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A relaxed hybrid steepest descent method for common solutions of generalized mixed equilibrium problems and fixed point problems

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Abstract

In the setting of Hilbert spaces, we introduce a relaxed hybrid steepest descent method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of a variational inequality for an inverse strongly monotone mapping and the set of solutions of generalized mixed equilibrium problems. We prove the strong convergence of the method to the unique solution of a suitable variational inequality. The results obtained in this article improve and extend the corresponding results.

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1. Introduction

Let H be a real Hilbert space, C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto the closed convex subset C. Let $S : C \to C$ be a *nonexpansive* mapping, that is, $||Sx - Sy|| \leq ||x - y||$ for all $x, y \in C$. We denote by $F(S)$ the set fixed point of S. If $C \subset H$ is nonempty, bounded, closed and convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty; see, for example, [1,2]. A mapping $f : C \to C$ is a *contraction* on C if there exists a constant $\eta \in (0, 1)$ such that $||f(x) - f(y)|| \leq \eta||x - y||$ for all $x, y \in C$. In addition, let $D : C \to H$ be a nonlinear mapping, $\phi : C \to \mathbb{R} \cup \{+\infty\}$ be a real-valued function and let $F : C \times C \to \mathbb{R}$ be a bifunction such that $C \cap \text{dom}\ \phi \neq \emptyset$, where $\mathbb{R}$ is the set of real numbers and dom $\phi = \{x \in C : \phi(x) < +\infty\}$.

The *generalized mixed equilibrium problem* for finding $x \in C$ such that

$$F(x, y) + \langle Dx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by GMEP(F, ϕ, D), that is,

$$\text{GMEP}\ (F, \varphi, D) = \{x \in C : F(x, y) + \langle Dx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C\}.$$

We find that if x is a solution of a problem (1.1), then $x \in \text{dom}\ \phi$.

If $D = 0$, then the problem (1.1) is reduced into the *mixed equilibrium problem* which is denoted by MEP(F, ϕ).

If $\phi = 0$, then the problem (1.1) is reduced into the *generalized equilibrium problem* which is denoted by GEP(F, D).

If $D = 0$ and $\phi = 0$, then the problem (1.1) is reduced into the *equilibrium problem* which is denoted by EP(F).

If $F = 0$ and $\phi = 0$, then the problem (1.1) is reduced into the *variational inequality problem* which is denoted by VI(C, D).

The generalized mixed equilibrium problems include, as special cases, some optimization problems, fixed point problems, variational inequality problems, Nash equilibrium problems in noncooperative games, equilibrium problem, Numerous problems in physics, economics and others. Some methods have been proposed to solve the problem (1.1); see, for instance, [3,4] and the references therein.

Definition 1.1. Let $B : C \to H$ be nonlinear mappings. Then, B is called

(1) *monotone* if $\langle Bx - By, x - y\rangle \geq 0, \forall x, y \in C$,

(2) *β-inverse-strongly monotone* if there exists a constant $\beta > 0$ such that

$$\langle Bx - By,\ x - y\rangle \geq \beta ||\ Bx - By\ ||^2, \quad \forall x,\ y \in C.$$

(3) A set-valued mapping $Q : H \to 2^H$ is called *monotone* if for all $x, y \in H, f \in Qx$ and $g \in Qy$ imply $\langle x- y, f - g\rangle \geq 0$. A monotone mapping $Q : H \to 2^H$ is called *maximal* if the graph $G(Q)$ of Q is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping Q is maximal if and only if for (x, f) Î $H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every (y, g) Î $G(Q)$ implies f Î Qx.

A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping defined on a real Hilbert space H:

$$\min_{x\in F} \frac{1}{2}\langle Ax,\ x\rangle - \langle x,\ b\rangle,$$

where F is the fixed point set of a nonexpansive mapping S defined on H and b is a given point in H.

A linear-bounded operator A is *strongly positive* if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax,\ x\rangle \geq \bar{\gamma}||x||^2, \quad \forall x \in H.$$

Recently, Marino and Xu [5] introduced a new iterative scheme by the *viscosity approximation method*:

$$x_{n+1} = \varepsilon_n \gamma f(x_n) + (1 - \varepsilon_n A)Sx_n. \tag{1.2}$$

They proved that the sequences $\{x_n\}$ generated by (1.2) converges strongly to the unique solution of the variational inequality

$$\langle \gamma fz - Az,\ x - z\rangle \leq 0, \quad \forall x \in F(S),$$

which is the optimality condition for the minimization problem:

$$\min_{x\in F(S)} \frac{1}{2}\langle Ax,\ x\rangle - h(x),$$

where h is a potential function for γf.

For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequalities for a ξ-inverse-strongly monotone mapping, Takahashi and Toyoda [6] introduced the following iterative scheme:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrary}, \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n)SP_C(x_n - \alpha_n Bx_n), \quad \forall n \geq 0, \end{cases} \tag{1.3}$$

where B is a ξ-inverse-strongly monotone mapping, $\{\gamma_n\}$ is a sequence in (0, 1), and $\{\alpha_n\}$ is a sequence in $(0, 2\xi)$. They showed that if $F(S) \cap \mathrm{VI}(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.3) converges weakly to some $z \in F(S) \cap \mathrm{VI}(C, B)$.

The method of the steepest descent, also known as The Gradient Descent, is the simplest of the gradient methods. By means of simple optimization algorithm, this popular method can find the local minimum of a function. It is a method that is widely popular among mathematicians and physicists due to its easy concept.

For finding a common element of $F(S) \cap \mathrm{VI}(C, B)$, let $S : H \to H$ be nonexpansive mappings, Yamada [7] introduced the following iterative scheme called the *hybrid steepest descent method*:

$$x_{n+1} = Sx_n - \alpha_n \mu BSx_n, \quad \forall n \geq 1, \tag{1.4}$$

where $x_1 = x \in H$, $\{\alpha_n\} \subset (0, 1)$, $B : H \to H$ is a strongly monotone and Lipschitz continuous mapping and μ is a positive real number. He proved that the sequence $\{x_n\}$ generated by (1.4) converged strongly to the unique solution of the $F(S) \cap \mathrm{VI}(C, B)$.

On the other hand, for finding an element of $F(S) \cap \mathrm{VI}(C, B) \cap \mathrm{EP}(F)$, Su et al. [8] introduced the following iterative scheme by the viscosity approximation method in Hilbert spaces: $x_1 \in H$

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \quad y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)SP_C(u_n - \lambda_n Bu_n), \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $\alpha_n \subset [0, 1)$ and $r_n \subset (0, \infty)$ satisfy some appropriate conditions. Furthermore, they prove $\{x_n\}$ and $\{u_n\}$ converge strongly to the same point $z \in F(S) \cap \mathrm{VI}(C, B) \cap \mathrm{EP}(F)$, where $z = P_{F(S)\cap \mathrm{VI}(C,B)\,\cap\,\mathrm{EP}(F)}f(z)$.

For finding a common element of $F(S) \cap \mathrm{GEP}(F, D)$, let C be a nonempty closed convex subset of a real Hilbert space H. Let D be a β-inverse-strongly monotone mapping of C into H, and let S be a nonexpansive mapping of C into itself, Takahashi and Takahashi [9] introduced the following iterative scheme:

$$\begin{cases} F(u_n, y) + \langle Dx_n,\ y - u_n\rangle + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x + (1-\alpha_n)u_n, \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n)Sy_n, \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\gamma_n\} \subset [0, 1]$ and $\{r_n\} \subset [0, 2\beta]$ satisfy some parameters controlling conditions. They proved that the sequence $\{x_n\}$ defined by (1.6) converges strongly to a common element of $F(S) \cap \mathrm{GEP}(F, D)$.

Recently, Chantarangsi et al. [10] introduced a new iterative algorithm using a viscosity hybrid steepest descent method for solving a common solution of a generalized mixed equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality problem in a real Hilbert space. Jaiboon [11] suggests and analyzes an iterative scheme based on the hybrid steepest descent method for finding a common element of the set of solutions of a system of equilibrium problems, the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problems for inverse strongly monotone mappings in Hilbert spaces.

In this article, motivated and inspired by the studies mentioned above, we introduce an iterative scheme using a relaxed hybrid steepest descent method for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality problems for inverse strongly monotone mapping in a real Hilbert space. Our results improve and extend the corresponding results of Jung [12] and some others.

2. Preliminaries

Throughout this article, we always assume H to be a real Hilbert space, and let C be a nonempty closed convex subset of H. For a sequence $\{x_n\}$, the notation of $x_n \rightharpoonup x$ and $x_n \to x$ means that the sequence $\{x_n\}$ converges weakly and strongly to x, respectively.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$||x - P_C x|| \leq ||x - y||, \quad \forall x \in C.$$

Such a mapping P_C from H onto C is called the metric projection.

The following known lemmas will be used in the proof of our main results.

