Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Systems of general nonlinear set-valued mixed variational inequalities problems in Hilbert spaces

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Abstract

In this paper, the existing theorems and methods for finding solutions of systems of general nonlinear set-valued mixed variational inequalities problems in Hilbert spaces are studied. To overcome the difficulties, due to the presence of a proper convex lower semi-continuous function, ϕ and a mapping g, which appeared in the considered problem, we have used some applications of the resolvent operator technique. We would like to point out that although many authors have proved results for finding solutions of the systems of nonlinear set-valued (mixed) variational inequalities problems, it is clear that it cannot be directly applied to the problems that we have considered in this paper because of ϕ and g.

2000 AMS Subject Classification: 47H05; 47H09; 47J25; 65J15.

Keywords: set-valued mixed variational inequalities, maximal monotone operator, resolvent operator, strongly monotone operator, Hausdorff metric

1. Introduction and preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot, \cdot\rangle$, and $||\cdot||$, respectively. Let $CB(H)$ be the family of all nonempty, closed, and bounded sets in H. Let A, $B : H \to CB(H)$ be nonlinear set-valued mappings, $g : H \to H$ be a single-valued mapping, and $\phi : H \to (-\infty, +\infty]$ be a proper convex lower semi-continuous function on H. For each fixed positive real numbers, ρ and η, we consider the following so-called *system of general nonlinear set-valued mixed variational inequalities problems*:

Find $x^*, y^* \in H$, $u^* \in Ay^*$, $v^* \in Bx^*$, such that

$$\begin{cases} \langle \rho u^* + x^* - g(y^*), g(x) - x^* \rangle + \varphi(g(x)) - \varphi(x^*) \geq 0, & \forall x \in H, g(x) \in H, \\ \langle \eta v^* + y^* - g(x^*), g(x) - y^* \rangle + \varphi(g(x)) - \varphi(y^*) \geq 0, & \forall x \in H, g(x) \in H. \end{cases} \tag{1.1}$$

We denote by $SGNSM(A, B, g, \phi, \rho, \eta)$, the set of all solutions (x^*, y^*, u^*, v^*) of the problem (1.1).

We shall now discuss several special cases of the problem (1.1).

Special cases of the problem (1.1) are as follows:

(I) If $g = I$ (: the identity operator), then, from the problem (1.1), we have the following *system of nonlinear set-valued mixed variational inequalities problems*:

Find x^*, $y^* \in H$, $u^* \in Ay^*$, $v^* \in Bx^*$, such that

$$\begin{cases} \langle \rho u^* + x^* - y^*, x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0, & \forall x \in H, \\ \langle \eta v^* + y^* - x^*, x - y^* \rangle + \varphi(x) - \varphi(y^*) \geq 0, & \forall x \in H. \end{cases} \tag{1.2}$$

We denote by $SNSM(A, B, \phi, \rho, \eta)$, the set of all solutions (x^*, y^*, u^*, v^*) of the problem (1.2).

(II) If K is a closed convex subset of H and $\phi(x) = \delta_K(x)$ for all $x \in K$, where δ_K is the indicator function of K defined by

$$\delta_K = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{otherwise}, \end{cases}$$

then, from the problem (1.1), we have the following *system of general nonlinear set-valued variational inequalities problems*:

Find x^*, $y^* \in K$, $u^* \in Ay^*$, $v^* \in Bx^*$, such that

$$\begin{cases} \langle \rho u^* + x^* - g(y^*), g(x) - x^* \rangle \geq 0, & \forall x \in H, g(x) \in K, \\ \langle \eta v^* + y^* - g(x^*), g(x) - y^* \rangle \geq 0, & \forall x \in H, g(x) \in K. \end{cases} \tag{1.3}$$

We denote by $SGNS(A, B, g, K, \rho, \eta)$, the set of all solutions (x^*, y^*, u^*, v^*) of the problem (1.3).

The problem (1.3) was recently introduced and studied by Noor [1], when A and B are single-valued mappings. Consequently, it was pointed out that such a problem includes a wide class of the system of variational inequalities problems and related optimization problems as special cases, and hence the results announced in [1] is very interesting.

