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Rodé's theorem on common fixed points of semigroup of nonexpansive mappings in CAT(0) spaces

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Abstract

We extend Rodé's theorem on common fixed points of semigroups of nonexpansive mappings in Hilbert spaces to the CAT(0) space setting.

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1 Introduction

In 1976, Lim [1] introduced a concept of convergence in a general metric space, called strong Δ-convergence. In [2], Kirk and Panyanak introduced a concept of convergence in a CAT(0) space, called Δ-convergence (see Section 2 for the definition). Moreover, they showed that many Banach space concepts and results which involve weak convergence can be extended to the CAT(0) space setting by using the Δ-convergence.

For each semigroup S, let $B(S)$ be the Banach space of all bounded real-valued mappings on S with supremum norm. A continuous linear functional $\mu \in B(S)^*$ (the dual space of $B(S)$) is called a *mean* on $B(S)$ if $\|\mu\| = \mu(1)$. For any $f \in B(S)$, we use the following notation:

$$\mu(f) = \mu_s(f(s)).$$

A mean μ on $B(S)$ is said to be *left invariant* [respectively, *right invariant*] if $\mu_s(f(ts)) = \mu_s(f(s))$ [respectively, $\mu_s(f(st)) = \mu_s(f(s))$] for all $f \in B(S)$ and for all $t \in S$. We will say that μ is an *invariant mean* if it is both left and right invariants. If $B(S)$ has an invariant mean, we call S an *amenable semigroup*. It is well known that every commutative semigroup is amenable [3]. For each $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for any $t \in S$, respectively. A net $\{\mu_\alpha\}$ of means on $B(S)$ is said to be *asymptotically invariant* if

$$\lim_\alpha(\mu_\alpha(l_s f) - \mu_\alpha(f)) = 0 = \lim_\alpha(\mu_\alpha(r_s f) - \mu_\alpha(f)).$$

In [4], Rodé proved the following:

Theorem 1.1. [4]*If S is an amenable semigroup, C is a closed convex subset of a Hilbert space H, $\mathcal{S} = \{T_s : s \in S\}$ is a nonexpansive semigroup on C such that the set $F(\mathcal{S})$of common fixed points of $\mathcal{S}$is nonempty and $\{\mu_\alpha\}$ is an asymptotically invariant net of means, then for each $x \in C$,$\{T_{\mu_\alpha}x\}$converges weakly to an element of $F(\mathcal{S})$.*

Further, for each $x \in C$, the limit point of $\{T_{\mu_\alpha} x\}$ is the same for all asymptotically invariant nets of means $\{\mu_\alpha\}$.

It is remarked that if S is amenable, then there is always an asymptotically strong invariant net of finite means, i.e., means that are convex combination of point evaluations. This follows from Proposition 3.3 in [5].

Development of this subject had been made by several authors [1,6-8]. The main purpose of this article is to extend this result of Rodé for a nonexpansive semigroup on a CAT(0) space in which the Δ-convergence plays the role of weak convergence.

2 Preliminaries

Let (X, d) be a metric space. A *geodesic* joining $x \in X$ to $y \in X$ is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t-t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image γ of c is called a *geodesic (or metric) segment* joining x and y. When it is unique, this geodesic is denoted $[x, y]$. Write $c(\alpha\, 0 + (1 - \alpha)l) = \alpha x \oplus (1 - \alpha)y$, and for $\alpha = \frac{1}{2}$, we write $\frac{1}{2}x \oplus \frac{1}{2}y$ as $\frac{x \oplus y}{2}$, the midpoint of x and y. The space X is said to be a *geodesic space* if every two points of X are joined by a geodesic.

Following [2], a metric space X is said to be a *CAT(0) space* if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. This latter property, which is what we referred to as the (CN) inequality, enables one to define the concept of nonpositive curvature in this situation, generalizing the same concept in Riemannian geometry. In fact (cf. [[9], p. 163]), the following are equivalent for a geodesic space X:

(i) X is a CAT(0) space.

(ii) X satisfies the **(CN) inequality**: If $x_0, x_1 \in X$ and $\frac{x_0 \oplus x_1}{2}$ is the midpoint of x_0 and x_1, then

$$d^2\left(y, \frac{x_0 \oplus x_1}{2}\right) \le \frac{1}{2} d^2(y, x_0) + \frac{1}{2} d^2(y, x_1) - \frac{1}{4} d^2(x_0, x_1), \quad \text{for all } y \in X.$$

(iii) X satisfies the **Law of cosine**: If $a = d(p, q)$, $b = d(p, r)$, $c = d(q, r)$ and ξ is the Alexandrov angle at p between $[p, q]$ and $[p, r]$, then $c^2 \ge a^2 + b^2 - 2ab \cos \xi$.

For any subset C of X, let $\pi = \pi_D$ be a nearest point projection mapping from C to a subset D. It is known by [[9], pp. 176-177] (see also [[10], Proposition 2.6]) that if D is closed and convex, the mapping π is well-defined, nonexpansive, and satisfies

$$d^2(x, y) \ge d^2(x, \pi x) + d^2(\pi x, y) \text{ for all } x \in C \text{ and } y \in D. \tag{1}$$

Definition 2.1. [[11], Definition 5.13] *A complete CAT(0) space X has the property of the nice projection onto geodesics (property (N) for short) if, given any geodesic segment $I \subset X$, it is the case that $\pi_I(m) \in [\pi_I(x), \pi_I(y)]$ for any x, y in X and $m \in [x, y]$.*

As noted in [11], we do not know of any example of a CAT(κ) space which does not enjoy the property (N).

Let S be a semigroup, C be a closed convex subset of a Hilbert space H, and for each s in S, T_s is a mapping from C into itself. Suppose $\{T_s x : s \in S\}$ is bounded for all $x \in C$. Let $x \in C$ and μ be a mean on $B(S)$. By Riesz's representation theorem, there exists a unique $x_0 \in C$ such that

$$\mu_s\langle T_s x, y\rangle = \langle x_0, y\rangle \qquad (2)$$

for all $y \in H$. Here $\langle\ ,\ \rangle$ denotes the inner product on H.

The following result is a mild generalization of a result of Kakavandi and Amini [[12], Lemma 2.1].

Lemma 2.2. *Let C be a closed convex subset of a CAT(0) space X and μ be a mean on $B(S)$. For a bounded function $h : S \to C$, define*

$$\varphi_\mu(y) := \mu_s(d^2(h(s), y))$$

for all $y \in X$. Then, ϕ_μ attains its unique minimum at a point of $\overline{co}\{h(s) : s \in S\}$.

For each $x \in C$, denote $\mathcal{S}(x) := \{T_s x : s \in S\}$. If $\mathcal{S}(x)$ is bounded, then by Lemma 2.2 we put

$$T_\mu(h) := argmin\{y \mapsto \mu_s(d^2(h(s), y))\},$$

and for $h(s)$ of the form $T_s x$, we write $T_\mu(h)$ shortly as $T_\mu x$.

Remark 2.3. *If X is a Hilbert space, then*

(i) $T_\mu x = x_0$ *where* x_0 *verifies (2), and*
(ii) $||x_0||^2 = \sup_{y\in X} (2\langle x_0, y\rangle - ||y||^2)$.