Lemma 2.1. *Let H be a real Hilbert spaces H. Then, the following identities hold:*

(i) *for each* $x \in H$ *and* $x^* \in C$, $x^* = P_C x \Leftrightarrow \langle x - x^*, y - x^* \rangle \leq 0$, $\forall y \in C$;
(ii) $P_C : H \to C$ *is nonexpansive, that is,* $||P_C x - P_C y|| \leq ||x - y||$, $\forall x, y \in H$;
(iii) P_C *is firmly nonexpansive, that is,* $||P_C x - P_C y||^2 \leq \langle P_C x - P_C y, x - y \rangle$, $\forall x, y \in H$;
(iv) $||tx + (1 - t)y||^2 = t||x||^2 + (1 - t)||y||^2 - t(1 - t)||x - y||^2$, $\forall t \in [0, 1]$, $\forall x, y \in H$;
(v) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y \rangle$.

Lemma 2.2. [2]*Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let B be a mapping of C into H. Let* $x^* \in C$. *Then, for* $\lambda > 0$,

$$x^* \in VI(C, B) \Leftrightarrow x^* = P_C(x^* - \lambda B x^*),$$

where P_C *is the metric projection of H onto C.*

Lemma 2.3. [2]*Let H be a Hilbert space, and let C be a nonempty closed convex subset of H. Let* $\beta > 0$, *and let* $A : C \to H$ *be* β*-inverse strongly monotone. If* $0 < \varrho \leq 2\beta$, *then* $I - \varrho A$ *is a nonexpansive mapping of C into H, where I is the identity mapping on H.*

Lemma 2.4. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, let* $S : C \to C$ *be a nonexpansive mapping, and let* $B : C \to H$ *be a* ξ*-inverse strongly monotone. If* $0 < \alpha_n \leq 2\xi$, *then* $S - \alpha_n BS$ *is a nonexpansive mapping in H.*

Proof. For any $x, y \in C$ and $0 < \alpha_n \le 2\xi$, we have

$$\begin{aligned}\|(S-\alpha_n BS)x-(S-\alpha_n BS)y\|^2 &= ||(Sx-Sy)-\alpha_n(BSx-BSy)||^2\\ &= ||Sx-Sy||^2-2\alpha_n\langle Sx-Sy,\ BSx-BSy\rangle+\alpha_n^2||BSx-BSy||^2\\ &\le ||x-y||^2-2\alpha_n\xi||BSx-BSy||+\alpha_n^2||BSx-BSy||^2\\ &= ||x-y||^2+\alpha_n(\alpha_n-2\xi)||BSx-BSy||^2\\ &\le ||x-y||^2.\end{aligned}$$

Hence, $S - \alpha_n BS$ is a nonexpansive mapping of C into H. □

Lemma 2.5. [13]*Let B be a monotone mapping of C into H and let $N_C w_1$ be the normal cone to C at $w_1 \in C$, that is, $N_C w_1 = \{w \in H : \langle w_1 - w_2, w\rangle \ge 0, \forall w_2 \in C\}$ and define a mapping Q on C by*

$$Qw_1 = \begin{cases} Bw_1 + N_C w_1, & w_1 \in C;\\ \emptyset, & w_1 \notin C.\end{cases}$$

Then, Q is maximal monotone and $0 \in Qw_1$ if and only if $w_1 \in VI(C, B)$.

Lemma 2.6. [14]*Each Hilbert space H satisfies Opial's condition, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty}||x_n-x|| < \liminf_{n\to\infty}||x_n-y||$$

holds for each $y \in H$ with $y \ne x$.

Lemma 2.7. [5]*Let C be a nonempty closed convex subset of H and let f be a contraction of H into itself with coefficient $\eta \in (0, 1)$ and A be a strongly positive linear-bounded operator on H with coefficient $\bar{\gamma} > 0$. Then, for $0 < \gamma < \frac{\bar{\gamma}}{\eta}$,*

$$\langle x-y,\ (A-\gamma f)x-(A-\gamma f)y\rangle \ge (\bar{\gamma}-\eta\gamma)||x-y||^2, \quad x,\ y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma}-\eta\gamma$.

Lemma 2.8. [5]*Assume A to be a strongly positive linear-bounded operator on H with coefficient $\bar{\gamma} > 0$and $0 < \rho \le ||A||^{-1}$. Then, $||I-\rho A|| \le 1-\rho\bar{\gamma}$.*

For solving the generalized mixed equilibrium problem and the mixed equilibrium problem, let us give the following assumptions for the bifunction F, the function ϕ and the set C:

(H1) $F(x, x) = 0, \forall x \in C$;
(H2) F is monotone, that is, $F(x, y) + F(y, x) \le 0\ \forall x, y \in C$;
(H3) for each $y \in C$, x α $F(x, y)$ is weakly upper semicontinuous;
(H4) for each $x \in C$, y α $F(x, y)$ is convex;
(H5) for each $x \in C$, y α $F(x, y)$ is lower semicontinuous;
(B1) for each $x \in H$ and λ >0, there exist abounded subset $G_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \backslash n\ G_x$,

$$F(z,\ y_x)+\varphi(y_x)-\varphi(z)+\tfrac{1}{\lambda}\langle y_x-z,\ z-x\rangle < 0; \tag{2.1}$$

(B2) C is a bounded set.

Lemma 2.9. [15]*Let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfies (H1)-(H5), and let $\phi : C \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semi continuous and convex function. Assume that either (B1) or (B2) holds. For $\lambda > 0$ and $x \in H$,* *define a mapping* $T_\lambda^{(F,\varphi)} : H \to C$*as follows:*

$$T_\lambda^{(F,\varphi)}(x) = \left\{ z \in C : \ F(z, \ y) + \varphi(y) - \varphi(z) + \frac{1}{\lambda}\langle y - z, \ z - x\rangle \geq 0, \ y \in C \right\}, \quad \forall z \in H.$$

Then, the following properties hold:

(i) *For each* $x \in H$, $T_\lambda^{(F,\varphi)}(x) \neq \emptyset$;
(ii) $T_\lambda^{(F,\varphi)}$*is single-valued;*
(iii) $T_\lambda^{(F,\varphi)}$*is firmly nonexpansive, that is, for any* $x, y \in H$,

$$||T_\lambda^{(F,\varphi)}x - T_\lambda^{(F,\varphi)}y||^2 \leq \left\langle T_\lambda^{(F,\varphi)}x - T_\lambda^{(F,\varphi)}y, \ x - y \right\rangle;$$

(iv) $F(T_\lambda^{(F,\varphi)}) = MEP(F, \ \varphi)$;
(v) $MEP(F, \phi)$ *is closed and convex.*

Lemma 2.10. [16]*Assume* $\{a_n\}$ *to be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n)a_n + c_n, \quad n \geq 0,$$

where $\{b_n\}$ *is a sequence in* (0, 1) *and* $\{c_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} b_n = \infty$,
(2) $\limsup_{n-\infty} \frac{c_n}{b_n} \leq 0$*or* $\sum_{n=1}^{\infty} |c_n| < \infty$

Then, $\lim_{n \to \infty} a_n = 0$.

3. Main results

In this section, we are in a position to state and prove our main results.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be bifunction from* $C \times C$ *to*$\mathbb{R}$ *satisfying (H1)-(H5), and let* $\phi : C \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function with either (B1) or (B2). Let B, D be two* ξ, β*-inverse strongly monotone mapping of C into H, respectively, and let* $S : C \to C$ *be a nonexpansive mapping. Let* $f : C \to C$ *be a contraction mapping with* $\eta \in (0, 1)$, *and let A be a strongly positive linear-bounded operator with* $\bar{\gamma} > 0$*and* $0 < \gamma < \frac{\bar{\gamma}}{\eta}$. *Assume that* $\Theta := F(S) \cap VI(C, B) \cap GMEP(F, \phi, D) \neq \emptyset$. *Let* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *be sequences generated by the following iterative algorithm:*

$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrary}, \\ u_n = T_{\lambda_n}^{(F,\varphi)}(x_n - \lambda_n Dx_n), \\ y_n = \beta_n \gamma f(x_n) + (I - \beta_n A)P_C(Su_n - \alpha_n BSu_n), \\ x_{n+1} = (1 - \delta_n)y_n + \delta_n P_C(Sy_n - \alpha_n BSy_n), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\delta n\}$ *and* $\{\beta_n\}$ *are two sequences in* (0, 1) *satisfying the following conditions:*

(C1) $\lim_{n \to \infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$,
(C2) $\{\delta_n\} \subset [0, b]$, *for some* $b \in (0, 1)$ *and* $\lim_{n \to \infty} |\delta_{n+1} - \delta_n| = 0$,
(C3) $\{\lambda_n\} \subset [c, d] \subset (0, 2\beta)$ *and* $\lim_{n \to \infty} |\lambda_{n+1} - \lambda_n| = 0$,
(C4) $\{\alpha_n\} \subset [e, g] \subset (0, 2\xi)$ *and* $\lim_{n \to \infty} |\alpha_{n+1} - \alpha_n| = 0$.