(III) If $A, B : H \to H$ are single-valued mappings, then, from the problem (1.1), we have the following *system of general nonlinear mixed variational inequalities problems*:

Find x^*, $y^* \in H$, such that

$$\begin{cases} \langle \rho Ay^* + x^* - y^*, x - x^* \rangle + \varphi(g(x)) - \varphi(x^*) \geq 0, & \forall x \in H, g(x) \in H, \\ \langle \eta Bx^* + y^* - x^*, x - y^* \rangle + \varphi(g(x)) - \varphi(y^*) \geq 0, & \forall x \in H, g(x) \in H. \end{cases} \tag{1.4}$$

We denote by $SGNM(A, B, g, \phi, \rho, \eta)$, the set of all solutions (x^*, y^*) of the problem (1.4).

This means, generally speaking, the class of system general nonlinear set-valued variational inequalities problems is more general and has had a great impact and influence in the development of several branches of pure, applied, and engineering sciences. For more information and results on the general variational inequalities problems, one may consult [2-18].

Inspired and motivated by the recent research going on in this area, in this paper, we consider the existence theorem and a method for finding solutions for the systems of nonlinear general set-valued mixed variational inequalities problems (1.1). Our results extend the results announced by Noor [1], from single-valued mappings to set-valued mappings, and hence include several related problems as spacial cases.

We need the following basic concepts and well-known results:

Definition 1.1. A mapping $g : H \to H$ is said to be:

(1) *monotone* if

$$\langle g(x) - g(y), x - y \rangle \geq 0, \quad \forall x, y \in H;$$

(2) *ν-strongly monotone* if there exists a constant $\nu > 0$, such that

$$\langle g(x) - g(y), x - y \rangle \geq \nu ||x - y||^2, \quad \forall x, y \in H.$$

Definition 1.2. A set-valued mapping $A : H \to 2^H$ is said to be *ν-strongly monotone* if there exists a constant $\nu > 0$, such that,

$$\langle w_1 - w_2, u_1 - u_2 \rangle \geq \nu ||u_1 - u_2||^2, \quad \forall u_1, u_2 \in H, w_1 \in Au_1, w_2 \in Au_2.$$

Definition 1.3. A set-valued mapping $A : H \to CB(H)$ is said to be *τ-Lipschitzian continuous* if there exists a constant $\tau > 0$, such that,

$$H(Au_1, Au_2) \leq \tau ||u_1 - u_2||, \quad \forall u_1, u_2 \in H,$$

where $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$.

Definition 1.4. A single-valued mapping $T : H \to H$ is said to be a *κ-Lipschitzian continuous mapping* if there exists a positive constant κ, such that,

$$||Tx - Ty|| \leq \kappa ||x - y||, \quad \forall x, y \in H.$$

In the case of $\kappa = 1$, the mapping T is known as a *nonexpansive mapping*.

Definition 1.5. [19] If M is a maximal monotone operator on H, then, for any $\lambda > 0$, the *resolvent operator associated with M* is defined as

$$J_M(u) = (I + \lambda M)^{-1}(u), \quad \forall u \in H.$$

It is well-known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. Furthermore, the resolvent operator is single-valued and nonexpansive. In particular, it is well-known that the subdifferential $\partial\phi$ of a proper convex lower semi-continuous function $\phi : H \to (-\infty, +\infty]$ is a maximal monotone operator.

Moreover, we have the following interesting characterization:

Lemma 1.6. [19]*The points u, $z \in H$ satisfy the inequality*

$$\langle u - z, x - u \rangle + \lambda\varphi(x) - \lambda\varphi(u) \geq 0, \quad \forall x \in H,$$

if and only if $u = J_\phi(z)$, where J_ϕ $(I + \lambda\partial\phi)^{-1}$ is the resolvent operator and $\lambda > 0$ is a constant.

