p_n\} = p_n$ Then, by (3.2), we have $p_n \leq \phi(p_n) \leq p_n$ and so $p_n = 0$ and (3.1) follows. By induction, we obtain

$$p(f^n(x_0), f^{n+1}(x_0)) \leq \varphi^n(p(x_0, f(x_0)))$$

or, equivalently,

$$p_n \leq \varphi^n(p_0)$$

for any $n \in \mathbf{N}$, Now, we have

$$p(f^n(x_0), f^{n+2}(x_0)) \leq p_n + p_{n+1} \leq \varphi^n(p_0) + \varphi^{n+1}(p_0) \to 0$$

as $n \to \infty$.

Similarly, we have

$$\begin{aligned} p(f^n(x_0), f^{n+3}(x_0)) &\leq p(f^n(x_0), f^{n+2}(x_0)) + p_{n+2} \\ &\leq p(f^n(x_0), f^{n+2}(x_0)) + \varphi^{p+2}(p_0) \to 0 \end{aligned}$$

as $n \to \infty$ and so, by induction, we obtain

$$p(f^n(x_0), f^{n+k}(x_0)) \to 0 \tag{3.3}$$

as $n \to \infty$ for any $k > 0$. Therefore, $\{f^n(x_0)\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $f^n(x_0) \to x^*$ as $n \to \infty$.

Now, we show that x^* is a fixed point. If (c_1) holds, then $f^{n+1}(x_0) \to f(x^*)$ and, by lower semi-continuity of $p(f^n(x_0), \cdot)$, we have

$$p(f^n(x_0), x^*) \le \liminf_{m\to\infty} p(f^n(x_0), f^m(x_0)) = \alpha_n,$$

$$p(f^n(x_0), f(x^*)) \le \liminf_{m\to\infty} p(f^n(x_0), f^{m+1}(x_0)) = \beta n$$

and α_n, $\beta_n \to 0$ as $n \to \infty$. Thus, by (3.3) and Lemma 1.2, we conclude that $f(x^*) = x^*$.

Now, suppose that (c_2) holds. Since $\{f^{n_k}(x_0)\}$ converges to x^* and f is $X_\le$-orbitally continuous, it follows that $f^{n_k+1}(x_0)$ converges to $f(x^*)$. Similarly, by lower semi-continuity of $p(f^n(x_0), \cdot)$, we conclude that $f(x^*) = x^*$. This completes the proof. □

Corollary 2.2. *Let* $(X, d, \le)$ *be an ordered metric space and* $f: X \to X$ *be an operator. Let* p *be a w-distance on* (X, d) *and suppose that*

(a) $X_\le \in I(f \times f)$;
(b) *there exists* $x_0 \in X$ *such that* $(x_0, f(x_0)) \in X_\le$;
(c) (c_1)) f *is orbitally continuous or*
(c_2) f *is orbitally* $X_\le$*-continuous and there exists a subsequence* $\{f^{n_k}(x_0)\}$*of* $\{f^n(x_0)\}$ *such that* $(f^{n_k}(x_0), x^*) \in X_\le$ *for any* $k \in \mathbf{N}$;
(d) *and there is a comparison function* $\phi : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *such that*

$$p(f(x), f(y)) \le \varphi(M_{xy})$$

for any $(x, y) \in X_\le$, *where*

$$M_{xy} = \max\{p(x,y), p(x,f(x)), p(y,f(y)), \frac{1}{2}(p(x,f(y)) + p(y,f(x)))\};$$

(e) *the metric* d *is complete;*
(f) *if* $(x, y) \in X_\le$ *and* $(y, z) \in X_\le$*.vskip 1 mm*

Then, $F_f \neq \emptyset$.