Then, $\{x_n\}$ *converges strongly to* $z \in \Theta$, *which is the unique solution of the variational inequality*

$$\langle \gamma f(z) - Az, \ x - z \rangle \leq 0, \quad \forall x \in \Theta. \tag{3.2}$$

Proof. We may assume, in view of $\beta_n \to 0$ as $n \to \infty$, that $\beta_n \in (0, ||A||^{-1})$. By Lemma 2.8, we obtain $||I - \beta_n A|| \leq 1 - \beta_n \bar{\gamma}$, $\forall_n \in \mathbb{N}$.

We divide the proof of Theorem 3.1 into six steps.

Step 1. We claim that the sequence $\{x_n\}$ is bounded.

Now, let $p \in \Theta$. Then, it is clear that

$$p = Sp = P_C(p - \alpha_n Bp) = T_{\lambda_n}^{(F,\varphi)}(p - \lambda_n Dp).$$

Let $u_n = T_{\lambda_n}^{(F,\varphi)}(x_n - \lambda_n Dx_n) \in \operatorname{dom} \varphi$, D be β-inverse strongly monotone and $0 \leq \lambda_n \leq 2\beta$. Then, we have

$$||u_n - p|| \leq ||x_n - p||. \tag{3.3}$$

Let $z_n = PC(Su_n - \alpha_n BSu_n)$ and $S - \alpha_n BS$ be a nonexpansive mapping. Then, we have from Lemma 2.4 that

$$||z_n - p|| \leq ||u_n - p|| \leq ||x_n - p|| \tag{3.4}$$

and

$$\begin{aligned} ||y_n - p|| &\leq \beta_n ||\gamma f(x_n) - Ap|| + ||1 - \beta_n A|| ||z_n - p|| \\ &\leq \beta_n ||\gamma f(x_n) - Ap|| + (1 - \beta_n \bar{\gamma}) ||z_n - p|| \\ &\leq \beta_n \gamma ||f(x_n) - f(p)|| + \beta_n ||\gamma f(p) - Ap|| + (1 - \beta_n \bar{\gamma}) ||x_n - p|| \\ &\leq \beta_n \gamma \eta ||x_n - p|| + \beta_n ||\gamma f(p) - Ap|| + (1 - \beta_n \bar{\gamma}) ||x_n - p|| \\ &= (1 - (\bar{\gamma} - \eta\gamma)\beta_n) ||x_n - p|| + \beta_n ||\gamma f(p) - Ap||. \end{aligned}$$

Similarly, and let $w_n = P_C(Sy_n - \alpha_n BSy_n)$ in (3.4). Then, we can prove that

$$||w_n - p|| \leq ||y_n - p|| \leq (1 - (\bar{\gamma} - \eta\gamma)\beta_n) ||x_n - p|| + \beta_n ||\gamma f(p) - Ap||, \tag{3.5}$$

which yields that

$$\begin{aligned} ||x_{n+1} - p|| &\leq (1 - \delta_n) ||y_n - p|| + \delta_n ||w_n - p|| \\ &\leq (1 - \delta_n) ||y_n - p|| + \delta_n ||y_n - p|| \\ &= ||y_n - p||| \\ &\leq (1 - (\bar{\gamma} - \eta\gamma)\beta_n) ||x_n - p|| + \beta_n ||\gamma f(p) - Ap|| \\ &= (1 - (\bar{\gamma} - \eta\gamma)\beta_n) ||x_n - p|| + \frac{(\bar{\gamma} - \eta\gamma)\beta_n}{(\bar{\gamma} - \eta\gamma)} ||\gamma f(p) - Ap|| \\ &\leq \max\left\{ ||x_n - p||, \ \frac{||\gamma f(p) - Ap||}{(\bar{\gamma} - \eta\gamma)} \right\} \\ &\leq \ldots \\ &\leq \max\left\{ ||x_1 - p||, \ \frac{||\gamma f(p) - Ap||}{(\bar{\gamma} - \eta\gamma)} \right\}, \quad \forall n \geq 1. \end{aligned}$$

This shows that $\{x_n\}$ is bounded. Hence, $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, $\{w_n\}$, $\{BSu_n\}$, $\{BSy_n\}$, $\{Az_n\}$ and $\{f(x_n)\}$ are also bounded.

We can choose some appropriate constant $M > 0$ such that

$$M \geq \max\left\{\sup_{n\geq 1}\{||BSu_n||\}, \sup_{n\geq 1}\{||BSy_n||\}, \sup_{n\geq 1}\{||\gamma f(x_n) - Az_n||\}, \sup_{n\geq 1}\{||u_n - x_n||\}, \sup_{n\geq 1}\{||w_n - y_n||\right\}. \tag{3.6}$$

Step 2. We claim that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$.

It follows from Lemma 2.9 that $u_{n-1} = T_{\lambda_{n-1}}^{(F,\varphi)}(x_{n-1} - \lambda_{n-1}Dx_{n-1})$ and $u_n = T_{\lambda_n}^{(F,\varphi)}(x_n - \lambda_n Dx_n)$ for all $n \geq 1$, and we get

$$F(u_{n-1}, y) + \varphi(y) - \varphi(u_{n-1}) + \langle Dx_{n-1}, y - u_{n-1}\rangle + \frac{1}{\lambda_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0, \quad \forall y \in C \tag{3.7}$$

and

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Dx_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.8}$$

Take $y = u_{n-1}$ in (3.8) and $y = u_n$ in (3.7), and then we have

$$F(u_{n-1}, u_n) + \varphi(u_n) - \varphi(u_{n-1}) + \langle Dx_{n-1}, u_n - u_{n-1}\rangle + \frac{1}{\lambda_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0$$

and

$$F(u_n, u_{n-1}) + \varphi(u_{n-1}) - \varphi(u_n) + \langle Dx_n, u_{n-1} - u_n\rangle + \frac{1}{\lambda_n}\langle u_{n-1} - u_n, u_n - x_n\rangle \geq 0.$$

Adding the above two inequalities, the monotonicity of F implies that

$$\langle Dx_n - Dx_{n-1}, u_{n-1} - u_n\rangle + \left\langle u_{n-1} - u_n, \frac{u_n - x_n}{\lambda_n} - \frac{u_{n-1} - x_{n-1}}{\lambda_{n-1}}\right\rangle \geq 0$$

and

$$\begin{aligned}
0 &\leq \left\langle u_{n-1} - u_n, \lambda_{n-1}(Dx_n - Dx_{n-1}) + \frac{\lambda_{n-1}}{\lambda_n}(u_n - x_n) - (u_{n-1} - x_{n-1})\right\rangle \\
&= \left\langle u_n - u_{n-1}, u_{n-1} - u_n + \left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right)u_n + (x_n - \lambda_{n-1}Dx_n) \right. \\
&\quad \left. - (x_{n-1} - \lambda_{n-1}Dx_{n-1}) - x_n + \frac{\lambda_{n-1}}{\lambda_n}x_n\right\rangle \\
&= \left\langle u_n - u_{n-1}, u_{n-1} - u_n + \left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right)(u_n - x_n) + (x_n - \lambda_{n-1}Dx_n) \right. \\
&\quad \left. - (x_{n-1} - \lambda_{n-1}Dx_{n-1})\right\rangle.
\end{aligned}$$

Without loss of generality, let us assume that there exists $c \in \mathbb{R}$ such that $\lambda_n > c > 0$, $\forall n \geq 1$. Then, we have

$$||u_n - u_{n-1}||^2 \leq ||u_n - u_{n-1}|| \left\{||x_n - x_{n-1}|| + \left|1 - \tfrac{\lambda_{n-1}}{\lambda_n}\right| ||u_n - x_n||\right\}$$

and hence,

$$\begin{aligned}||u_n - u_{n-1}|| &\le ||x_n - x_{n-1}|| + \frac{1}{\lambda_n}|\lambda_n - \lambda_{n-1}|||u_n - x_n|| \\ &\le ||x_n - x_{n-1}|| + \frac{1}{c}|\lambda_n - \lambda_{n-1}|M.\end{aligned} \tag{3.9}$$

Since $S - \alpha_n BS$ is nonexpansive for each $n \ge 1$, we have

$$\begin{aligned}||z_n - z_{n-1}|| &= ||P_C(Su_n - \alpha_n BSu_n) - P_C(Su_{n-1} - \alpha_{n-1}BSu_{n-1})|| \\ &\le ||(Su_n - \alpha_n BSu_n) - (Su_{n-1} - \alpha_{n-1}BSu_{n-1})|| \\ &= ||(Su_n - \alpha_n BSu_n) - (Su_{n-1} - \alpha_n BSu_{n-1}) + (\alpha_{n-1} - \alpha_n)BSu_{n-1}|| \\ &\le ||(Su_n - \alpha_n BSu_n) - (Su_{n-1} - \alpha_n BSu_{n-1})|| + |\alpha_{n-1} - \alpha_n|||BSu_{n-1}|| \\ &\le ||u_n - u_{n-1}|| + |\alpha_{n-1} - \alpha_n|||BSu_{n-1}||.\end{aligned} \tag{3.10}$$

Substituting (3.9) into (3.10), we obtain

$$||z_n - z_{n-1}|| \le \ ||x_n - x_{n-1}|| + \frac{1}{c}|\lambda_n - \lambda_{n-1}|M + |\alpha_{n-1} - \alpha_n|||BSu_{n-1}||. \tag{3.11}$$