The property of the resolvent operator J_ϕ presented in Lemma 1.6 plays an important role in developing the numerical methods for solving the system of general nonlinear set-valued mixed variational inequalities problems. In fact, assuming that $g : H \to H$ is a surjective mapping and by applying Lemma 1.6, one can easily prove the following result:

Lemma 1.7. *If $g : H \to H$ is a surjective mapping, then the problem* (1.1) *is equivalent to the following problem:*

Find x^*, $y^* \in H$, $u^* \in Ay^*$, $v^* \in Bx^*$, such that,

$$\begin{cases} x^* = J_\varphi[g(y^*) - \rho u^*], \\ y^* = J_\varphi[g(x^*) - \eta v^*], \end{cases} \tag{1.5}$$

where $J_\phi = (I + \partial\phi)^{-1}$.

The equivalent formulation (1.5) enables us to suggest an explicit iterative method for solving the system of general nonlinear set-valued mixed variational inequalities problem (1.1), as we show in the next section. Of course, we hope to use the Lemma 1.7 to obtain our results in this paper, and hence, from now on, we assume that the mapping $g : H \to H$ is a surjection.

In order to prove our main results, the next lemma is very important.

Lemma 1.8. [20]*Let* $B_1, B_2 \in CB(H)$ *and* $r > 1$ *be any real number. Then, for all* $b_1 \in B_1$, *there exists* $b_2 \in B_2$, *such that* $||b_1 - b_2|| \le rH(B_1, B_2)$.

2. Main results

We begin with some observations that are guidelines to a method for proving the main results in this paper.

Remark 2.1. If $(x^*, y^*, u^*, v^*) \in SGNSM(A, B, g, \phi, \rho, \eta)$, then it follows from (1.5) that

$$\begin{cases} x^* = (1-t)x^* + tJ_\varphi[g(y^*) - \rho u^*], \quad \forall t \in [0,1], \\ y^* = J_\varphi[g(x^*) - \eta v^*], \end{cases}$$

From Remark 2.1, we suggest a method for finding a solution for the problem (2.1), as following iterative procedures:

Let $\{\varepsilon_n\}$ be a sequence of positive real numbers with $\varepsilon_n \to 0$ as $n \to \infty$ and $t \in (0, 1]$ be fixed. For any $x_0, y_0 \in H$, pick $u_0 \in Ay_0$ and let

$$x_1 = (1-t)x_0 + tJ_\varphi[g(y_0) - \rho u_0].$$

Then take $v_1 \in Bx_1$ and let

$$y_1 = J_\varphi[g(x_1) - \eta v_1].$$

Now, by Lemma 1.8, there exists $u_1 \in Ay_1$, such that

$$||u_0 - u_1|| \le (1 + \varepsilon_1)H(Ay_0, Ay_1).$$

Take

$$x_2 = (1-t)x_1 + tJ_\varphi[g(y_1) - \rho u_1].$$

Similarly, by Lemma 1.8, there exists $v_2 \in Bx_2$, such that

$$||v_1 - v_2|| \le (1 + \varepsilon_2)H(Bx_1, Bx_2).$$

Take

$$y_2 = J_\varphi[g(x_2) - \eta v_2].$$

Inductively, we have the following algorithm:

Algorithm 1. Let $\{\varepsilon_n\}$ be a sequence of nonnegative real numbers with $\varepsilon_n \to 0$ as $n \to \infty$ and $t \in (0, 1]$ be a fixed constant. For any $x_0, y_0 \in H$, compute the sequences $\{x_n\}$, $\{y_n\} \subset H$, $\{u_n\} \subset \bigcup_{n=0}^{\infty} Ay_n$ and $\{v_n\} \subset \bigcup_{n=1}^{\infty} Bx_n$ generated by the iterative processes:

$$\begin{cases} x_{n+1} = (1-t)x_n + tJ_\varphi[g(y_n) - \rho u_n], \\ y_{n+1} = J_\varphi[g(x_{n+1}) - \eta v_{n+1}], \\ \text{where } u_n \in Ay_n \text{ and } v_n \in Bx_n \text{ satisfying} \\ ||u_{n-1} - u_n|| \le (1 + \varepsilon_n)H(Ay_{n-1}, Ay_n), \\ ||v_n - v_{n+1}|| \le (1 + \varepsilon_{n+1})H(Bx_n, Bx_{n+1}). \end{cases} \tag{2.1}$$

We now state and prove the existence theorem of a solution for the problem (1.1).