Proof. (i): Let x_0 be such that $\mu_s\langle T_s x, y\rangle = \langle x_0, y\rangle$ for all $y \in X$. We have $\phi_\mu(x_0) = \phi_\mu(0) + ||x_0||^2 - 2\langle x_0, x_0\rangle = \phi_\mu(0) - ||x_0||^2 \le \phi_\mu(0) + ||T_\mu x||^2 - 2\langle x_0, T_\mu x\rangle = \phi_\mu(T_\mu x)$. Therefore, $x_0 = T_\mu x$.

(ii): For any $x, y \in X$, we know that $||T_s x - y||^2 = ||T_s x||^2 - 2\langle T_s x, y\rangle + ||y||^2$. By the linearity of μ and (2), we have $\mu_s(||T_s x - y||^2) = \mu_s(||T_s x||^2) - 2\langle x_0, y\rangle + ||y||^2$. Thus, $\inf_{y\in X} \mu_s(||T_s x - y||^2) = \mu_s(||T_s x||^2) - \sup_{y\in X} (2\langle x_0, y\rangle - ||y||^2)$. On the other hand, by (i), $\inf_{y\in X} \mu_s(||T_s x - y||^2) = \mu_s(||T_s x - x_0||^2) = \mu_s(||T_s x||^2) - 2\mu_s\langle T_s x, x_0\rangle + ||x_0||^2 = \mu_s(||T_s x||^2) - ||x_0||^2$. Hence, $||x_0||^2 = \sup_{y\in X} (2\langle x_0, y\rangle - ||y||^2)$. ■

Let C be a closed convex subset of a CAT(0) space X and S a semigroup. We say that the set $\mathcal{S}(S) := \{T_s : s \in S\}$ is a *nonexpansive semigroup* on C if

(i) $T_s : C \to C$ is a nonexpansive mapping, i.e., $d(T_s x, T_s y) \le d(x, y)$ for all $x, y \in X$, for all $s \in S$,
(ii) the mapping $s \to T_s x$ is bounded for all $x \in C$, and
(iii) $T_{ts} = T_t T_s$, for all $s, t \in S$.

We denote by $F(\mathcal{S})$ the set of all common fixed points of mappings in $\mathcal{S}(S)$, i.e., $F(\mathcal{S}) := \bigcup_{s\in S} F(T_s)$, where $F(T_s) := \{x \in C : T_s x = x\}$ is the set of fixed points of T_s.

For any bounded net $\{x_\alpha\}$ in a closed convex subset C of a CAT(0) space X, put

$$r(x, \{x_\alpha\}) = \limsup_\alpha d(x, x_\alpha)$$

for each $x \in C$. The *asymptotic radius* of $\{x_\alpha\}$ on C is given by

$$r(C, \{x_\alpha\}) = \inf_{x \in C} r(x, \{x_\alpha\}),$$

and the *asymptotic center* of $\{x_\alpha\}$ in C is the set

$$A(C, \{x_\alpha\}) = \{x \in C : r(x, \{x_\alpha\}) = r(C, \{x_\alpha\})\}.$$

It is known that in a complete CAT(0) space, $A(C, \{x_\alpha\})$ consists of exactly one point and $A(X, \{x_\alpha\}) = A(C, \{x_\alpha\})$ (cf. [2]).

Remark 2.4. *(i) Let D, E be directions and* $\nu : E \to D$. *If* $\{x_{\nu(\beta)} : \beta \in E\}$ *is a subnet of a bounded net* $\{x_\alpha : \alpha \in D\}$, *then* $r(C, \{x_{\nu(\beta)}\}) \le r(C, \{x_\alpha\})$.

(ii) Let C be a closed convex subset of a CAT(0) space X, $T : C \to C$ *a nonexpansive mapping and* $x \in C$. *If* $\{T^n x\}$ *is bounded and if* $z \in A(C, \{T^n x\})$, *then* $z \in F(T)$.

Proof. (i) Let $\alpha_0 \in D$. By the definition of subnets, there exists $\beta_0 \in E$ such that $\nu(\beta) \succcurlyeq \alpha_0$ for all $\beta \succcurlyeq \beta_0$. For each $x \in C$, we have $\sup_{\alpha \succcurlyeq \alpha_0} d(x, x_\alpha) \ge \sup_{\beta \succcurlyeq \beta_0} d(x, x_{\nu(\beta)})$. Thus, $\sup_{\alpha \succcurlyeq \alpha_0} d(x, x_\alpha) \ge \inf_{\beta_1} \sup_{\beta \succcurlyeq \beta_1} d(x, x_{\nu(\beta)})$, and this holds for all α_0. Hence, $r(x, \{x_\alpha\}) = \inf_{\alpha_0} \sup_{\alpha \succcurlyeq \alpha_0} d(x, x_\alpha) \ge r(x, x_{\nu(\beta)})$, and this holds for all $x \in C$. Consequently, $r(C, \{x_\alpha\}) = \inf_{x \in C} r(x, \{x_\alpha\}) \ge \inf_{x \in C} r(x, x_{\nu(\beta)}) = r(C, \{x_{\nu(\beta)}\})$.

(ii) Since T is nonexpansive, $\limsup_n d^2(T^n x, Tz) = \limsup_n d^2(TT^n x, Tz) \le \limsup_n d^2(T^n x, z)$.

As every asymptotic center is unique, we have $z = Tz$. □

Definition 2.5. [[2], Definition 3.3] *A net* $\{x_\alpha\}$ *in X is said to* Δ*-converge to* $x \in X$ *if x is the unique asymptotic center of* $\{u_\beta\}$ *for every subnet* $\{u_\beta\}$ *of* $\{x_\alpha\}$. *In this case, we write* $\Delta\text{-}\lim_\alpha x_\alpha = x$ *and call x the* Δ*-limit of* $\{x_\alpha\}$.

Proposition 2.6. [[2], Proposition 3.4] *Every bounded net in X has a* Δ*-convergent subnet.*

Remark 2.7. *(i) Let D be a direction,* $\{x_\alpha : \alpha \in D\}$ *a net in X and* $x \in X$. *If* $\limsup_\alpha d(x, x_\alpha) > \rho$ *for some* $\rho > 0$, *then there exists a subnet* $\{x_{\beta_\alpha}\}$ *of* $\{x_\alpha\}$ *such that* $d(x, x_{\beta_\alpha}) \ge \rho$ *for all* α.