From (3.1), we have

$$\begin{aligned}||y_n - y_{n-1}|| &= ||\beta_n \gamma f(x_n) + (I - \beta_n A)z_n - \beta_{n-1}\gamma f(x_{n-1}) - (I - \beta_{n-1}A)z_{n-1}|| \\ &= ||\beta_n \gamma (f(x_n) - f(x_{n-1})) + (\beta_n - \beta_{n-1})\gamma f(x_{n-1}) \\ &\quad + (I - \beta_n A)(z_n - z_{n-1}) - (\beta_n - \beta_{n-1})Az_{n-1}|| \\ &= ||\beta_n \gamma (f(x_n) - f(x_{n-1})) + (\beta_n - \beta_{n-1})(\gamma f(x_{n-1}) - Az_{n-1}) \\ &\quad + (I - \beta_n A)(z_n - z_{n-1})|| \\ &\le \beta_n \gamma ||f(x_n) - f(x_{n-1})|| + |\beta_n - \beta_{n-1}|||\gamma f(x_{n-1}) - Az_{n-1}|| \\ &\quad + (I - \beta_n A)||z_n - z_{n-1}|| \\ &\le \beta_n \gamma \eta ||x_n - x_{n-1}|| + |\beta_n - \beta_{n-1}|||\gamma f(x_{n-1}) - Az_{n-1}|| \\ &\quad + (1 - \beta_n \bar{\gamma})||z_n - z_{n-1}||.\end{aligned} \tag{3.12}$$

Substituting (3.11) into (3.12) yields

$$\begin{aligned}||y_n - y_{n-1}|| &\le \beta_n \gamma \eta ||x_n - x_{n-1}|| + |\beta_n - \beta_{n-1}|||\gamma f(x_{n-1}) - Az_{n-1}|| \\ &\quad + (1 - \beta_n \bar{\gamma})\left\{||x_n - x_{n-1}|| + \frac{1}{c}|\lambda_n - \lambda_{n-1}|M + |\alpha_{n-1} - \alpha_n|||BSu_{n-1}||\right\} \\ &= (1 - (\bar{\gamma} - \gamma\eta)\beta_n)||x_n - x_{n-1}|| + |\beta_n - \beta_{n-1}|||\gamma f(x_{n-1}) - Az_{n-1}|| \\ &\quad + \frac{(1 - \beta_n \bar{\gamma})}{c}|\lambda_n - \lambda_{n-1}|M + (1 - \beta_n \bar{\gamma})|\alpha_{n-1} - \alpha_n|||BSu_{n-1}||.\end{aligned} \tag{3.13}$$

Since $w_n = P_C(Sy_n - \alpha_n BSy_n)$ and $S - \alpha_n BS$ is nonexpansive mapping, we have

$$\begin{aligned}||w_n - w_{n-1}|| &= ||P_C(Sy_n - \alpha_n BSy_n) - P_C(Sy_{n-1} - \alpha_{n-1}BSy_{n-1})|| \\ &\le ||(Sy_n - \alpha_n BSy_n) - (Sy_{n-1} - \alpha_{n-1}BSy_{n-1})|| \\ &= ||(Sy_n - \alpha_n BSy_n) - (Sy_{n-1} - \alpha_n BSy_{n-1}) + (\alpha_{n-1} - \alpha_n)BSy_{n-1}|| \\ &\le ||y_n - y_{n-1}|| + |\alpha_{n-1} - \alpha_n|||BSy_{n-1}||.\end{aligned} \tag{3.14}$$

Also, from (3.1) and (3.13), we have

$$\begin{aligned}
||x_{n+1}-x_n|| &= ||(1-\delta_n)y_n+\delta_n w_n-\{(1-\delta_{n-1})y_{n-1}+\delta_{n-1}w_{n-1}\}||\\
&= ||(1-\delta_n)(y_n-y_{n-1})+\delta_n(w_n-w_{n-1})+(\delta_n-\delta_{n-1})(w_{n-1}-y_{n-1})||\\
&\le (1-\delta_n)||y_n-y_{n-1}||+\delta_n||w_n-w_{n-1}||+|\delta_n-\delta_{n-1}|||w_{n-1}-y_{n-1}||\\
&\le (1-\delta_n)||y_n-y_{n-1}||+\delta_n\{||y_n-y_{n-1}||+|\alpha_{n-1}-\alpha_n|||BSy_{n-1}||\}\\
&\quad +|\delta_n-\delta_{n-1}|||w_{n-1}-y_{n-1}||\\
&= ||y_n-y_{n-1}||+\delta_n|\alpha_{n-1}-\alpha_n|||BSy_{n-1}||+|\delta_n-\delta_{n-1}|||w_{n-1}-y_{n-1}||\\
&\le (1-(\bar{\gamma}-\gamma\eta)\beta_n)||x_n-x_{n-1}||+|\beta_n-\beta_{n-1}|||\gamma f(x_{n-1})-Az_{n-1}||\\
&\quad +\frac{(1-\beta_n\bar{\gamma})}{c}|\lambda_n-\lambda_{n-1}|M+(1-\beta_n\bar{\gamma})|\alpha_{n-1}-\alpha_n|||BSu_{n-1}||\\
&\quad +\delta_n|\alpha_{n-1}-\alpha_n|||BSy_{n-1}||+|\delta_n-\delta_{n-1}|||w_{n-1}-y_{n-1}||\\
&\le (1-(\bar{\gamma}-\gamma\eta)\beta_n)||x_n-x_{n-1}||+\Big\{|\beta_n-\beta_{n-1}|+\frac{(1-\beta_n\bar{\gamma})}{c}|\lambda_n-\lambda_{n-1}|\\
&\quad +(1-\beta_n\bar{\gamma}+\delta_n)|\alpha_{n-1}-\alpha_n|+|\delta_n-\delta_{n-1}|\Big\}M.
\end{aligned} \tag{3.15}$$

Set $b_n=(\bar{\gamma}-\gamma\eta)\beta_n$ and

$$c_n=\left\{|\beta_n-\beta_{n-1}|+\tfrac{(1-\beta_n\bar{\gamma})}{c}|\lambda_n-\lambda_{n-1}|+(1-\beta_n\bar{\gamma}+\delta_n)|\alpha_{n-1}-\alpha_n|+|\delta_n-\delta_{n-1}|\right\}M.$$

Then, we have

$$||x_{n+1}-x_n||\le(1-b_n)||x_n-x_{n-1}||+c_n,\quad \forall n\ge 0. \tag{3.16}$$

From the conditions (C1)-(C4), we find that

$$\lim_{n\to\infty} b_n=0,\quad \sum_{n=0}^{\infty} b_n=\infty \quad\text{and}\quad \limsup_{n\to\infty} c_n\le 0.$$

Therefore, applying Lemma 2.10 to (3.16), we have

$$\lim_{n\to\infty}||x_{n+1}-x_n||=0. \tag{3.17}$$

Step 3. We claim that $\lim_{n\to\infty}||Sw_n - w_n|| = 0$.
For any $p\in\Theta$ and Lemma 2.4, we obtain

$$\begin{aligned}
||z_n-p||^2 &= ||P_C(Su_n-\alpha_n BSu_n)-P_C(p-\alpha_n Bp)||^2\\
&\le ||(Su_n-\alpha_n BSu_n)-(p-\alpha_n Bp)||^2\\
&= ||(Su_n-\alpha_n BSu_n)-(Sp-\alpha_n BSp)||^2\\
&\le ||x_n-p||^2+(\alpha_n^2-2\alpha_n\xi)||BSu_n-Bp||^2.
\end{aligned} \tag{3.18}$$

From (3.1) and (3.18), we have

$$\begin{aligned}
\|y_n-p\|^2 &= ||\beta_n(\gamma f(x_n)-Ap)+(I-\beta_n A)(z_n-p)||^2\\
&= ||(I-\beta_n A)(z_n-p)||^2+\beta_n^2||\gamma f(x_n)-Ap||^2\\
&\quad +2\beta_n\langle (I-\beta_n A)(z_n-p),\ \gamma f(x_n)-Ap\rangle\\
&\le (1-\beta_n\bar{\gamma})^2||z_n-p||^2+\beta_n^2||\gamma f(x_n)-Ap||^2\\
&\quad +2\beta_n\langle (I-\beta_n A)(z_n-p),\ \gamma f(x_n)-Ap\rangle\\
&\le (1-\beta_n\bar{\gamma})^2\left\{||x_n-p||^2+(\alpha_n^2-2\alpha_n\xi)||BSu_n-Bp||^2\right\}\\
&\quad +\beta_n^2||\gamma f(x_n)-Ap||^2+2\beta_n\langle (I-\beta_n A)(z_n-p),\ \gamma f(x_n)-Ap\rangle\\
&= (1-\beta_n\bar{\gamma})^2||x_n-p||^2+(1-\beta_n\bar{\gamma})^2(\alpha_n^2-2\alpha_n\xi)||BSu_n-Bp||^2\\
&\quad +\beta_n^2||\gamma f(x_n)-Ap||^2+2\beta_n\langle (I-\beta_n A)(z_n-p),\ \gamma f(x_n)-Ap\rangle\\
&\le ||x_n-p||^2+(1-\beta_n\bar{\gamma})^2(\alpha_n^2-2\alpha_n\xi)||BSu_n-Bp||^2\\
&\quad +\beta_n^2||\gamma f(x_n)-Ap||^2+2\beta_n\langle (I-\beta_n A)(z_n-p),\ \gamma f(x_n)-Ap\rangle.
\end{aligned} \tag{3.19}$$