Theorem 2.2. *Let H be a real Hilbert space. Let $A : H \to CB(H)$ be ν_A-strongly monotone and Lipschitz continuous mapping with a constant τ_A and $B : H \to CB(H)$ be ν_B-strongly monotone and Lipschitz continuous mapping with a constant τ_B. Let $g : H \to H$ be ν_g-strongly monotone and Lipschitz continuous mapping with a constant τ_g. Put*

$$p = \sqrt{1 - 2\nu_g + \tau_g^2}.$$

If the following conditions are satisfied:

(i) $p \in [0, \delta)$, *where* $\delta = \min\left\{\frac{\nu_A^2}{\tau_A^2}, \frac{\nu_B^2}{\tau_B^2}\right\}$,

(ii) $\left|\rho - \frac{\nu_A}{\tau_A^2}\right| < \frac{\sqrt{\nu_A^2 - p\tau_A^2}}{\tau_A^2}$ *and* $\left|\eta - \frac{\nu_B}{\tau_B^2}\right| < \frac{\sqrt{\nu_B^2 - p\tau_B^2}}{\tau_B^2}$,

then $SGNSM(A, B, g, \phi, \rho, \eta) \neq \emptyset$. Moreover, the sequence $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$ defined by (2.1) converge strongly to x^*, y^*, u^*, and v^*, respectively, where $(x^*, y^*, u^*, v^*) \in SGNSM(A, B, g, \phi, \rho, \eta)$.

Proof. Firstly, by (2.1), we have

$$\begin{aligned}
&||x_{n+1} - x_n|| \\
= &\ ||(1-t)x_n + tJ_\varphi[g(y_n) - \rho u_n] - (1-t)x_{n-1} - tJ_\varphi[g(y_{n-1}) - \rho u_{n-1}]|| \\
\le &\ (1-t)||x_n - x_{n-1}|| + t||g(y_n) - \rho u_n - g(y_{n-1}) + \rho u_{n-1}|| \\
\le &\ (1-t)||x_n - x_{n-1}|| \\
&+ t\left[||y_n - y_{n-1} - [g(y_n) - g(y_{n-1})]|| + ||y_n - y_{n-1} - (\rho u_n - \rho u_{n-1})||\right].
\end{aligned} \tag{2.2}$$

Now, we compute

$$\begin{aligned}
&||y_n - y_{n-1} - [g(y_n) - g(y_{n-1})]||^2 \\
= &\ ||y_n - y_{n-1}||^2 - 2\langle g(y_n) - g(y_{n-1}), y_n - y_{n-1}\rangle + ||g(y_n) - g(y_{n-1})||^2 \\
\le &\ ||y_n - y_{n-1}||^2 - 2\nu_g||y_n - y_{n-1}||^2 + ||g(y_n) - g(y_{n-1})||^2 \\
\le &\ ||y_n - y_{n-1}||^2 - 2\nu_g||y_n - y_{n-1}||^2 + \tau_g^2||y_n - y_{n-1}||^2 \\
= &\ p^2||y_n - y_{n-1}||^2
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&||y_n - y_{n-1} - (\rho u_n - \rho u_{n-1})||^2 \\
= &\ ||y_n - y_{n-1}||^2 - 2\rho\langle u_n - u_{n-1}, y_n - y_{n-1}\rangle + \rho^2||u_n - u_{n-1}||^2 \\
\le &\ ||y_n - y_{n-1}||^2 - 2\rho\nu_A||y_n - y_{n-1}||^2 + \rho^2||u_n - u_{n-1}||^2 \\
\le &\ (1 - 2\rho\nu_A)||y_n - y_{n-1}||^2 + \rho^2[(1+\varepsilon_n)H(Au_n, Au_{n-1})]^2 \\
\le &\ (1 - 2\rho\nu_A)||u_n - u_{n-1}||^2 + \rho^2(1+\varepsilon_n)^2\tau_A^2||y_n - y_{n-1}||^2 \\
= &\ q_n^2||y_n - y_{n-1}||^2,
\end{aligned} \tag{2.4}$$