(ii) Let $\{x_\alpha\}$ *be a net in X. Then,* $\{x_\alpha\}$ Δ*-converges to* $x \in X$ *if and only if every subnet* $\{x_{\alpha'}\}$ *of* $\{x_\alpha\}$ *has a subnet* $\{x_{\alpha''}\}$ *which* Δ*-converges to x.*

Proof. (i): For each $\alpha \in D$, we have $\sup_{\alpha' \succcurlyeq \alpha} d(x, x_{\alpha'}) > \rho$. Thus there exists $\beta_\alpha \succ \alpha$ such that $d(x, x_{\beta_\alpha}) \ge \rho$, and this holds for all α. Set a set $E = \{\beta_\alpha : \alpha \in D\}$. Clearly, E is a direction, and define $\nu : E \to D$ by $\nu(\beta_\alpha) = \beta_\alpha$. Let $\alpha_0 \in D$, thus $\nu(\beta_\alpha) \succcurlyeq \alpha_0$ for all $\beta_\alpha \succcurlyeq \beta_{\alpha_0}$ and this shows that $\{x_{\beta_\alpha}\}$ is a subnet of $\{x_\alpha\}$ satisfying $d(x, x_{\beta_\alpha}) \ge \rho$ for all α.

ii): It is easy to see that if $\{x_\alpha\}$ Δ-converges to x, then every subnet of $\{x_\alpha\}$ also Δ-converges to x. On the other hand, suppose $\{x_\alpha\}$ does not Δ-converge to x. Thus, there exists a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that $x \notin A(C, \{x_\beta\})$, and so $\limsup_\beta d(x, x_\beta) > \rho > r(C, \{x_\beta\})$ for some $\rho > 0$. By (i), there exists a subnet $\{x_{\gamma_\beta}\}$ of $\{x_\beta\}$ satisfying $d(x, x_{\gamma_\beta}) \ge \rho$ for all β. By assumption, there exists a subnet $\{x_{(\gamma_\beta)_\eta}\}$ of $\{x_{\gamma_\beta}\}$ Δ-converging to x. Using Remark 2.4, $\rho \le \limsup_\gamma d(x, x_{(\gamma_\beta)_\eta}) = r(C, \{x_{(\gamma_\beta)_\eta}\}) \le r(C, \{x_{\gamma_\beta}\}) \le r(C, \{x_\beta\})$, a contradiction. □

In [13], Berg and Nikolaev introduced a concept of quasilinearization. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then, quasilinearization is defined as a map $\langle , \rangle : (X \times X) \times (X \times X)$ by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}d^2(a,d) + \frac{1}{2}d^2(b,c) - \frac{1}{2}d^2(a,c) - \frac{1}{2}d^2(b,d)$$

for all $a, b, c, d \in X$. Recently, Kakavandi and Amini [14] introduced a concept of w-convergence: a sequence $\{x_n\}$ is said to *w-converge* to $x \in X$ if $\lim_{n\to\infty}\langle \overrightarrow{x_n x}, \overrightarrow{ab} \rangle = 0$ for all $a, b \in X$.

Proposition 2.8. [[14], Proposition 2.5] *For sequences in a complete CAT(0) space X, w-convergence implies Δ-convergence (to the same limit).*

A simple example shows that the converse of this proposition does not hold:

Example 2.9. *Consider an $\mathbb{R}$-tree in $\mathbb{R}^\infty$ defined as follow: Let $\{e_n\}$ be the standard basis, $x_0 = e_1 = (1, 0, 0, 0,...)$, and for each n, let $x_n = x_0 + e_{n+1}$. An $\mathbb{R}$-tree is formed by the segments $[x_1, x_n]$ for $n \geq 0$. It is easy to see that $\{x_n\}$ Δ-converges to x_1. But $\{x_n\}$ does not w-converge to x_1 since $\langle \overrightarrow{x_n x_1}, \overrightarrow{x_0 x_1} \rangle = -1$for all $n \geq 2$.*

Thus, a bounded sequence does not necessary contain an w-convergent subsequence.

3 Main results

3.1 Δ-convergence

Lemma 3.1. [[12], Lemma 3.1] *If C is a closed convex subset of a CAT(0) space X and $T : C \to C$ is a nonexpansive mapping, then $F(T)$ is closed and convex.*

Lemma 3.2. [[12], Proposition 3.2] *Let C be a closed convex subset of a CAT(0) space X and S an amenable semigroup. If $\mathcal{S}(S)$is a nonexpansive semigroup on C, then the following conditions are equivalent.*

(i) $\mathcal{S}(x)$is bounded for some $x \in C$;
(ii) $\mathcal{S}(x)$is bounded for all $x \in C$;
(iii) $F(\mathcal{S}) \neq \emptyset$.

Proposition 3.3. [[12], Theorem 3.3] *Let C be a closed convex subset of a complete CAT(0) space X, S an amenable semigroup, and $\mathcal{S}(S)$a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \emptyset$. Then, $T_\mu x \in F(\mathcal{S})$for any invariant mean μ on $B(S)$.*

We now let S be a commutative semigroup and define a partial order $\succcurlyeq$ on S by $s \succcurlyeq t$ if $s = t$ or there exists $u \in S$ such that $s = ut$. When $s \succcurlyeq t$ but $s \neq t$, we simply write $s \succ t$. We can see that $(S, \succcurlyeq)$ is a directed set. Examples of such S are the usual ordered sets $(\mathbb{N} \cup \{0\}, +, \geq)$ and $(\mathbb{R}^+ \cup \{0\}, +, \geq)$. The following fact is well known:

Proposition 3.4. *Let μ be a right invariant mean on $B(S)$. Then,*

$$\sup_s \inf_t f(ts) \leq \mu(f(s)) \leq \inf_s \sup_t f(ts)$$

for each $f \in B(S)$. Similarly, let μ be a left invariant mean on $B(S)$. Then,

$$\sup_s \inf_t f(st) \leq \mu(f(s)) \leq \inf_s \sup_t f(st)$$

for each $f \in B(S)$.

Remark 3.5. *If $\lim_s f(s) = a$ for some $a \in \mathbb{R}$ and $\{s'\}$ is a subnet of $\{s\}$ satisfying $s' \succ s$ for each s, then*

$$\mu_{s'}(f(s')) = a.$$

Proof. This is an easy consequence of Proposition 3.4 since $\mu_{s'}(f(s')) = \mu_s(f(s')) = \lim_s f(s') = a$. ■

Proposition 3.6. [[12], Proposition 4.1] *Let C be a closed convex subset of a complete CAT(0) space X, S a commutative semigroup, and* $\mathcal{S}(S)$ *a nonexpansive semigroup on C with* $F(\mathcal{S}) \neq \emptyset$*. Then, for each* $x \in C$*, the net* $\{\pi T_s x\}_{s\in S}$ *converges to a point Px in* $F(\mathcal{S})$*, where* $\pi = \pi_{F(\mathcal{S})} : C \to F(\mathcal{S})$ *is the nearest point projection.*

Proposition 3.7. *Let C be a closed convex subset of a complete CAT(0) space X, S a commutative semigroup, and* $\mathcal{S}(S)$ *a nonexpansive semigroup on C with* $F(\mathcal{S}) \neq \emptyset$*. Then, for any invariant mean* μ *on B(S),* $T_\mu x = \lim_s \pi T_s x = Px$ *for all* $x \in C$*.*

Proof. Fix $x \in C$ and let $\varepsilon > 0$. From Proposition 3.6, we see that there exists $s_0 \in S$ such that $d(\pi T_s x, Px) < \varepsilon$ for all $s \succcurlyeq s_0$. We know by Proposition 3.3 that $T_\mu x \in F(\mathcal{S})$. So, $d(Px, T_s x) \le d(Px, \pi T_s x) + d(\pi T_s x, T_s x) < d(\pi T_s x, T_s x) + \varepsilon \le d(T_\mu x, T_s x) + \varepsilon$ for all $s \succcurlyeq s_0$. Since $\{T_s x : s \in S\}$ is bounded by Lemma 3.2, there exists $M > 0$ such that $d(T_\mu x, T_s x) < M$ for all $s \in S$. Therefore, $d^2(Px, T_s x) \le d^2(T_\mu x, T_s x) + 2M\varepsilon + \varepsilon^2$ for each $s \succcurlyeq s_0$. Since μ is an invariant mean, we have $\mu_s(d^2(Px, T_s x)) = \mu_s(d^2(Px, T_{ss_0}x)) \le \mu_s(d^2(T_\mu x, T_{ss_0}x)) + 2M\varepsilon + \varepsilon^2 = \mu_s(d^2(T_\mu x, T_s x)) + 2M\varepsilon + \varepsilon^2$ for any $\varepsilon > 0$. By the argminimality of $T_\mu x$ (see Lemma 2.2), $T_\mu x = Px$. □

