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A new modified block iterative algorithm for uniformly quasi-φ-asymptotically nonexpansive mappings and a system of generalized mixed equilibrium problems

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Abstract

In this paper, we introduce a new modified block iterative algorithm for finding a common element of the set of common fixed points of an infinite family of closed and uniformly quasi-**φ**-asymptotically nonexpansive mappings, the set of the variational inequality for an **α**-inverse-strongly monotone operator, and the set of solutions of a system of generalized mixed equilibrium problems. We obtain a strong convergence theorem for the sequences generated by this process in a 2-uniformly convex and uniformly smooth Banach space. Our results extend and improve ones from several earlier works.
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Keywords: modified block iterative algorithm, inverse-strongly monotone operator, variational inequality, a system of generalized mixed equilibrium problem, uniformly quasi-**φ**-asymptotically nonexpansive mapping

1 Introduction

Let C be a nonempty closed convex subset of a real Banach space E with $||\cdot||$ and let E^* be the dual space of E. Let $\{f_i\}_{i \in \Gamma} : C \times C \to \mathbb{R}$ be a bifunction, $\{\phi_i\}_{i \in \Gamma} : C \to \mathbb{R}$ be a real-valued function, and $\{B_i\}_{i \in \Gamma} : C \to E^*$ be a monotone mapping, where Γ is an arbitrary index set. The *system of generalized mixed equilibrium problems* is to find $x \in C$ such that

$$f_i(x, y) + \langle B_i x, y - x \rangle + \varphi_i(y) - \varphi_i(x) \geq 0, \quad i \in \Gamma, \ \forall y \in C. \tag{1.1}$$

If Γ is a singleton, then problem (1.1) reduces to the *generalized mixed equilibrium problem*, which is to find $x \in C$ such that

$$f(x, y) + \langle Bx, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions to (1.2) is denoted by GMEP(f, B, ϕ), i.e.,

$$\text{GMEP}(f, B, \varphi) = \{x \in C : f(x, y) + \langle Bx, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C\}. \tag{1.3}$$

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If $B \equiv 0$, the problem (1.2) reduces into the *mixed equilibrium problem for f*, denoted by MEP (f, ϕ), which is to find $x \in C$ such that

$$f(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.4}$$

If $f \equiv 0$, the problem (1.2) reduces into the *mixed variational inequality* of Browder type, denoted by VI(C, B, ϕ), which is to find $x \in C$ such that

$$\langle Bx, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.5}$$

If $B \equiv 0$ and $\phi \equiv 0$ the problem (1.2) reduces into the *equilibrium problem for f*, denoted by EP(f), which is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{1.6}$$

If $f \equiv 0$, the problem (1.4) reduces into the *minimize problem*, denoted by Argmin(ϕ), which is to find $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.7}$$

The above formulation (1.5) was shown in [1] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP(f). In other words, the EP(f) is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of EP(f); see, for example [1,2] and references therein. Some solution methods have been proposed to solve the EP(f); see, for example, [1-15] and references therein.

The *normalized duality mapping* $J : E \rightarrow 2^{E*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \| x \|^2, \| x^* \| = \| x \|\}$$

for all $x \in E$. If E is a Hilbert space, then $J = I$, where I is the identity mapping. Consider the functional defined by

$$\phi(x, y) = \| x \|^2 - 2\langle x, Jy \rangle + \| y \|^2, \quad \forall x, y \in E. \tag{1.8}$$

As well known that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \rightarrow C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. It is obvious from the definition of function φ that

$$(\| x \| - \| y \|)^2 \leq \phi(x, y) \leq (\| x \| + \| y \|)^2, \quad \forall x, y \in E. \tag{1.9}$$

If E is a Hilbert space, then $\varphi(x, y) = ||x - y||^2$, for all $x, y \in E$. On the other hand, the *generalized projection* [16] $\Pi_C : E \rightarrow C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\varphi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x), \tag{1.10}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\varphi(x, y)$ and strict monotonicity of the mapping J (see, for example, [16-20]).

Remark 1.1. If E is a reflexive, strictly convex, and smooth Banach space, then for x, $y \in E$, $\varphi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\varphi(x, y) = 0$ then $x = y$. From (1.8), we have $||x|| = ||y||$. This implies that $\langle x, Jy \rangle = ||x||^2 = ||Jy||^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [18,20] for more details.

Let C be a closed convex subset of E, a mapping $T : C \rightarrow C$ is said to be *L-Lipschitz continuous* if $||Tx - Ty|| \leq L||x - y||$, $\forall x, y \in C$ and a mapping T is said to be *nonexpansive* if $||Tx - Ty|| \leq ||x - y||$, $\forall x, y \in C$. A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a point p in C is said to be an *asymptotic fixed point* of T [21] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \to \infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$.