where $q_n = \sqrt{1 - 2\rho\nu_A + \rho^2(1+\varepsilon_n)^2\tau_A^2}$. Substituting (2.3) and (2.4) into (2.2), we have

$$||x_{n+1} - x_n|| \le (1-t)||x_n - x_{n-1}|| + t(p + q_n)||y_n - y_{n-1}||, \quad \forall n \ge 1. \tag{2.5}$$

Now, since $y_{n+1} = J_\phi[g(x_{n+1}) - \eta v_{n+1}]$ and the resolvent operator J_ϕ is nonexpansive, we have

$$\begin{aligned} &||y_n - y_{n-1}|| \\ \le\ & ||[g(x_n) - \eta v_n] - [g(x_{n-1}) - \eta v_{n-1}]|| \\ \le\ & ||x_n - x_{n-1} - [g(x_n) - g(x_{n-1})]|| + ||x_n - x_{n-1} - (\eta v_n - \eta v_{n-1})||, \quad \forall n \ge 1. \end{aligned}$$

Using the same lines as in (2.3) and (2.4), we know that

$$||y_n - y_{n-1}|| \le (p + r_n)||x_n - x_{n-1}||, \quad \forall n \ge 1, \tag{2.6}$$

where $r_n = \sqrt{1 - 2\eta v_B + \eta^2(1 + \varepsilon_n)^2 \tau_B^2}$. Substituting (2.6) into (2.5), we have

$$\begin{aligned} ||x_{n+1} - x_n|| &\le (1 - t)||x_n - x_{n-1}|| + t(p + q_n)(p + r_n)||x_n - x_{n-1}|| \\ &= \left[1 - t\left(1 - (p + q_n)(p + r_n)\right)\right] ||x_n - x_{n-1}||, \quad \forall n \ge 1. \end{aligned} \tag{2.7}$$

Observe that

$$\lim_{n\to\infty} q_n = \sqrt{1 - 2\rho v_A + \rho^2 \tau_A^2} =: q \tag{2.8}$$

and

$$\lim_{n\to\infty} r_n = \sqrt{1 - 2\eta v_B + \eta^2 \tau_B^2} =: r. \tag{2.9}$$

Consequently, by the conditions (i) and (ii), we have $\Delta =: (p + q)(p + r) <1$.

Now, let $s \in (\Delta, 1)$ be a fixed real number. Then, by (2.8) and (2.9), there exists a positive integer, N, such that $(p + q_n)(p + r_n) < s$ for all $n \ge N$. Then, by (2.7), we have

$$||x_{n+1} - x_n|| \le \kappa ||x_n - x_{n-1}||, \quad \forall n \ge N, \tag{2.10}$$

where $\kappa := 1 - t(1 - s)$. Then it follows from (2.10) that

$$||x_{n+1} - x_n|| \le \kappa^{n-N} ||x_{N+1} - x_N||, \quad \forall n \ge N.$$

Hence it follows that

$$||x_m - x_n|| \le \sum_{i=n}^{m-1} ||x_{i+1} - x_i|| \le \sum_{i=n}^{m-1} \kappa^{i-N} ||x_{N+1} - x_N||, \quad \forall m \ge n > N. \tag{2.11}$$