In order to obtain the Rodé's theorem (Theorem 1.1) in the framework of CAT(0) spaces, we need to restrict the asymptotically invariant nets of means $\{\mu_\alpha\}$ to those that satisfy an additional condition: for each $t \in S$,

$$\mu_{\alpha_s}(d^2(T_s x, y)) - \mu_{\alpha_s}(d^2(T_{st} x, y)) \to 0 \text{ uniformly for } y \in C. \tag{3}$$

In the Hilbert space setting, condition (3) is not required because the weak convergence can obtain from (2) directly.

Lemma 3.8. *Let X be a complete CAT(0) space that has property (N), C be a closed convex subset of X, S a commutative semigroup, and* $\mathcal{S}(S)$ *a nonexpansive semigroup on C with* $F(\mathcal{S}) \neq \emptyset$*. Suppose* $\{\mu_\alpha\}$ *is an asymptotically invariant nets of means on B(S) satisfying condition (3). If* $\{T_{\mu_\alpha} x\}$ Δ*-converges to* x_0*, then* $x_0 \in F(\mathcal{S})$*.*

Proof. First, we show that, for each $r \in S$,

$$\lim_\alpha d(T_{\mu_\alpha} x, T_r T_{\mu_\alpha} x) = 0. \tag{4}$$

If this is not the case, there must be some $\delta > 0$ so that for each α, there exists $\alpha' \succ \alpha$ satisfying $d(T_{\mu_{\alpha'}} x, T_r T_{\mu_{\alpha'}} x) \ge \delta$. Put $\varepsilon = \frac{\delta^2}{2}$. Since the asymptotically invariant net $\{\mu_\alpha\}$ satisfies (3), there exists α_0 for which for each $\alpha \succcurlyeq \alpha_0$, $\varphi_{\mu_\alpha}(T_r T_{\mu_\alpha} x) = \mu_{\alpha_s}(d^2(T_s x, T_r T_{\mu_\alpha} x)) < \mu_{\alpha_s}(d^2(T_r T_s x, T_r T_{\mu_\alpha} x)) + \varepsilon \le \mu_{\alpha_s}(d^2(T_s x, T_{\mu_\alpha} x)) + \varepsilon = \varphi_{\mu_\alpha}(T_{\mu_\alpha} x) + \varepsilon$. Set $w = \frac{T_{\mu_{\alpha'_0}} x \oplus T_r T_{\mu_{\alpha'_0}} x}{2}$. By the (CN) inequality, the following in equalities hold for each $s \in S$:

$$\begin{aligned} d^2(T_s x, w) &\le \frac{1}{2} d^2(T_s x, T_{\mu_{\alpha'_0}} x) + \frac{1}{2} d^2(T_s x, T_r T_{\mu_{\alpha'_0}} x) - \frac{1}{4} d^2(T_{\mu_{\alpha'_0}} x, T_r T_{\mu_{\alpha'_0}} x) \\ &\le \frac{1}{2} d^2(T_s x, T_{\mu_{\alpha'_0}} x) + \frac{1}{2} d^2(T_s x, T_r T_{\mu_{\alpha'_0}} x) - \frac{\delta^2}{4}. \end{aligned}$$

Therefore,

$$\begin{aligned}\varphi_{\mu_{\alpha'_0}}(w) &\leq \frac{1}{2}\varphi_{\mu_{\alpha'_0}}(T_{\mu_{\alpha'_0}}x) + \frac{1}{2}\varphi_{\mu_{\alpha'_0}}(T_rT_{\mu_{\alpha'_0}}x) - \frac{\delta^2}{4} \\ &< \varphi_{\mu_{\alpha'_0}}(T_{\mu_{\alpha'_0}}x) + \frac{\varepsilon}{2} - \frac{\delta^2}{4} \\ &= \varphi_{\mu_{\alpha'_0}}(T_{\mu_{\alpha'_0}}x),\end{aligned}$$

which is a contradiction and thus (4) holds.

Next, we show that $x_0 \in F(\mathcal{S})$. We suppose on the contrary that $x_0 \notin F(\mathcal{S})$. Thus, for some $r \in S$, $T_rx_0 \neq x_0$, i.e., $d(x_0, T_rx_0) := \gamma > 0$. Since $\{T_{\mu_\alpha}x\} \subset \overline{co}\{T_sx\}$, it is bounded. We can get an $M > 0$ so that $d(T_{\mu_\alpha}x, x_0) \leq M$ for all α. We let $0 < \varepsilon < \min\{\frac{\gamma^2}{16M}, 2M\}$. From (4), there exists α_0 with the property that $d(T_rT_{\mu_\alpha}x, T_{\mu_\alpha}x) < \varepsilon$ for all $\alpha \succcurlyeq \alpha_0$. Now, for each $\alpha \succcurlyeq \alpha_0$, $d(T_{\mu_\alpha}x, T_rx_0) \leq d(T_{\mu_\alpha}x, T_rT_{\mu_\alpha}x) + d(T_rT_{\mu_\alpha}x, T_rx_0) < d(T_{\mu_\alpha}x, x_0) + \varepsilon$. Thus, $d^2(T_{\mu_\alpha}x, T_rx_0) < d^2(T_{\mu_\alpha}x, x_0) + 2\varepsilon d(T_{\mu_\alpha}x, x_0) + \varepsilon^2$. Let $w = \frac{x_0 \oplus T_rx_0}{2}$. Using the (CN) inequality, we see that

$$\begin{aligned}d^2(T_{\mu_\alpha}x, w) &\leq \frac{1}{2}d^2(T_{\mu_\alpha}x, x_0) + \frac{1}{2}d^2(T_{\mu_\alpha}x, T_rx_0) - \frac{1}{4}d^2(x_0, T_rx_0) \\ &\leq \frac{1}{2}d^2(T_{\mu_\alpha}x, x_0) + \frac{1}{2}(d^2(T_{\mu_\alpha}x, x_0) + 2\varepsilon M + \varepsilon^2) - \frac{\gamma^2}{4} \\ &= d^2(T_{\mu_\alpha}x, x_0) + \varepsilon M + \frac{\varepsilon^2}{2} - \frac{\gamma^2}{4}\end{aligned}$$

for all $\alpha \succcurlyeq \alpha_0$. Consequently,

$$\begin{aligned}\limsup_\alpha d^2(T_{\mu_\alpha}x, w) &\leq \limsup_\alpha d^2(T_{\mu_\alpha}x, x_0) + \varepsilon M + \frac{\varepsilon^2}{2} - \frac{\gamma^2}{4} \\ &< \limsup_\alpha d^2(T_{\mu_\alpha}x, x_0),\end{aligned}$$

contradicting to the fact that $\{x_0\}$ is the center of $\{T_{\mu_\alpha}x\}$. Therefore, $T_rx_0 = x_0$ for all $r \in S$, and this shows that $x_0 \in F(\mathcal{S})$ as desired. □