Theorem 2.3. *Let* $(X, d, \le)$ *be an ordered metric space and* $f: X \to X$ *be an operator. Let* p *be a w-distance on* (X, d) *and suppose that*

(a) $X_\le \in I(f \times f)$;
(b) *There exists* $x_0 \in X$ *such that* $(x_0, f(x_0)) \in X_\le$;
(c) (c_1) f *is orbitally continuous or*
(c_2) f *is orbitally* $X_\le$*-continuous and there exists a subsequence* $\{f^{n_k}(x_0)\}$*of* $\{f^n(x_0)\}$ *such that* $(f^{n_k}(x_0), x^*) \in X_\le$ *for any* $k \in \mathbf{N}$;
(d) *there is a comparison function* $\phi : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *such that*

$$p(f(x), f(y)) \le \varphi(M_{xy})$$

for any $(x, y) \in X_{\leq}$, *where*

$$M_{xy} = \max\{p(x,y), p(x,f(x)), p(y,f(y)), \frac{1}{2}(p(x,f(y)) + p(y,f(x)))\};$$

(e) *the metric d is complete;*
(f) *if* $x, y \in X$ *with* $(x, y) \notin X_{\leq}$, *then there exists* $c(x, y) \in X$ *such that* $(x, c(x, y)) \in X_{\leq}$ *and* $(y, c(x, y)) \in X_{\leq}$.

Then, f is PO.

Proof. According to Theorem 2.1, there exists $x^* \in X$ such that $f(x^*) = x^*$. Take $x \in X$.

If $(x, x_0) \in X_{\leq}$, then $(f^n(x), f^n(x0)) \in X_{\leq}$ and so

$$p(f^n(x_0), f^n(x)) \leq \varphi^n(p(x_0, x)), \quad p(f^n(x_0), x^*) \leq \varphi^n(p(x_0, x^*))$$

for any $n \in \mathbf{N}$. Thus, by Lemma 1.2, $f^n(x) \to x^*$ as $n \to \infty$.

If $(x, x_0) \notin X_{\leq}$, then there exists $z \in X$ such that $(x, z) \in X_{\leq}$ and $(x_0, z) \in X_{\leq}$ and so

$$p(f^n(x_0), x^*) \leq \varphi^n(p(x_0, x^*)), \quad p(f^n(x_0), f^n(z)) \leq \varphi^n(p(x_0, z))$$

for any $n \in N$. Thus, by Lemma 1.2, we have $f^n(z) \to x^*$ as $n \to \infty$. Also, since $(x, z) \in X_{\leq}$, we have $f^n(z) \to x^*$ as $n \to \infty$. Consequently, $f^n(x) \to x^*$ as $n \to \infty$.

Now, if there exist $y \in X$ such that $f(y) = y$, then

$$p(f^n(y), x^*) \leq \varphi^n(p(y, x^*)), \quad p(f^n(y), y) \leq \varphi^n(p(y, y))$$

and so, by Lemma 2.1, $y = x^*$, i.e., $F_f = \{x^*\}$. This completes the proof. □

Corollary 2.4. *Let* $(X, d, \leq)$ *be an ordered metric space and* $f : X \to X$ *be an operator. Let p be a w-distance on* (X, d) *and suppose that*

(a) *if* $x, y \in X$ *with* $(x, y)X_{\leq}$ *there exists* $c(x, y) \in X$ *such that* $(x, c(x, y)) \in X_{\leq}$ *and* $(y, c(x, y)) \in X_{\leq}$;
(b) $X_{\leq} \in I(f \times f)$;
(c) *There exists* $x_0 \in X$ *such that* $(x_0, f(x_0)) \in X_{\leq}$;
(d) (d_1) *f is orbitally continuous or*
(d_2) *f is orbitally* $X_{\leq}$*-continuous and there exists a subsequence* $\{f^{n_k}(x_0)\}$*of* $\{f^n(x_0)\}$ *such that* $(f^{n_k}(x_0), x^*) \in X_{\leq}$ *for any* $k \in \mathbf{N}$;
(e) *there is a comparison function* $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ *such that*

$$p(f(x), f(y)) \leq \varphi(M_{xy})$$

for any $(x, y) \in X_{\leq}$, *where*

$$M_{xy} = \max\{p(x,y), p(x,f(x)), p(y,f(y)), \frac{1}{2}(p(x,f(y)) + p(y,f(x)))\};$$

(f) *the metric d is complete,*

Then, f is PO.