Since $\kappa <1$, it follows from (2.11) that $||x_m - x_n|| \to 0$ as $n \to \infty$, which implies that $\{x_n\}$ is a Cauchy sequence in H. Consequently, by (2.6), it follows that $\{y_n\}$ is a Cauchy sequence in H. Moreover, since A is a τ_A- Lipschitz continuous mapping, and B is a τ_B-Lipschitz continuous mapping, we also know that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences, respectively. Thus there exist $x^*, y^*, u^*, v^* \in H$, such that $x_n \to x^*$, $y_n \to y^*$, $u_n \to u^*$, and $v_n \to v^*$ as $n \to \infty$. Moreover, by applying the continuity of the mappings A, B, g, and J_ϕ to (2.1), we have

$$\begin{cases} x^* = J_\varphi[g(y^*) - \rho u^*], \\ y^* = J_\varphi[g(x^*) - \eta v^*]. \end{cases}$$

Hence, from Lemma 1.7, it follows that $(x^*, y^*, u^*, v^*) \in SGNSM(A, B, g, \phi, \rho, \eta)$.

Finally, we prove that $u^* \in Ay^*$ and $v^* \in Bx^*$. Indeed, we have

$$\begin{aligned} d(u^*, Ay^*) &= \inf\{\|u^* - z\| : z \in Ay^*\} \\ &\le \|u^* - u_n\| + d(u_n, Ay^*) \\ &\le \|u^* - u_n\| + H(Ay_n, Ay^*) \\ &\le \|u^* - u_n\| + \tau_A\|y_n - y^*\| \to 0 \quad (n \to \infty). \end{aligned}$$

That is, $d(u^*, Ay^*) = 0$. Hence, since $Ay^* \in CB(H)$, we must have $u^* \in Ay^*$. Similarly, we can show that $v^* \in Bx^*$. This completes the proof.

Remark 2.3. Theorem 2.2 not only gives the conditions for the existence of a solution for the problem (1.1) but also provides an iterative algorithm to find such a solution for any initial points $x_0, y_0 \in H$.

Using Theorem 2.2, we can obtain the following results:

(I) If $g = I$ (: the identity mapping), then from Algorithm 1, we have the following:

Algorithm 2. Let $\{\varepsilon_n\}$ be a sequence of nonnegative real numbers with $\varepsilon_n \to 0$. Let $t \in (0, 1]$ be a fixed constant. For any $x_0, y_0 \in H$, compute the sequences $\{x_n\}, \{y_n\} \subset H$, $\{u_n\} \subset \bigcup_{n=0}^{\infty} Ay_n$ and $\{v_n\} \subset \bigcup_{n=1}^{\infty} Bx_n$ generated by the iterative processes:

$$\begin{cases} x_{n+1} = (1-t)x_n + tJ_\varphi[y_n - \rho u_n], \\ y_{n+1} = J_\varphi[x_{n+1} - \eta v_{n+1}], \end{cases} \tag{2.12}$$

where $u_n \in Ay_n$ and $v_n \in Bx_n$ satisfy the following:

$$\begin{aligned} \|u_{n-1} - u_n\| &\le (1+\varepsilon_n)H(Ay_{n-1}, Ay_n), \\ \|v_n - v_{n+1}\| &\le (1+\varepsilon_{n+1})H(Bx_n, Bx_{n+1}). \end{aligned}$$

Corollary 2.4. *Let H be a real Hilbert space. Let $A : H \to CB(H)$ be v_A-strongly monotone and Lipschitz continuous mapping with a constant τ_A, and $B : H \to CB(H)$ be v_B-strongly monotone and Lipschitz continuous mapping with a constant τ_B. If*

$$\rho \in \left(0, \frac{2v_A}{\tau_A^2}\right), \quad \eta \in \left(0, \frac{2v_B}{\tau_B^2}\right),$$

then $SNSM(A, B, \phi, \rho, \eta) \neq \emptyset$. Moreover, the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$ defined by (2.12) *converge strongly to x^*, y^*, u^* and v^*, respectively, where $(x^*, y^*, u^*, v^*) \in SNSM(A, B, \phi, \rho, \eta)$.*

Proof. If $g = I$ (: the identity operator), we know that the constant p defined in Theorem 2.2 is vanished. Thus the result follows immediately.