From (3.1), (3.5), (3.19) and Lemma 2.1(iv), we have

$$\begin{aligned} ||x_{n+1}-p||^2 &\le (1-\delta_n)||y_n-p||^2+\delta_n||w_n-p||^2 \\ &\le (1-\delta_n)||y_n-p||^2+\delta_n||y_n-p||^2 \\ &\le ||y_n-p||^2 \\ &\le ||x_n-p||^2+(1-\beta_n\bar{\gamma})^2(\alpha_n^2-2\alpha_n\xi)||BSu_n-Bp||^2 \\ &\quad +\beta_n^2||\gamma f(x_n)-Ap||^2+2\beta_n\langle (I-\beta_nA)(z_n-p),\ \gamma f(x_n)-Ap\rangle. \end{aligned} \tag{3.20}$$

It follows that

$$\begin{aligned} (1-\beta_n\bar{\gamma})^2(2g\xi-e^2)||BSu_n-Bp||^2 &\le (1-\beta_n\bar{\gamma})^2(2\alpha_n\xi-\alpha_n^2)||BSu_n-Bp||^2 \\ &\le ||x_n-p||^2-||x_{n+1}-p||^2+\beta_n^2||\gamma f(x_n)-Ap||^2 \\ &\quad +2\beta_n\langle (I-\beta_nA)(z_n-p),\ \gamma f(x_n)-Ap\rangle \\ &\le ||x_n-x_{n+1}||(||x_n-p||+||x_{n+1}-p||)+\beta_n^2||\gamma f(x_n)-Ap||^2 \\ &\quad +2\beta_n\langle (I-\beta_nA)(z_n-p),\ \gamma f(x_n)-Ap\rangle. \end{aligned} \tag{3.21}$$

From condition (C1) and (3.17), we obtain

$$\lim_{n\to\infty}||BSu_n-Bp||=0. \tag{3.22}$$

From $w_n = PC(Sy_n - \alpha_n BSy_n)$, (3.19) and Lemma 2.4, we have

$$\begin{aligned} ||w_n-p||^2 &= ||P_C(Sy_n-\alpha_nBSy_n)-P_C(p-\alpha_nBp)||^2 \\ &\le ||(Sy_n-\alpha_nBSy_n)-(p-\alpha_nBp)||^2 \\ &= ||(Sy_n-\alpha_nBSy_n)-(Sp-\alpha_nBSp)||^2 \\ &\le ||y_n-p||^2+(\alpha_n^2-2\alpha_n\xi)||BSy_n-Bp||^2 \\ &\le \Big\{||x_n-p||^2+(1-\beta_n\bar{\gamma})^2(\alpha_n^2-2\alpha_n\xi)||BSu_n-Bp||^2 \\ &\quad +\beta_n^2||\gamma f(x_n)-Ap||^2+2\beta_n\langle (I-\beta_nA)(z_n-p),\ \gamma f(x_n)-Ap\rangle\Big\} \\ &\quad +(\alpha_n^2-2\alpha_n\xi)||BSy_n-Bp||.^2 \end{aligned} \tag{3.23}$$

Using (3.1), (3.19) and (3.23), we obtain

$$\begin{aligned} ||x_{n+1}-p||^2 &\le (1-\delta_n)||y_n-p||^2+\delta_n||w_n-p||^2 \\ &\le (1-\delta_n)\Big\{||x_n-p||^2+(1-\beta_n\bar{\gamma})^2(\alpha_n^2-2\alpha_n\xi)||BSu_n-Bp||^2 \\ &\quad +\beta_n^2||\gamma f(x_n)-Ap||^2+2\beta_n\langle (I-\beta_nA)(z_n-p),\ \gamma f(x_n)-Ap\rangle\Big\} \\ &\quad +\delta_n\Big\{||x_n-p||^2+(1-\beta_n\bar{\gamma})^2(\alpha_n^2-2\alpha_n\xi)||BSu_n-Bp||^2 \\ &\quad +\beta_n^2||\gamma f(x_n)-Ap||^2+2\beta_n\langle (I-\beta_nA)(z_n-p),\ \gamma f(x_n)-Ap\rangle] \\ &\quad +(\alpha_n^2-2\alpha_n\xi)||BSy_n-Bp||^2\Big\} \\ &= ||x_n-p||^2+(1-\beta_n\bar{\gamma})^2(\alpha_n^2-2\alpha_n\xi)||BSu_n-Bp||^2 \\ &\quad +\beta_n^2||\gamma f(x_n)-Ap||^2+2\beta_n\langle (I-\beta_nA)(z_n-p),\ \gamma f(x_n)-Ap\rangle \\ &\quad +(\alpha_n^2-2\alpha_n\xi)\delta_n||BSy_n-Bp||.^2 \end{aligned} \tag{3.24}$$

It follows that

$$\begin{aligned} (2g\xi-e^2)b||BSy_n-Bp||^2 &\le ||x_n-x_{n+1}||(||x_n-p||+||x_{n+1}-p||) \\ &\quad +(1-\beta_n\bar{\gamma})^2(\alpha_n^2-2\alpha_n\xi)||BSu_n-Bp||^2+\beta_n^2||\gamma f(x_n)-Ap||^2 \\ &\quad +2\beta_n\langle (I-\beta_nA)(z_n-p),\ \gamma f(x_n)-Ap\rangle. \end{aligned} \tag{3.25}$$

From condition (C1), (3.17) and (3.22), we obtain

$$\lim_{n\to\infty} ||BSy_n - Bp|| = 0. \tag{3.26}$$

Since P_C is firmly nonexpansive, we have

$$\begin{aligned} ||w_n - p||^2 &= ||P_C(Sy_n - \alpha_n BSy_n) - P_C(p - \alpha_n Bp)||^2 \\ &\le \langle (Sy_n - \alpha_n BSy_n) - (p - \alpha_n Bp),\ w_n - p\rangle \\ &= \frac{1}{2}\left\{||(Sy_n - \alpha_n BSy_n) - (p - \alpha_n Bp)||^2 + ||w_n - p||^2 \right. \\ &\quad \left. -||(Sy_n - \alpha_n BSy_n) - (p - \alpha_n Bp) - (w_n - p)||^2\right\} \\ &\le \frac{1}{2}\left\{||y_n - p||^2 + ||w_n - p||^2 - ||(Sy_n - w_n) - \alpha_n(BSy_n - Bp)||^2\right\} \\ &\le \frac{1}{2}(||x_n - p||^2 + (1-\beta_n\bar{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)||BSu_n - Bp||^2 \\ &\quad + \beta_n^2||\gamma f(x_n) - Ap||^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p),\ \gamma f(x_n) - Ap\rangle) \\ &\quad + \frac{1}{2}\left\{||w_n - p||^2 - ||Sy_n - w_n||^2 \right. \\ &\quad \left. -\alpha_n^2||BSy_n - Bp||^2 + 2\alpha_n\langle Sy_n - w_n, BSy_n - Bp\rangle\right\}. \end{aligned} \tag{3.27}$$

Hence, we have

$$\begin{aligned} ||w_n - p||^2 &\le ||x_n - p||^2 - ||Sy_n - w_n||^2 + (1-\beta_n\bar{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)||BSu_n - Bp||^2 \\ &\quad + \beta_n^2||\gamma f(x_n) - Ap||^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p),\ \gamma f(x_n) - Ap\rangle \\ &\quad + 2\alpha_n||Sy_n - w_n||||BSy_n - Bp||. \end{aligned} \tag{3.28}$$

Using (3.24) and (3.28), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &\le (1-\delta_n)||y_n - p||^2 + \delta_n||w_n - p||^2 \\ &\le (1-\delta_n)\{||x_n - p||^2 + (1-\beta_n\bar{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)||BSu_n - Bp||^2 \\ &\quad + \beta_n^2||\gamma f(x_n) - Ap||^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p),\ \gamma f(x_n) - Ap\rangle\} \\ &\quad + \delta_n\{||x_n - p||^2 - ||Sy_n - w_n||^2 \\ &\quad + (1-\beta_n\bar{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)||BSu_n - Bp||^2 + 2\alpha_n||Sy_n - w_n||||BSy_n - Bp|| \\ &\quad + \beta_n^2||\gamma f(x_n) - Ap||^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p),\ \gamma f(x_n) - Ap\rangle\} \\ &= ||x_n - p||^2 - \delta_n||Sy_n - w_n||^2 \\ &\quad + (1-\beta_n\bar{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)||BSu_n - Bp||^2 + 2\alpha_n\delta_n||Sy_n - w_n||||BSy_n - Bp|| \\ &\quad + \beta_n^2||\gamma f(x_n) - Ap||^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p),\ \gamma f(x_n) - Ap\rangle. \end{aligned} \tag{3.29}$$