Theorem 3.9. *Let X be a complete CAT(0) space that has Property (N), C be a closed convex subset of X, S a commutative semigroup, and $\mathcal{S}(S)$ a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \emptyset$. Suppose $\{\mu_\alpha\}$ is an asymptotically invariant net of means on B(S) satisfying condition (3). Then, $\{T_{\mu_\alpha}x\}$ Δ-converges to $Px \in F(\mathcal{S})$ for all $x \in C$. Here, Px is defined in Proposition 3.6.*

Proof. Let $x \in C$ and $\{\mu_{\alpha'}\}$ be any subnet of $\{\mu_\alpha\}$. There exists a subnet $\{\mu_{\alpha''}\}$ of $\{\mu_{\alpha'}\}$ such that $\{\mu_{\alpha''}\}$ w*-converges to μ for some invariant mean μ on $B(S)$. By Proposition 3.7, $T_\mu x = Px$. Since the net $\{T_{\mu_{\alpha''}}x\} \subset \overline{co}\{T_sx : s \in S\}$, it is bounded. Then by Proposition 2.6, there exists a subnet $\{\mu_{\alpha_\beta}\}$ of $\{\mu_{\alpha''}\}$ such that $\{T_{\mu_{\alpha_\beta}}x\}$ Δ-converges to some $x_0 \in C$. By Lemma 3.8, $x_0 \in F(\mathcal{S})$.

We show $x_0 = T_\mu x$ by splitting the proof into three steps.

Step 1. If $\overline{T_{\mu_{\alpha_\beta}}x} := \mathit{argmin}\{y \mapsto \mu_{\beta_s}(d^2(T_{ss_0}x, y))\}$, then $\overline{T_{\mu_{\alpha\beta}}x} \in \overline{co}\{T_sx\}_{s \succcurlyeq s_0}$.

Suppose $\overline{T_{\mu_{\alpha_\beta}}x} \notin \overline{co}\{T_sx\}_{s \succcurlyeq s_0}$, by (1), $d^2(T_{ss_0}x, \overline{T_{\mu_{\alpha_\beta}}x}) \geq d^2(T_{ss_0}x, \pi\overline{T_{\mu_{\alpha_\beta}}x}) + d^2(\overline{T_{\mu_{\alpha_\beta}}x}, \pi\overline{T_{\mu_{\alpha_\beta}}x})$ for each $s \in S$ where $\pi : C \to \overline{co}\{T_sx\}_{s \succcurlyeq s_0}$ is the nearest point projection. Thus,

$\mu_{\alpha_\beta}(d^2(T_{ss_0}x, \overline{T_{\mu_{\alpha_\beta}}x})) \geq \mu_{\alpha_\beta}(d^2(T_{ss_0}x, \pi\overline{T_{\mu_{\alpha_\beta}}x})) + d^2(\overline{T_{\mu_{\alpha_\beta}}x}, \pi\overline{T_{\mu_{\alpha_\beta}}x}) > \mu_{\alpha_\beta}(d^2(T_{ss_0}x, \pi\overline{T_{\mu_{\alpha_\beta}}x}))$. This impossibility shows that $\overline{T_{\mu_{\alpha_\beta}}x} \in \overline{co}\{T_s x\}_{s \succcurlyeq s_0}$.

Step 2. $\lim_\beta d(T_{\mu_{\alpha_\beta}}x, \overline{T_{\mu_{\alpha_\beta}}x}) = 0$.

If this does not hold, there must be some $\eta > 0$ so that for each β, there exists $\beta' \succ \beta$ satisfying $d(T_{\mu_{\alpha_{\beta'}}}x, \overline{T_{\mu_{\alpha_{\beta'}}}x}) \geq \eta$. Put $\varepsilon = \frac{\eta^2}{2}$. Since the asymptotically invariant net $\{\mu_\beta\}$ satisfies (3), there exists β_0 such that $|\mu_{\alpha_{\beta_s}}(d^2(T_s x, \overline{T_{\mu_{\alpha_\beta}}x})) - \mu_{\alpha_{\beta_s}}(d^2(T_{ss_0}x, \overline{T_{\mu_{\alpha_\beta}}x}))| < \varepsilon$ for each $\beta \succcurlyeq \beta_0$. We suppose first that $\mu_{\alpha_{\beta'_{0_s}}}(d^2(T_{ss_0}x, \overline{T_{\mu_{\alpha_{\beta'_0}}}x})) \leq \mu_{\alpha_{\beta'_{0_s}}}(d^2(T_s x, T_{\mu_{\alpha_{\beta'_0}}}x))$. Set $w = \frac{T_{\mu_{\alpha_{\beta'_0}}}x \oplus \overline{T_{\mu_{\alpha_{\beta'_0}}}x}}{2}$. By (CN) inequality, the following inequalities hold for each $s \in S$:

$$\begin{aligned} d^2(T_s x, w) &\leq \frac{1}{2}d^2(T_s x, T_{\mu_{\alpha_{\beta'_0}}}x) + \frac{1}{2}d^2(T_s x, \overline{T_{\mu_{\alpha_{\beta'_0}}}x}) - \frac{1}{4}d^2(T_{\mu_{\alpha_{\beta'_0}}}x, \overline{T_{\mu_{\alpha_{\beta'_0}}}x}) \\ &\leq \frac{1}{2}d^2(T_s x, T_{\mu_{\alpha_{\beta'_0}}}x) + \frac{1}{2}d^2(T_s x, \overline{T_{\mu_{\alpha_{\beta'_0}}}x}) - \frac{\eta^2}{4}. \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi_{\mu_{\alpha_{\beta'_0}}}(w) &\leq \frac{1}{2}\varphi_{\mu_{\alpha_{\beta'_0}}}(T_{\mu_{\alpha_{\beta'_0}}}x) + \frac{1}{2}\varphi_{\mu_{\alpha_{\beta'_0}}}(\overline{T_{\mu_{\alpha_{\beta'_0}}}x}) - \frac{\eta^2}{4} \\ &< \frac{1}{2}\varphi_{\mu_{\alpha_{\beta'_0}}}(T_{\mu_{\alpha_{\beta'_0}}}x) + \frac{1}{2}\mu_{\alpha_{\beta'_{0_s}}}(d^2(T_{ss_0}x, \overline{T_{\mu_{\alpha_{\beta'_0}}}x})) + \frac{\varepsilon}{2} - \frac{\eta^2}{4} \\ &\leq \varphi_{\mu_{\alpha_{\beta'_0}}}(T_{\mu_{\alpha_{\beta'_0}}}x) + \frac{\varepsilon}{2} - \frac{\eta^2}{4} = \varphi_{\mu_{\alpha_{\beta'_0}}}(T_{\mu_{\alpha_{\beta'_0}}}x), \end{aligned}$$

contradicting to the argminimality of $T_{\mu_{\alpha_{\beta'_0}}}x$. In case $\mu_{\alpha_{\beta'_{0_s}}}(d^2(T_s x, T_{\mu_{\alpha_{\beta'_0}}}x)) < \mu_{\alpha_{\beta'_{0_s}}}(d^2(T_{ss_0}x, \overline{T_{\mu_{\alpha_{\beta'_0}}}x}))$, we can show in the same way that $\mu_{\alpha_{\beta'_{0_s}}}(d^2(T_{ss_0}x, w)) < \mu_{\alpha_{\beta'_{0_s}}}(d^2(T_{ss_0}x, \overline{T_{\mu_{\alpha_{\beta'_0}}}x}))$ for some w which also leads to a contradiction.