(II) If the function $\phi(\cdot)$ is the indicator function of a closed convex set K in H, then it is well-known that $J_\phi = P_K$, the projection operator of H onto the closed convex set K (see [2]). Then, from Algorithm 1, we have the following:

Algorithm 3. Let $\{\varepsilon_n\}$ be a sequence of nonnegative real numbers with $\varepsilon_n \to 0$ as $n \to \infty$. Let $t \in (0, 1]$ be a fixed constant. For any $x_0, y_0 \in K$, compute the sequences $\{x_n\}, \{y_n\} \subset K$, $\{u_n\} \subset \bigcup_{n=0}^{\infty} Ay_n$, and $\{v_n\} \subset \bigcup_{n=1}^{\infty} Bx_n$ generated by the iterative processes:

$$\begin{cases} x_{n+1} = (1-t)x_n + tP_K[g(y_n) - \rho u_n], \\ y_{n+1} = P_K[g(x_{n+1}) - \eta v_{n+1}], \\ \text{where } u_n \in Ay_n \text{ and } v_n \in Bx_n \text{ satisfying} \\ \|u_{n-1} - u_n\| \le (1+\varepsilon_n)H(Ay_{n-1}, Ay_n), \\ \|v_n - v_{n+1}\| \le (1+\varepsilon_{n+1})H(Bx_n, Bx_{n+1}). \end{cases} \tag{2.13}$$

Corollary 2.5. *Let K be a closed convex subset of a real Hilbert space H. Let $A : K \to CB(H)$ be ν_A-strongly monotone and Lipschitz continuous mapping with a constant τ_A, and $B : K \to CB(H)$ be ν_B-strongly monotone and Lipschitz continuous mapping with a constant τ_B. Let $g : K \to K$ be a ν_g-strongly monotone and Lipschitz continuous mapping with a constant τ_g and satisfying $K \subset g(H)$.*

Put

$$p = \sqrt{1 - 2\nu_g + \tau_g^2}.$$

If the following conditions are satisfied:

(i) $p \in [0, \delta)$, *where* $\delta = \min\left\{\frac{\nu_A^2}{\tau_A^2}, \frac{\nu_B^2}{\tau_B^2}\right\}$,

(ii) $\left|\rho - \frac{\nu_A}{\tau_A^2}\right| < \frac{\sqrt{\nu_A^2 - p\tau_A^2}}{\tau_A^2}$, *and* $\left|\eta - \frac{\nu_B}{\tau_B^2}\right| < \frac{\sqrt{\nu_B^2 - p\tau_B^2}}{\tau_B^2}$,

then $SGNS(A, B, g, K, \rho, \eta) \neq \emptyset$. Moreover, the sequence $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$ defined by (2.13) *converge strongly to x^*, y^*, u^* and v^*, respectively, where $(x^*, y^*, u^*, v^*) \in SGNS(A, B, g, K, \rho, \eta)$.*

Remark 2.6. Corollary 2.5 is an extension of the results announced by Noor [1] from single-valued mappings to set-valued mappings.

(III) If A, $B : H \to H$ are single-valued mappings, then, from Algorithm 1, we have the following:

Algorithm 4. Let $t \in (0, 1]$ be a fixed constant. For any x_0, $y_0 \in H$, compute the sequences $\{x_n\}$, $\{y_n\} \subset H$ by the iterative processes:

$$\begin{cases} x_{n+1} = (1 - t)x_n + tJ_\varphi[g(y_n) - \rho Ay_n], \\ y_{n+1} = J_\varphi[g(x_{n+1}) - \eta Bx_{n+1}]. \end{cases} \tag{2.14}$$

Corollary 2.7. *Let H be a real Hilbert space. Let $A : H \to H$ be ν_A-strongly monotone and Lipschitz continuous mapping with a constant τ_A, and $B : H \to H$ be ν_B-strongly monotone and Lipschitz continuous mapping with a constant τ_B. Let $g : H \to H$ be ν_g-strongly monotone and Lipschitz continuous mapping with a constant τ_g. Put*

$$p = \sqrt{1 - 2\nu_g + \tau_g^2}.$$

If the following conditions are satisfied:

(i) $p \in [0, \delta)$, *where* $\delta = \min\left\{\frac{\nu_A^2}{\tau_A^2}, \frac{\nu_B^2}{\tau_B^2}\right\}$,

(ii) $\left|\rho - \frac{\nu_A}{\tau_A^2}\right| < \frac{\sqrt{\nu_A^2 - p\tau_A^2}}{\tau_A^2}$, *and* $\left|\eta - \frac{\nu_B}{\tau_B^2}\right| < \frac{\sqrt{\nu_B^2 - p\tau_B^2}}{\tau_B^2}$,

then $SGNM(A, B, g, \phi, \rho, \eta) \neq \emptyset$. Moreover, the sequences $\{x_n\}$ and $\{y_n\}$ defined by (2.14) *converge strongly to x^* and y^*, respectively, where $(x^*, y^*) \in SGNM(A, B, g, \phi, \rho, \eta)$.*

Remark 2.8. Under the assumption of Corollary 2.7, the solution of $SGNM(A, B, g, \phi, \rho, \eta)$ is unique, that is, there is a unique $(x^*, y^*) \in H \times H$ such that $(x^*, y^*) \in SGNM(A, B, g, \phi, \rho, \eta)$. Indeed, if (x^*, y^*) and (x', y') are elements of $SGNM(A, B, g, \phi, \rho, \eta)$. Put

$$q = \sqrt{1 - 2\rho\nu_A + \rho^2\tau_A^2}, \quad r = \sqrt{1 - 2\eta\nu_B + \eta^2\tau_B^2}.$$

Replacing x_{n+1} by x^*, x_n by x', y_n by y^*, and y_{n-1} by y', then, following the lines proof given in Theorem 2.2, we know that

$$||y^* - y'|| \leq (p + r)||x^* - x'|| \tag{2.15}$$

and

$$||x^* - x'|| \leq \left[1 - t\left(1 - (p + q)(p + r)\right)\right] ||x^* - x'||. \tag{2.16}$$

By the conditions (i), (ii), and (2.16), we must have $x^* = x'$. Consequently, by (2.15), we also have $y^* = y'$.

Remark 2.9. Recall that a mapping $A : H \to H$ is said to be:

(1) *μ-cocoercive* if there exists a constant μ >0 such that

$$\langle Ax - Ay, x - y\rangle \geq \mu||Ax - Ay||^2, \quad \forall x, y \in H,$$

(2) *relaxed μ-cocoercive* if there exists a constant μ >0 such that

$$\langle Ax - Ay, x - y\rangle \geq (-\mu)||Ax - Ay||^2, \quad \forall x, y \in H,$$

(3) *relaxed (μ, ν)-cocoercive* if there exist constants μ, ν >0 such that

$$\langle Ax - Ay, x - y\rangle \geq (-\mu)||Ax - Ay||^2 + \nu||x - y||^2, \quad \forall x, y \in H.$$

It is easy to see that the class of the relaxed (μ, ν)- cocoercive mappings is the most general one. However, it is worth noting that if the mapping A is relaxed (μ, ν)-cocoercive, and τ-Lipschitz continuous mapping satisfying $\nu - \mu\tau^2$ >0, then A is a ($\nu - \mu\tau^2$)-strongly monotone. Hence, the result appeared in Corollary 2.7 can be also applied to the class of the relaxed cocoercive mappings. In the conclusion, for a suitable and appropriate choice of the mappings A, B, g, and ϕ, Theorem 2.2 includes many important known results given by some authors as special cases.

Acknowledgements
Yeol Je Cho was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00050). Narin Petrot was supported by the Centre of Excellence in Mathematics, the commission on Higher Education, Thailand.

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Both authors contributed equally in this paper. They read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 30 December 2010 Accepted: 11 August 2011 Published: 11 August 2011

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doi:10.1186/1687-1812-2011-31
Cite this article as: Agarwal *et al.*: **Systems of general nonlinear set-valued mixed variational inequalities problems in Hilbert spaces.** *Fixed Point Theory and Applications* 2011 **2011**:31.