It follows that

$$\begin{aligned} b||Sy_n - w_n||^2 \le \delta_n||Sy_n - w_n||^2 &\le ||x_n - x_{n+1}||(||x_n - p|| + ||x_{n+1} - p||) \\ &\quad + (1-\beta_n\bar{\gamma})^2(\alpha_n^2 - 2\alpha_n\xi)||BSu_n - Bp||^2 + 2\alpha_n\delta_n||Sy_n - w_n||||BSy_n - Bp| \\ &\quad + \beta_n^2||\gamma f(x_n) - Ap||^2 + 2\beta_n\langle (I - \beta_n A)(z_n - p),\ \gamma f(x_n) - Ap\rangle. \end{aligned} \tag{3.30}$$

From the condition (C1), (3.17), (3.22) and (3.26), we obtain

$$\lim_{n\to\infty} ||Sy_n - w_n|| = 0. \tag{3.31}$$

Note that

$$\begin{aligned}||y_n - p||^2 &\le (1-\beta_n\bar{\gamma})^2||z_n - p||^2 + \beta_n^2||\gamma f(x_n) - Ap||^2 + 2\beta_n\langle (I-\beta_nA)(z_n - p),\ \gamma f(x_n) - Ap\rangle \\ &\le (1-\beta_n\bar{\gamma})^2||u_n - p||^2 + \beta_n^2||\gamma f(x_n) - Ap||^2 + 2\beta_n\langle (I-\beta_nA)(z_n - p),\ \gamma f(x_n) - Ap\rangle \\ &\le (1-\beta_n\bar{\gamma})^2\{||x_n - p||^2 + \lambda_n(\lambda_n - 2\beta)||Dx_n - Dp||^2\} + \beta_n^2||\gamma f(x_n) - Ap||^2 \\ &\quad + 2\beta_n\langle (I-\beta_nA)(z_n - p),\ \gamma f(x_n) - Ap\rangle \\ &\le ||x_n - p||^2 + (1-\beta_n\bar{\gamma})^2\lambda_n(\lambda_n - 2\beta)||Dx_n - Dp||^2 + \beta_n^2||\gamma f(x_n) - Ap||^2 \\ &\quad + 2\beta_n\langle (I-\beta_nA)(z_n - p),\ \gamma f(x_n) - Ap\rangle. \end{aligned} \tag{3.32}$$

From (3.1) and (3.32), we can compute

$$\begin{aligned}||x_{n+1} - p||^2 &\le (1-\delta_n)||y_n - p||^2 + \delta_n||w_n - p||^2 \\ &\le (1-\delta_n)||y_n - p||^2 + \delta_n||y_n - p||^2 \\ &= ||y_n - p||^2 \\ &\le ||x_n - p||^2 + (1-\beta_n\bar{\gamma})^2\lambda_n(\lambda_n - 2\beta)||Dx_n - Dp||^2 \\ &\quad + \beta_n^2||\gamma f(x_n) - Ap||^2 + 2\beta_n\langle (I-\beta_nA)(z_n - p),\ \gamma f(x_n) - Ap\rangle. \end{aligned} \tag{3.33}$$

It follows that

$$\begin{aligned}(1-\beta_n\bar{\gamma})^2 d(2\beta - c)||Dx_n - Dp||^2 &\le ||x_n - x_{n+1}||(||x_n - p|| + ||x_{n+1} - p||) + \beta_n^2||\gamma f(x_n) - Ap||^2 \\ &\quad + 2\beta_n\langle (I-\beta_nA)(z_n - p),\ \gamma f(x_n) - Ap\rangle, \end{aligned} \tag{3.34}$$

which implies that

$$\lim_{n\to\infty} ||Dx_n - Dp|| = 0. \tag{3.35}$$

In addition, from the firmly nonexpansivity of $T_{\lambda_n}^{(F,\varphi)}$, we have

$$\begin{aligned}||u_n - p||^2 &= ||T_{\lambda_n}^{(F,\varphi)}(x_n - \lambda_nDx_n) - T_{\lambda_n}^{(F,\varphi)}(p - \lambda_nDp)||^2 \\ &\le \langle (x_n - \lambda_nDx_n) - (p - \lambda_nDp),\ u_n - p\rangle \\ &= \frac{1}{2}\{||(x_n - \lambda_nDx_n) - (p - \lambda_nDp)||^2 + ||u_n - p||^2 \\ &\quad - ||(x_n - \lambda_nDx_n) - (p - \lambda_nDp) - (u_n - p)||^2\} \\ &\le \frac{1}{2}\left\{||x_n - p||^2 + ||u_n - p||^2 - ||x_n - u_n - \lambda_n(Dx_n - Dp)||^2\right\} \\ &= \frac{1}{2}\left\{||x_n - p||^2 + ||u_n - p||^2 - ||x_n - u_n||^2 \right. \\ &\quad \left. +2\lambda_n\langle x_n - u_n,\ Dx_n - Dp\rangle - \lambda_n^2||Dx_n - Dp||^2\right\}. \end{aligned}$$

Hence, we obtain

$$||u_n - p||^2 \le ||x_n - p||^2 - ||x_n - u_n||^2 + 2\lambda_n||x_n - u_n||||Dx_n - Dp||. \tag{3.36}$$

Substituting (3.36) into (3.32) to get

$$\begin{aligned}||y_n - p||^2 &\le (1-\beta_n\bar{\gamma})^2||u_n - p||^2 + \beta_n^2||\gamma f(x_n) - Ap||^2 + 2\beta_n\langle (I-\beta_nA)(z_n - p),\ \gamma f(x_n) - Ap\rangle \\ &\le (1-\beta_n\bar{\gamma})^2\left\{||x_n - p||^2 - ||x_n - u_n||^2 + 2\lambda_n||x_n - u_n||||Dx_n - Dp||\right\} \\ &\quad + \beta_n^2||\gamma f(x_n) - Ap||^2 + 2\beta_n\langle (I-\beta_nA)(z_n - p),\ \gamma f(x_n) - Ap\rangle \\ &\le ||x_n - p||^2 - (1-\beta_n\bar{\gamma})^2||x_n - u_n||^2 + 2(1-\beta_n\bar{\gamma})^2\lambda_n||x_n - u_n||||Dx_n - Dp|| \\ &\quad + \beta_n^2||\gamma f(x_n) - Ap||^2 + 2\beta_n\langle (I-\beta_nA)(z_n - p),\ \gamma f(x_n) - Ap\rangle \end{aligned} \tag{3.37}$$

and hence,

$$\begin{aligned}||x_{n+1} - p||^2 &\leq ||y_n - p||^2 \\ &\leq ||x_n - p||^2 - (1 - \beta_n\bar{\gamma})^2||x_n - u_n||^2 \\ &\quad + 2(1 - \beta_n\bar{\gamma})^2\lambda_n||x_n - u_n||||Dx_n - Dp|| \\ &\quad + \beta_n^2||\gamma f(x_n) - Ap||^2 + 2\beta_n\langle(I - \beta_n A)(z_n - p),\ \gamma f(x_n) - Ap\rangle.\end{aligned} \tag{3.38}$$

It follows that

$$\begin{aligned}(1 - \beta_n\bar{\gamma})^2||x_n - u_n||^2 &\leq ||x_{n+1} - x_n||(||x_{n+1} - p|| + ||x_n - p||) \\ &\quad + 2(1 - \beta_n\bar{\gamma})^2\lambda_n||x_n - u_n||||Dx_n - Dp|| + \beta_n^2||\gamma f(x_n) - Ap||^2 \\ &\quad + 2\beta_n\langle(I - \beta_n A)(z_n - p),\ \gamma f(x_n) - Ap\rangle.\end{aligned} \tag{3.39}$$

This together with $||x_{n+1} - x_n|| \to 0$, $||Dx_n - D_p|| \to 0$, $\beta_n \to 0$ as $n \to \infty$ and the condition on λ_n implies that

$$\lim_{n\to\infty} ||x_n - u_n|| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{||x_n - u_n||}{\lambda_n} = 0. \tag{3.40}$$

Consequently, from (3.17) and (3.40)

$$||x_{n+1} - u_n|| \leq ||x_{n+1} - x_n|| + ||x_n - u_n|| \to 0 \quad \text{as } n \to \infty. \tag{3.41}$$

From (3.1) and condition (C1), we have

$$||y_n - z_n|| = ||\beta_n\gamma f(x_n) + (1 - \beta_n A)z_n - z_n|| \leq \beta_n||\gamma f(x_n) - Az_n|| \to 0 \quad \text{as } n \to \infty \tag{3.42}$$

Since $S - \alpha_n BS$ is nonexpansive mapping(Lemma 2.4), we have

$$\begin{aligned}||w_n - z_n|| &= ||P_C(Sy_n - \alpha_n BSy_n) - P_C(Su_n - \alpha_n BSu_n)|| \\ &\leq ||(S - \alpha_n BS)y_n - (S - \alpha_n BS)u_n|| \\ &\leq ||y_n - u_n||.\end{aligned} \tag{3.43}$$