Step 3. $x_0 = T_\mu x$.

We suppose on the contrary and let $\eta := d(x_0, T_\mu x) > 0$. Let $I = [T_\mu x, x_0]$ and $\pi_I : C \to I$ be the nearest point projection onto I. Since $\{T_s x\}$ is bounded, there exists $M > 0$ such that $d(T_s x, \pi_I(T_s x)) \leq M$ for all $s \in S$. Set $N_0 > \frac{4(M+\eta)}{5\eta}$ and $\rho = \frac{\eta}{5N_0}$. Suppose there exists $s_0 \in S$ such that $d(\pi_I(T_s x), x_0) \geq 2\rho$ for all $s \succcurlyeq s_0$. We know, by Step 1, that $\overline{T_{\mu_{\alpha_\beta}}x} \in \overline{co}\{T_s x\}_{s \succcurlyeq s_0}$. Let $A := \{y \in C: d(\pi_I(y), x_0) > 2\rho\}$. Using property (N), A is convex and $\overline{co}\{T_s x\}_{s \succcurlyeq s_0} \subset \bar{A} \subset \{y \in C : d(\pi_1(y), x_0) \geq 2\rho\}$ and thus $d(\pi_I(\overline{T_{\mu_{\alpha_\beta}}x}), x_0) \geq 2\rho$. By Step 2, $\lim_\beta d(T_{\mu_{\alpha_\beta}}x, \overline{T_{\mu_{\alpha_\beta}}x}) = 0$. Choose β_0, using the nonexpansiveness of π_I, so that $d(\pi_I(T_{\mu_{\alpha_\beta}}x), \pi_I(\overline{T_{\mu_{\alpha_\beta}}x})) < \rho$ for all $\beta \succcurlyeq \beta_0$. Thus, $d(\pi_I(T_{\mu_{\alpha_\beta}}x), x_0) > \rho$ for all $\beta \succcurlyeq \beta_0$. But then $x_0 \notin \overline{co}\{T_{\mu_{\alpha_\beta}}x\}_{\beta \succcurlyeq \beta_0}$ which contradicts to the fact that x_0 is the Δ - limit of $\{T_{\mu_{\alpha_\beta}}x\}$. Therefore, there must be a subnet {s'} of S such that $s' \succ s$ for all s and

$$d(\pi_I(T_{s'}x), x_0) < 2\rho = \frac{2\eta}{5N_0} \tag{5}$$

for all s'. Hence,

$$d(\pi_I(T_{s'}x), T_\mu x) = \eta - d(\pi_I(T_{s'}x), x_0) > \eta - \frac{2\eta}{5N_0}. \tag{6}$$

By the property of N_0, $5\eta^2 N_0 > 4\eta M + 4\eta^2$ and so

$$\eta^2 - \frac{4\eta^2}{5N_0} > \frac{4\eta M}{5N_0}. \tag{7}$$

From (5), (6), and (7),

$$\begin{aligned} d^2(\pi_I(T_{s'}x), T_\mu x) &> \eta^2 - \frac{4\eta^2}{5N_0} + (\frac{2\eta}{5N_0})^2 > \frac{4\eta M}{5N_0} + (\frac{2\eta}{5N_0})^2 \\ &> 2d(x_0,\ \pi_I(T_{s'}x))d(T_{s'}x,\ \pi_I(T_{s'}x)) + d^2(x_0,\ \pi_I(T_{s'}x)). \end{aligned}$$

Using (1),

$$\begin{aligned} d^2(T_{s'}x, T_\mu x) &\geq d^2(\pi_I(T_{s'}x), T_\mu x) + d^2(\pi_I(T_{s'}x), T_{s'}x) \\ &> d^2(x_0, \pi_I(T_{s'}x)) + 2d(x_0, \pi_I(T_{s'}x))d(T_{s'}x, \pi_I(T_{s'}x)) + d^2(\pi_I(T_{s'}x), T_{s'}x) \\ &= (d(x_0, \pi_I(T_{s'}x)) + d(T_{s'}x, \pi_I(T_{s'}x)))^2 \\ &\geq d^2(T_{s'}x, x_0) \end{aligned}$$

for all s'. Since the points x_0 and $T_\mu x$ belong to the set $F(\mathcal{S})$, the nets $\{d^2(T_s x, x_0)\}$ and $\{d^2(T_s x, T_\mu x)\}$ are decreasing. So, $\lim_s d^2(T_s x, x_0)$ and $\lim_s d^2(T_s x, T_\mu x)$ exist. Hence, $\phi_\mu(T_\mu x) = \lim_s d^2(T_s x, T_\mu x) = \lim_{s'} d^2(T_s x, T_\mu x) = \mu_{s'}(d^2(T_s x, T_\mu x)) \geq \mu_{s'}(d^2(T_s x, x_0)) = \lim_{s'} d^2(T_s x, x_0) = \lim_s d^2(T_s x, x_0) = \phi_\mu(x_0)$, a contradiction. Thus, $x_0 = T_\mu x$.

The above argument shows that, for every subnet $\{\mu_{\alpha'}\}$ of $\{\mu_\alpha\}$, there exists a subnet $\{\mu_{\alpha_\beta}\}$ of $\{\mu_{\alpha'}\}$ such that $\{T_{\mu_{\alpha_\beta}}x\}$ Δ-converges to $T_\mu x (= Px)$. By Remark 2.7 (ii), $\{T_{\mu_\alpha}x\}$ Δ-converges to Px. □

It is an interesting open problem to determine whether Theorem 3.9 remains valid when the semigroup is amenable but not commutative.

3.2 Applications

Proposition 3.10. *Let C be a closed convex subset of a complete CAT(0) space X and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $S = (\mathbb{N} \cup \{0\}, +)$, $\mathcal{S}(S) = \{T^n : n \in S\}$, $\Lambda = \mathbb{N}$ or $\mathbb{R}^+$ and $\beta_{\lambda k} \geq 0$ be such that $\sum_{k \in S} \beta_{\lambda k} = 1$ for all $\lambda \in \Lambda$. Suppose for all $k \in S$,*

$$\lim_{\lambda \to \infty} \beta_{\lambda k} = 0 \tag{8}$$

and for each $m \in S$,

$$\lim_{\lambda \to \infty} \sum_{k=m}^{\infty} |\beta_{\lambda k} - \beta_{\lambda(k-m)}| = 0. \tag{9}$$

For any $f = (a_0, a_1, \ldots) \in B(S)$ let $\mu_\lambda(f) = \sum_{k=0}^{\infty} \beta_{\lambda k} a_k$. Then for each $x \in C$, $\{T_{\mu_\lambda}x\}$ Δ-converges to z for some z in $F(T)$.