Corollary 2.5. *Let* $(X, d, \leq)$ *be an ordered metric space and* $f : X \to X$ *be an operator. Let p be a w-distance on* (X, d) *and suppose that*

(a) *if* $x, y \lfloor X$ *with* $(x, y)X_{\leq}$, *then there exists* $c(x, y) \in X$ *such that* $(x, c(x, y)) \in X_{\leq}$ *and* $(y, c(x, y)) \in X_{\leq}$*;*
(b) *if* $(x, y) \in X_{\leq}$ *and* $(y, z) \in X_{\leq}$, *then* $(x, z) \in X_{\leq}$*;*
(c) *f is orbitally continuous* (iv) *there is a comparison function* $\phi : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *such that*

$$p(f(x), f(y)) \leq \varphi(M_{xy})$$

for any $(x, y) \in X_{\leq}$, *where*

$$M_{xy} = \max\{p(x, y), p(x, f(x)), p(y, f(y)), \frac{1}{2}(p(x, f(y)) + p(y, f(x)))\};$$

(d) *the metric d is complete,*

Then, f is PO.

Corollary 2.6. *Let* $(X, d, \leq)$ *be an ordered metric space and* $f : X \to X$ *be an operator. Let p be a w-distance on* (X, d) *and suppose that*

(a) *if* $x, y \in X$ *with* $(x, y)X_{\leq}$, *then there exists* $c(x, y) \in X$ *such that* $(x, c(x, y)) \in X_{\leq}$ *and* $(y, c(x, y)) \in X_{\leq}$*;*
(b) $X_{\leq} \in I(f \times f)$ *;*
(c) *there exists* $x_0 \in X$ *such that* $(x_0, f(x_0)) \in X_{\leq}$*;*
(d) *if* $(x, y) \in X_{\leq}$ *and* $(y, z) \in X_{\leq}$, *then* $(x, z) \in X_{\leq}$*;*
(e) (e_1) *f is orbitally continuous or*
(e_2) *f is orbitally* $X_{\leq}$*-continuous and there exists a subsequence* $\{f^{n_k}(x_0)\}$ *of* $\{f^n(x_0)\}$ *such that* $(f^{n_k}(x_0), x^*) \in X_{\leq}$ *for any* $k \in \mathbf{N}$*;*
(f) *there is a comparison function* $\phi : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *such that*

$$p(f(x), f(y)) \leq \varphi(M_{xy})$$

for any $(x, y) \in X_{\leq}$, *where*

$$M_{xy} = \max\{p(x, y), p(x, f(x)), p(y, f(y)), \frac{1}{2}(p(x, f(y)) + p(y, f(x)))\};$$

(g) *the metric d is complete,*

Then, f is PO.

Corollary 2.7. *Let* $(X, d, \leq)$ *be an ordered metric space and* $f : X \to X$ *be an operator. Let p be a w-distance on* (X, d) *and suppose that*

(a) *if* $x, y \in X$ *with* $(x, y)X_{\leq}$, *then there exists* $c(x, y) \in X$ *such that* $(x, c(x, y)) \in X_{\leq}$ *and* $(y, c(x, y)) \in X_{\leq}$;

(b) f *is increasing or decreasing;*

(c) *there exists* $x_0 \in X$ *such that* $(x_0, f(x_0)) \in X_{\leq}$;

(d) (d_1) f *is orbitally continuous or*

(d_2) f *is orbitally* $X_{\leq}$*-continuous and there exists a subsequence* $\{f^{n_k}(x_0)\}$*of* $\{f^n(x_0)\}$ *such that* $(f^{n_k}(x_0), x^*) \in X_{\leq}$*for any* $k \in \mathbf{N}$;

(e) *there is a comparison function* $\phi : \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *such that*

$$p(f(x), f(y)) \leq \varphi(M_{xy})$$

for any $(x, y) \in X_{\leq}$, *where*

$$M_{xy} = \max\{p(x, y), p(x, f(x)), p(y, f(y)), \frac{1}{2}(p(x, f(y)) + p(y, f(x)))\};$$

(f) *the metric* d *is complete,*

Then, f *is PO.*

Acknowledgements
The authors would like to thank the referees and area editor Professor Simeon Reich for giving useful suggestions and comments for the improvement of this article. Y. J. Cho was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00050).

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Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 14 March 2011 Accepted: 11 August 2011 Published: 11 August 2011

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doi:10.1186/1687-1812-2011-30
Cite this article as: Graily *et al.*: **Generalization of fixed point theorems in ordered metric spaces concerning generalized distance.** *Fixed Point Theory and Applications* 2011 **2011**:30.