Next, we will show that $||x_n - y_n|| \to 0$ as $n \to \infty$.
We consider $x_{n+1} - y_n = \delta_n(w_n - y_n) = \delta n(w_n - z_n + z_n - y_n)$.
From (3.43), we have

$$\begin{aligned}||x_{n+1} - y_n|| &\leq \delta_n(||w_n - z_n|| + ||z_n - y_n||) \\ &\leq \delta_n(||y_n - u_n|| + ||z_n - y_n||) \\ &\leq \delta_n(||x_{n+1} - y_n|| + ||x_{n+1} - u_n|| + ||z_n - y_n||).\end{aligned} \tag{3.44}$$

From the condition (C2), (3.41) and (3.42), it follows that

$$||x_{n+1} - y_n|| \leq \frac{\delta_n}{1 - \delta_n}(||x_{n+1} - u_n|| + ||z_n - y_n||) \leq \frac{b}{1 - b}(||x_{n+1} - u_n|| + ||z_n - y_n||) \to 0 \tag{3.45}$$

From (3.17) and (3.45), we obtain

$$||x_n - y_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - y_n|| \to 0 \quad \text{as } n \to \infty. \tag{3.46}$$

We observe that

$$\begin{aligned}||Sw_n - w_n|| &\leq ||Sw_n - Sz_n|| + ||Sz_n - Sy_n|| + ||Sy_n - w_n|| \\ &\leq ||w_n - z_n|| + ||z_n - y_n|| + ||Sy_n - w_n|| \\ &\leq ||y_n - u_n|| + ||z_n - y_n|| + ||Sy_n - w_n|| \\ &\leq ||y_n - x_n|| + ||x_n - u_n|| + ||z_n - y_n|| + ||Sy_n - w_n||.\end{aligned} \tag{3.47}$$

Consequently, we obtain

$$\lim_{n\to\infty} ||Sw_n - w_n|| = 0. \tag{3.48}$$

Step 4. We prove that the mapping $P_\Theta(\gamma f + (I - A))$ has a unique fixed point.

Let f be a contraction of C into itself with coefficient $\eta \in (0, 1)$. Then, we have

$$\begin{aligned}\left\|P_\Theta(\gamma f + (I-A))(x) - P_\Theta(\gamma f + (I-A))(y)\right\| &\le ||(\gamma f + (I - A))(x) - (\gamma f + (I - A))(y)|| \\ &\le \gamma ||f(x) - f(y)|| + ||I - A|| \, ||x - y|| \\ &\le \gamma\eta ||x - y|| + (1 - \bar{\gamma})||x - y|| \\ &= (1 - (\bar{\gamma} - \eta\gamma))||x - y||, \quad \forall x, y \in C.\end{aligned}$$

Since $0 < 1 - (\bar{\gamma} - \eta\gamma) < 1$, it follows that $P_\Theta(\gamma f + (I - A))$ is a contraction of C into itself. Therefore, by the Banach Contraction Mapping Principle, it has a unique fixed point, say $z \in C$, that is,

$$z = P_\Theta(\gamma f + (I - A))(z).$$

Step 5. We claim that $q \in F(S) \cap VI(C, B) \cap GMEP(F, \phi, D)$.

First, we show that $q \in F(S)$.

Assume $q \notin F(S)$. Since $w_{n_i} \rightharpoonup q$ and $q \neq Sq$, based on Opial's condition (Lemma 2.6), it follows that

$$\begin{aligned}\liminf_{i\to\infty} ||w_{n_i} - q|| &< \liminf_{i\to\infty} ||w_{n_i} - Sq|| \\ &\le \liminf_{i\to\infty} \{||w_{n_i} - Sw_{n_i}|| + ||Sw_{n_i} - Sq||\} \\ &= \liminf_{i\to\infty} ||Sw_{n_i} - Sq|| \\ &\le \liminf_{i\to\infty} ||w_{n_i} - q||.\end{aligned}$$

This is a contradiction. Thus, we have $q \in F(S)$.

Next, we prove that $q \in GMEP(F, \phi, D)$.

From Lemma 2.9 that $u_n = T_{\lambda_n}^{(F,\varphi)}(x_n - \lambda_n D x_n)$ for all $n \ge 1$ is equivalent to

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Dx_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

From (H2), we also have

$$\varphi(y) - \varphi(u_n) + \langle Dx_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \ge -F(u_n, y) \ge F(y, u_n).$$

Replacing n by n_i, we obtain

$$\varphi(y) - \varphi(u_{n_i}) + \langle Dx_{n_i}, y - u_{n_i}\rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \ge F(y, u_{n_i}). \tag{3.49}$$

Let $y_t = t_y + (1 - t)q$ for all $t \in (0, 1]$ and $y \in C$. Since $y \in C$ and $q \in C$, we obtain $y_t \in C$. Hence, from (3.49), we have

$$\begin{aligned}\langle y_t - u_{n_i}, Dy_t\rangle &\geq \langle y_t - u_{n_i}, Dy_t\rangle - \varphi(y_t) + \varphi(u_{n_i}) - \langle Dx_{n_i}, y_t - u_{n_i}\rangle \\ &\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle + F(y_t, u_{n_i}) \\ &\geq \langle y_t - u_{n_i}, Dy_t - Du_{n_i}\rangle + \langle y_t - u_{n_i}, Du_{n_i} - Dx_{n_i}\rangle - \varphi(y_t) \\ &\quad + \varphi(u_{n_i}) - \{y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\} + F(y_t, u_{n_i}).\end{aligned} \tag{3.50}$$

Since $||u_{n_i} - x_{n_i}|| \to 0$, $i \to \infty$ we obtain $||Du_{n_i} - Dx_{n_i}|| \to 0$. Furthermore, by the monotonicity of D, we have

$$\langle y_t - u_{n_i}, Dy_t - Du_{n_i}\rangle \geq 0.$$

Hence, from (H4), (H5) and the weak lower semicontinuity of ϕ, $\frac{u_{n_i}-x_{n_i}}{\lambda_{n_i}} \to 0$ and $u_{n_i} \to q$, we have

$$\langle y_t - q, Dy_t\rangle \geq -\varphi(y_t) + \varphi(q) + F(y_t, q) \quad \text{as } i \to \infty. \tag{3.51}$$

From (H1), (H4) and (3.51), we also get

$$\begin{aligned}0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, q) + t\varphi(y) + (1-t)\varphi(q) - \varphi(y_t) \\ &= t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)[F(y_t, q) + \varphi(q) - \varphi(y_t)] \\ &\leq t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y_t - q, Dy_t\rangle \\ &= t[F(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)t\langle y - q, Dy_t\rangle.\end{aligned}$$

Dividing by t, we get

$$F(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - q, Dy_t\rangle \geq 0.$$

Letting $t \to 0$ in the above inequality, we arrive that, for each $y \in C$,

$$F(q, y) + \varphi(y) - \varphi(q) + \langle y - q, Dq\rangle \geq 0.$$

This implies that $q \in \text{GMEP}(F, \phi, D)$.

Finally, we prove that $q \in \text{VI}(C, B)$.

We define the maximal monotone operator:

$$Qq_1 = \begin{cases} Bq_1 + N_C q_1, & q_1 \in C, \\ \emptyset, & q_1 \notin C. \end{cases}$$

Since B is ξ-inverse strongly monotone and by condition (C4), we have

$$\langle Bx - By, x - y\rangle \geq \xi||Bx - -By||^2 \geq 0.$$

Then, Q is maximal monotone. Let $(q_1, q_2) \in G(Q)$. Since $q_2 - Bq_1 \in N_C q_1$ and $w_n \in C$, we have $\langle q_1 - w_n, q_2 - Bq_1\rangle \geq 0$. On the other hand, from $w_n = P_C(Sy_n - \alpha_n BSy_n)$, we have

$$\langle q_1 - w_n, w_n - (Sy_n - \alpha_n BSy_n)\rangle \geq 0,$$

that is,

$$\left\langle q_1 - w_n, \frac{w_n - Sy_n}{\alpha_n} + BSy_n\right\rangle \geq 0.$$

Therefore, we obtain

$$\begin{aligned}\langle q_1 - w_{n_i}, q_2\rangle &\geq \langle q_1 - w_{n_i}, Bq_1\rangle \\ &\geq \langle q_1 - w_{n_i}, Bq_1\rangle - \left\langle q_1 - w_{n_i}, \frac{w_{n_i} - Sy_{n_i}}{\alpha_{n_i}} + BSy_{n_i}\right\rangle \\ &= \left\langle q_1 - w_{n_i}, Bq_1 - BSy_{n_i} - \frac{w_{n_i} - Sy_{n_i}}{\alpha_{n_i}}\right\rangle \\ &= \langle q_1 - w_{n_i}, Bq_1 - Bw_{n_i}\rangle + \langle q_1 - w_{n_i}, Bw_{n_i} - BSy_{n_i}\rangle \\ &\quad - \left\langle q_1 - w_{n_i}, \frac{w_{n_i} - Sy_{n_i}}{\alpha_{n_i}}\right\rangle \\ &\geq \langle q_1 - w_{n_i}, Bw_{n_i} - BSy_{n_i}\rangle - \left\langle q_1 - w_{n_i}, \frac{w_{n_i} - Sy_{n_i}}{\alpha_{n_i}}\right\rangle.\end{aligned} \tag{3.52}$$