In particular, if X is a Hilbert space, we have $\sum_{k=0}^{\infty} \beta_{\lambda k} T^k x$ converges weakly to z for some z in $F(T)$ as $\lambda \to \infty$.

Proof. For each $m \in S$, $|\mu_\lambda(f) - \mu_\lambda(r_m f)| = \left|\sum_{k=0}^{\infty} \beta_{\lambda k} a_k - \sum_{k=0}^{\infty} \beta_{\lambda k} a_{k+m}\right| \le \sum_{k=0}^{m-1} |\beta_{\lambda k}|\,|a_k| + \sum_{k=m}^{\infty} \left|\beta_{\lambda k} - \beta_{\lambda(k-m)}\right| |a_k|$. By (8) and (9), we have $\lim_{\lambda\to\infty} |\mu_\lambda(f) - \mu_\lambda(r_m f)| = 0$, and this shows that the net $\{\mu_\lambda\}$ is asymptotically invariant. Let $x \in C$ and consider a_k of the form $a_k = d^2(T^k x, y)$ where $y \in C$. We see that $\{\mu_\lambda\}$ satisfies (3). By Theorem 3.9, we have $\{T_{\mu_\lambda} x\}$ Δ - converges to z for some z in $F(T)$.

In Hilbert spaces, by a well-known result in probability theory, we know that

$$\sum_{k=0}^{\infty} \beta_{\lambda k} \left\| T^k x - \sum_{k=0}^{\infty} \beta_{\lambda k} T^k x \right\|^2 \le \sum_{k=0}^{\infty} \beta_{\lambda k} \left\| T^k x - y \right\|^2$$

for all $y \in C$. So we have $T_{\mu_\lambda} x = \sum_{k=0}^{\infty} \beta_{\lambda k} T^k x$. □

Corollary 3.11 (Bailon Ergodic Theorem). *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to z for some z in $F(T)$ as $n \to \infty$.

Proof. Let $\Lambda = \mathbb{N}$ and put, for $\lambda \in \Lambda$ and $k \in S = (\mathbb{N} \cup \{0\}, +)$,

$$\beta_{\lambda k} = \begin{cases} \frac{1}{\lambda}, \ k \le \lambda - 1, \\ 0, \ k > \lambda - 1. \end{cases}$$

The result now follows from Proposition 3.10. □

Corollary 3.12. [[15], Theorem 3.5.1] *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then, for any $x \in C$, $S_r x = (1 - r) \sum_{k=0}^{\infty} r^k T^k x$converges weakly to z for some z in $F(T)$ as $r \uparrow 1$.*

Proof. Let $\Lambda = \mathbb{R}^+$ and put, for $\lambda \in \Lambda$ and $k \in S = (\mathbb{N} \cup \{0\}, +)$,

$$\beta_{\lambda k} = \frac{(\lambda - 1)^k}{\lambda^{k+1}}.$$

Taking $r = \frac{\lambda - 1}{\lambda}$, Proposition 3.10 implies the desired result.

Let $S = (\mathbb{R}^+ \cup \{0\}, +)$ and C be a closed convex subset of a Hilbert space H. Then, a family $\mathcal{S}(S) = \{T(s) : s \in S\}$ is said to be a *continuous nonexpansive semigroup* on C if $\mathcal{S}(S)$ satisfies the following:

(i) $T(s) : C \to C$ is a nonexpansive mapping for all $s \in S$,
(ii) $T(t + s)x = T(t)T(s)x$ for all $x \in C$ and $t, s \in S$,
(iii) for each $x \in C$, the mapping $s \to T(s)x$ is continuous, and
(iv) $T(0)x = x$ for all $x \in C$.

Proposition 3.13. *Let C be a closed convex subset of a Hilbert space H. Let $S = (\mathbb{R}^+ \cup \{0\}, +)$, $\mathcal{S}(S)$be a continuous nonexpansive semigroup on C with $F(\mathcal{S}) \neq \emptyset$, $\Lambda = \mathbb{R}^+$ and g_λ be a density function on S,i.e., $g_\lambda \ge 0$ and $\int_0^\infty g_\lambda(s) ds = 1$for all $\lambda \in \Lambda$. Suppose g_λ satisfies the following properties. for each $h \in S$,*

$$\lim_{\lambda\to\infty} g_\lambda(s) = 0 \tag{10}$$

uniformly on [0, *h*] *and*

$$\lim_{\lambda\to\infty} \int_h^\infty |g_\lambda(s) - g_\lambda(s-h)|ds = 0. \tag{11}$$

Then, for any $x \in C$,

$$\int_0^\infty g_\lambda(s)T(s)xds$$

converges weakly to some $z \in F(\mathcal{S})$*as* $\lambda \to \infty$.

Proof. For $f \in B(S)$ we define $\mu_\lambda(f) = \int_0^\infty g_\lambda(s)f(s)ds$ for all $\lambda > 0$. Thus, μ_λ is a mean on $B(S)$. For any $h \in S$ we consider

$$\begin{aligned} |\mu_\lambda(f) - \mu_\lambda(r_h f)| &= \left|\int_0^\infty g_\lambda(s)f(s)ds - \int_0^\infty g_\lambda(s)f(s+h)ds\right| \\ &\le \int_0^h |g_\lambda(s)||f(s)|ds + \int_h^\infty |g_\lambda(s) - g_\lambda(s-h)||f(s)|ds. \end{aligned}$$

By (10) and (11), $\lim_\lambda|\mu_\lambda(f) - \mu_\lambda(r_h f)| = 0$. So, $\{\mu_\lambda\}$ is asymptotically invariant. For each $z \in C$, let $f(s) = ||z - T(s)x||^2$. We see that $\{\mu_\lambda\}$ satisfies (3). For each $x \in C$, we know that

$$\int_0^\infty g_\lambda(s)\left\|\int_0^\infty g_\lambda(s)T(s)xds - T(s)x\right\|^2 ds \le \int_0^\infty g_\lambda(s)\left\|y - T(s)x\right\|^2 ds$$

for all $y \in C$. Thus, $T_{\mu_\lambda}x = \int_0^\infty g_\lambda(s)T(s)xds$. By Theorem 3.9, we have $\int_0^\infty g_\lambda(s)T(s)xds$ converges weakly to some $z \in F(\mathcal{S})$ as $\lambda \to \infty$. □

Corollary 3.14. [[15], Theorem 3.5.2] *Let C be a closed convex subset of a Hilbert space H. Suppose* $S = (\mathbb{R}^+ \cup \{0\}, +)$ *and* $\mathcal{S}(S)$*be a continuous nonexpansive semigroup on C with* $F(\mathcal{S}) \neq \emptyset$. *Then, for any* $x \in C$,

$$S_\lambda x = \frac{1}{\lambda}\int_0^\lambda T(s)xds$$

converges weakly to some $z \in F(\mathcal{S})$*as* $\lambda \to \infty$.

Proof. Let $\Lambda = \mathbb{R}^+$ and put, for $\lambda \in \Lambda$ and $s \in S$, $g_\lambda(s) = \frac{1}{\lambda}\chi_{[0,\lambda]}$. The result now follows from Proposition 3.13. □

Corollary 3.15. [[15], Theorem 3.5.3] *Let C be a closed convex subset of a Hilbert space H. Suppose* $S = (\mathbb{R}^+ \cup \{0\}, +)$ *and* $\mathcal{S}(S)$*be a continuous nonexpansive semi-group on C with* $F(\mathcal{S}) \neq \emptyset$. *Then, for any* $x \in C$,

$$r\int_0^\infty e^{-rs}T(s)xds$$

converges weakly to some $z \in F(\mathcal{S})$*as* $r \downarrow 0$.