Noting that $||w_{n_i} - Sy_{n_i}|| \to 0$ as $i \to \infty$, we obtain

$$\langle q_1 - q, q_2\rangle \geq 0.$$

Since Q is maximal monotone, we obtain that $q \in Q^{-1}0$, and hence $q \in \mathrm{VI}(C, B)$. This implies $q \in \Theta$. Since $z = P_\Theta(\gamma f + (I - A))(z)$, we have

$$\limsup_{n\to\infty} \langle \gamma f(z) - Az, x_n - z\rangle = \lim_{i\to\infty} \langle \gamma f(z) - Az, x_{n_i} - z\rangle = \langle \gamma f(z) - Az, q - z\rangle \leq 0 \tag{3.53}$$

On the other hand, we have

$$\begin{aligned}\langle \gamma f(z) - Az, y_n - z\rangle &= \langle \gamma f(z) - Az, y_n - x_n\rangle + \langle \gamma f(z) - Az, x_n - z\rangle \\ &\leq ||\gamma f(z) - Az||\, ||y_n - x_n|| + \langle \gamma f(z) - Az, x_n - z\rangle.\end{aligned}$$

From (3.46) and (3.53), we obtain that

$$\limsup_{n\to\infty} \langle \gamma f(z) - Az, y_n - z\rangle \leq 0. \tag{3.54}$$

Step 6. Finally, we claim that $x_n \to z$, where $z = P_\Theta(\gamma f + (I - A))(z)$.

We note that

$$\begin{aligned}\|y_n - z\|^2 &= ||(I - \beta_n A)(z_n - z) + \beta_n(\gamma f(x_n) - Az)||^2 \\ &\leq ||(I - \beta_n A)(z_n - z)||^2 + 2\beta_n\langle(\gamma f(x_n) - Az), (I - \beta_n A)(z_n - z) + \beta_n(\gamma f(x_n) - Az)\rangle \\ &= ||(I - \beta_n A)(z_n - z)||^2 + 2\beta_n\langle(\gamma f(x_n) - Az), y_n - z\rangle \\ &\leq ||I - \beta_n A||^2||z_n - z||^2 + 2\beta_n\gamma\langle f(x_n) - f(z), y_n - z\rangle + 2\beta_n\langle\gamma f(z) - Az, y_n - z\rangle \\ &\leq (1 - \beta_n\bar{\gamma})^2||z_n - z||^2 + 2\beta_n\gamma\eta||x_n - z||\, ||y_n - z|| + 2\beta_n\langle\gamma f(z) - Az, y_n - z\rangle \\ &\leq (1 - \beta_n\bar{\gamma})^2||x_n - z||^2 + \beta_n\gamma\eta(||x_n - z||^2 + ||y_n - z||^2) + 2\beta_n\langle\gamma f(z) - Az, y_n - z\rangle \\ &= (1 - 2\beta_n\bar{\gamma} + \beta_n^2\bar{\gamma}^2 + \beta_n\gamma\eta)||x_n - z||^2 + \beta_n\gamma\eta||y_n - z||^2 + 2\beta_n\langle\gamma f(z) - Az, y_n - z\rangle\end{aligned} \tag{3.55}$$

which implies that

$$\begin{aligned}||y_n - z||^2 &\leq \left(1 - \frac{(2\bar{\gamma} - \gamma\eta)\beta_n}{1 - \gamma\eta\beta_n}\right)||x_n - z||^2 \\ &\quad + \frac{\beta_n}{1 - \gamma\eta\beta_n}\left[\beta_n\bar{\gamma}^2||x_n - z||^2 + 2\langle\gamma f(z) - Az, y_n - z\rangle\right].\end{aligned} \tag{3.56}$$

On the other hand, we have

$$\begin{aligned} \|x_{n+1}-z\|^2 &\le \|y_n-z\|^2 \\ &\le \left(1-\frac{(2\bar{\gamma}-\gamma\eta)\beta_n}{1-\gamma\eta\beta_n}\right)\|x_n-z\|^2 \\ &\quad + \frac{\beta_n}{1-\gamma\eta\beta_n}\left[\beta_n\bar{\gamma}^2\|x_n-z\|^2 + 2\langle\gamma f(z)-Az,\ y_n-z\rangle\right] \\ &\le \left(1-\frac{(2\bar{\gamma}-\gamma\eta)\beta_n}{1-\gamma\eta\beta_n}\right)\|x_n-z\|^2 \\ &\quad + \frac{\beta_n}{1-\gamma\eta\beta_n}\left[2\langle\gamma f(z)-Az, y_n-z\rangle + \beta_n\bar{\gamma}^2K\right], \end{aligned} \tag{3.57}$$

where K is an appropriate constant such that $K \ge \sup_{n\ge 1}\{\|x_n - z\|^2\}$.
Set $l_n = \frac{(2\bar{\gamma}-\gamma\eta)\beta_n}{1-\gamma\eta\beta_n}$ and $e_n = \frac{\beta_n}{1-\gamma\eta\beta_n}\left[2\langle\gamma f(z)-Az, y_n-z\rangle + \beta_n\bar{\gamma}^2K\right]$. Then, we have

$$\|x_{n+1}-z\|^2 \le (1-b_n)\|x_n-z\|^2 + c_n, \quad \forall n \ge 0. \tag{3.58}$$

From the condition (C1) and (3.54), we see that

$$\lim_{n\to\infty} l_n = 0, \quad \sum_{n=0}^{\infty} l_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} e_n \le 0.$$

Therefore, applying Lemma 2.10 to (3.58), we get that $\{x_n\}$ converges strongly to $z \in \Theta$.

This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, let B be ξ-inverse-strongly monotone mapping of C into H, and let $S : C \to C$ be a nonexpansive mapping. Let $f : C \to C$ be a contraction mapping with $\eta \in (0, 1)$, and let A be a strongly positive linear-bounded operator with $\bar{\gamma} > 0$and $0 < \gamma < \frac{\bar{\gamma}}{\eta}$. Assume that $\Theta := F(S) \cap VI(C, B) \ne \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequence generated by the following iterative algorithm:*

$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrary}, \\ y_n = \beta_n\gamma f(x_n) + (I-\beta_n A)P_C(Sx_n - \alpha_n BSx_n), \\ x_{n+1} = (1-\delta_n)y_n + \delta_n P_C(Sy_n - \alpha_n BSy_n), \quad \forall n \ge 1, \end{cases}$$

where $\{\delta_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$,
(C2) $\{\delta_n\} \subset [0, b]$, *for some* $b \in (0, 1)$ *and* $\lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$,
(C3) $\{\alpha_n\} \subset [e, g] \subset (0, 2\xi)$ *and* $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$.

Then, $\{x_n\}$ converges strongly to $z \in \Theta$, which is the unique solution of the variational inequality

$$\langle\gamma f(z)-Az,\ x-z\rangle \le 0, \quad \forall x \in \Theta. \tag{3.59}$$

Proof. Put $F(x, y) = \phi = D = 0$ for all $x, y \in C$ and $\lambda_n = 1$ for all $n \ge 1$ in Theorem 3.1, we get $u_n = x_n$. Hence, $\{x_n\}$ converges strongly to $z \in \Theta$, which is the unique solution of the variational inequality (3.59). □

Corollary 3.3. [12]*Let C be a nonempty closed convex subset of a real Hilbert space H and let F be bifunction from $C \times C$ to $\mathbb{R}$ satisfying (H1)-(H5). Let $S : C \to C$ be a nonexpansive mapping and let $f : C \to C$ be a contraction mapping with $\eta \in (0, 1)$. Assume that $\Theta := F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequence generated by the following iterative algorithm:*

$$\begin{cases} x_1 = x \in C \quad \text{chosen arbitrary}, \\ y_n = \beta_n f(x_n) + (1 - \beta_n) S T^F_{\lambda_n} x_n, \\ x_{n+1} = (1 - \delta_n) y_n + \delta_n S y_n, \quad \forall n \geq 1, \end{cases} \tag{3.60}$$

where $\{\delta_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ satisfying the following conditions:

(C1) $\lim_{n \to \infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$,
(C2) $\{\delta_n\} \subset [0, b]$, *for some* $b \in (0, 1)$ *and* $\lim_{n \to \infty} |\delta_{n+1} - \delta_n| = 0$,
(C3) $\lim_{n \to \infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then, $\{x_n\}$ converges strongly to $z \in \Theta$.

Proof. Put $\phi = D = 0$, $\gamma = 1$, $A = I$ and $\alpha_n = 0$ in Theorem 3.1. Then, we have $P_C(Su_n) = Su_n$ and $P_C(Sy_n) = Sy_n$. Hence, $\{x_n\}$ generated by (3.60) converges strongly to $z \in \Theta$. □

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All authors contribute equally and significantly in this research work. All authors read and approved the final manuscript.

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