Proof. Let $\Lambda = \mathbb{R}^+$ and put, for $\lambda \in \Lambda$ and $s \in S$, $g_\lambda(s) = \frac{1}{\lambda}e^{-\frac{1}{\lambda}s}$. Again, we can then apply Proposition 3.13 by taking $r = \frac{1}{\lambda}$. □

By using Lemma 2.2, we can obtain a strong convergence theorem in Hilbert spaces stated as Theorem 3.17 below.

Proposition 3.16. *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Given $x \in C$ and let $r = r(C, \{T^n x\})$. Let z be the unique asymptotic center of $\{T^n x\}$. For each $n \in \mathbb{N}$, define*

$$\Pi_n := \left\{ p = \{\beta_{nk}\}_{k \geq n} \subset [0, 1] : \sum_{k \geq n} \beta_{nk} = 1 \right\}$$

and

$$V_n := \sup_{p \in \Pi_n} \sum_{k \geq n} \beta_{nk} \left\| T^k x - \bar{x}_n \right\|^2$$

where $\bar{x}_n = \sum_{k \geq n} \beta_{nk} T^k x$. If $V := \lim_{n \to \infty} V_n$, then $V = r^2$.

Proof. Given $\varepsilon > 0$. Since $z \in A(C, \{T^n x\})$, by Remark 2.4 (ii), $z \in F(T)$. Choose $n_\varepsilon \in \mathbb{N}$ such that $||T^n x - z|| < r + \varepsilon$ for all $n \geq n_\varepsilon$. Fix $n \geq n_\varepsilon$ and let $p = \{\beta_{nk}\}_{k \geq n} \in \Pi_n$. Thus,

$$\sum_{k \geq n} \beta_{nk} \left\| T^k x - \bar{x}_n \right\|^2 \leq \sum_{k \geq n} \beta_{nk} \left\| T^k x - z \right\|^2 < (r + \varepsilon)^2.$$

So $V_n = \sup_{p \in \Pi_n} \sum_{k \geq n} \beta_{nk} \left\| T^k x - \bar{x}_n \right\|^2 < (r + \varepsilon)^2$. Letting $n \to \infty$, $V = \lim_{n \to \infty} V_n \leq (r + \varepsilon)^2$ for any $\varepsilon > 0$. Hence, $V \leq r^2$.

Next, we show that $r^2 \leq V$. Indeed, since $z \in \overline{co}\{T^k x\}_{k \geq n}$ for all $n \in \mathbb{N}$, there exists a sequence $\{\bar{x}_n\}$ with $\bar{x}_n \in co\{T^k x\}_{k \geq n}$ for each n and $\bar{x}_n \to z$ as $n \to \infty$. Put $\bar{x}_n = \sum_{k \geq n} \beta_{nk} T^k x$. Since $\{T^n x\}$ is bounded, there exists $M > 0$ such that $\left\| T^k x - \bar{x}_n \right\| + \left\| T^k x - z \right\| \leq M$. For each $\varepsilon > 0$, choose $n_\varepsilon \in \mathbb{N}$ such that $\|\bar{x}_n - z\| < \varepsilon$, $V_n < V + \varepsilon$ for all $n \geq n_\varepsilon$, and $||T^k x - z|| > r - \varepsilon$ for all $k \geq n_\varepsilon$. Thus for any $n \geq n_\varepsilon$,

$$\begin{aligned} \sum_{k \geq n} \beta_{nk} | ||T^k x - \bar{x}_n||^2 - ||T^k x - z||^2 | \\ &= \sum_{k \geq n} \beta_{nk} | ||T^k x - \bar{x}_n|| - ||T^k x - z|| | (||T^k x - \bar{x}_n|| + ||T^k x - z||) \\ &= \sum_{k \geq n} \beta_{nk} ||\bar{x}_n - z|| (||T^k x - \bar{x}_n|| + ||T^k x - z||) \leq \varepsilon M. \end{aligned}$$

Hence,

$$\begin{aligned} (r - \varepsilon)^2 &< \sum_{k \geq n} \beta_{nk} ||T^k x - z||^2 \\ &= \sum_{k \geq n} \beta_{nk} ||T^k x - z||^2 + \sum_{k \geq n} \beta_{nk} ||T^k x - \bar{x}_n||^2 - \sum_{k \geq n} \beta_{nk} ||T^k x - \bar{x}_n||^2 \\ &\leq V_n + \sum_{k \geq n} \beta_{nk} | ||T^k x - \bar{x}_n||^2 - ||T^k x - z||^2 | < V + \varepsilon + \varepsilon M. \end{aligned}$$

So $(r - \varepsilon)^2 < V + \varepsilon + \varepsilon M$ for any $\varepsilon > 0$. This implies $r^2 \leq V$.

Theorem 3.17. *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$,. Suppose z, Π_n, V, and $\bar{x}_n$be defined as in Proposition 3.16. If the sequence $\{\bar{x}_n\}$satisfies*

$$\lim_{n\to\infty}\left(\sum_{k\geq n}\beta_{nk}||T^k x-\bar{x}_n||^2\right)=V,\tag{12}$$

then $\{\bar{x}_n\}$*converges (strongly) to* $z\in F(T)$ *as* $n\to\infty$.

Proof. Suppose for some $\varepsilon>0$, there exists a subsequence $\{\bar{x}_{n_l}\}$ of $\{\bar{x}_n\}$ such that $\left\|\bar{x}_{n_l}-z\right\|\geq\varepsilon$ for all $l\in\mathbb{N}$. For each $y\in C$ and $n\in\mathbb{N}$, define $\varphi_n(y):=\sum_{k\geq n}\beta_{nk}||T^k x-y||^2$. Let $0<\delta<\frac{\varepsilon^2}{8}$. By (12) and $z\in A(C,\{T^k x\})$, we choose n_δ such that $r^2-\delta=V-\delta<\varphi_{n_l}(\bar{x}_{n_l})<V+\delta=r^2+\delta$ for all $n_l\geq n_\delta$ and $||T^k x-z||^2<r^2+\delta$ for all $k\geq n_\delta$. Fix $l\geq n_\delta$ and let $\omega=\frac{\bar{x}_{n_l}+z}{2}$. By the Parallelogram law, we have for each $k\geq n_l$,

$$||T^k x-\omega||^2=\frac{1}{2}||T^k x-\bar{x}_{n_l}||^2+\frac{1}{2}||T^k x-z||^2-\frac{1}{4}||\bar{x}_{n_l}-z||^2.$$

Hence,

$$\varphi_{n_l}(\omega)<\frac{1}{2}(r^2+\delta)+\frac{1}{2}(r^2+\delta)-\frac{1}{4}\varepsilon^2<r^2-\delta<\varphi_{n_l}(\bar{x}_{n_l}).$$

Using Lemma 2.2, we see that this contradicts to the minimality of $\varphi_{n_l}(\bar{x}_{n_l})$. □

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Authors' contributions

All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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