A mapping T from C into itself is said to be *relatively nonexpansive* [22-24] if $\widetilde{F(T)} = F(T)$ and $\varphi(p, Tx) \leq \varphi(p, x)$ for all $x \in C$ and $p \in F(T)$. T is said to be *relatively quasi-nonexpansive* if $F(T) \neq \varnothing$ and $\varphi(p, Tx) \leq \varphi(p, x)$ for all $x \in C$ and $p \in F(T)$. T is said to be *φ-nonexpansive*, if $\varphi(Tx, Ty) \leq \varphi(x, y)$ for $x, y \in C$. T is said to be *quasi-φ-asymptotically nonexpansive* if $F(T) \neq \varnothing$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ such that $\varphi(p, T^n x) \leq k_n \varphi(p, x)$ for all $n \geq 1$ $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [25-27].

We note that the class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [25-29] which requires the strong restriction: $F(T) = \widetilde{F(T)}$. A mapping T is said to be *closed* if for any sequence $\{x_n\} \subset C$ with $x_n \rightarrow x$ and $Tx_n \rightarrow y$, then $Tx = y$. It is easy to know that each relatively nonexpansive mapping is closed.

Definition 1.2. (*Chang et al.* [30]) (1) Let $\{T_i\}_{i=1}^{\infty} : C \rightarrow C$ be a sequence of mapping. $\{T_i\}_{i=1}^{\infty}$ is said to be *a family of uniformly quasi-φ-asymptotically nonexpansive mappings*, if $\mathcal{F} := \cap_{i=1}^{\infty} F(T_i) \neq \varnothing$, and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ such that for each $i \geq 1$

$$\phi(p, T_i^n x) \leq k_n \phi(p, x), \quad \forall p \in \mathcal{F}, \ x \in C, \ \forall n \geq 1. \tag{1.11}$$

(2) A mapping $T : C \rightarrow C$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\| T^n x - T^n \gamma \| \leq L \| x - \gamma \|, \quad \forall x, \gamma \in C. \tag{1.12}$$

Recall that let $A : C \rightarrow E^*$ be a mapping. Then A is called
(i) *monotone* if

$$\langle Ax - A\gamma, x - \gamma \rangle \geq 0, \quad \forall x, \gamma \in C,$$

(ii) *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - A\gamma, x - \gamma \rangle \geq \alpha \| Ax - A\gamma \|^2, \quad \forall x, \gamma \in C.$$

Remark 1.3. It is easy to see that an α-inverse-strongly monotone is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

In 2004, Matsushita and Takahashi [31] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1 - \alpha_n) JT x_n), \tag{1.13}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0, 1]$, T is a relatively nonexpansive mapping and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They proved that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

In 2005, Matsushita and Takahashi [28] proposed the following hybrid iteration method (it is also called the CQ method) with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) JT x_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{1.14}$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

In 2008, Iiduka and Takahashi [32] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator A in a 2-uniformly convex and uniformly smooth Banach space E : $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \tag{1.15}$$

for every $n = 1, 2, 3, \ldots$, where Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.15) converges weakly to some element of VI(A, C). Takahashi and Zembayashi [33,34] studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Banach spaces.

In 2009, Wattanawitoon and Kumam [14] using the idea of Takahashi and Zembayashi [33] extended the notion from relatively nonexpansive mappings or φ-nonexpansive mappings to two relatively quasi-nonexpansive mappings and also proved some strong convergence theorems to approximate a common fixed point of relatively quasi-nonexpansive mappings and the set of solutions of an equilibrium problem in the framework of Banach spaces. Cholamjiak [35] studied the following iterative algorithm:

$$\begin{cases} z_n = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \\ y_n = J^{-1}(\alpha_n J x_n + \beta_n JT x_n + \gamma_n JS z_n), \\ u_n \in C \text{ such that } f(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{1.16}$$

where J is the duality mapping on E. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$. Then, he proved that $\{x_n\}$ converges strongly to $q = \Pi_F x_0$, where $F := F(T) \cap F(S) \cap$ EP$(f) \cap$ VI(A, C).

In 2010, Saewan et al. [29] introduced a new hybrid projection iterative scheme which is difference from the algorithm (1.16) of Cholamjiak in [[35], Theorem 3.1] for two relatively quasi-nonexpansive mappings in a Banach space. Motivated by the results of Takahashi and Zembayashi [34], Cholumjiak and Suantai [36] proved the following strong convergence theorem by the hybrid iterative scheme for approximation of common fixed point of countable families of relatively quasi-nonexpansive mappings in a uniformly convex and uniformly smooth Banach space: $x_0 \in E$, $x_1 = \Pi_{C_1} x_0$, $C_1 = C$

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n), \\ u_{n,i} = T_{r_{m,n}}^{f_m} T_{r_{m-1,n}}^{f_{m-1}} \cdots T_{r_{1,n}}^{f_1} y_{n,i}, \\ C_{n+1} = \{z \in C_n : \sup_{i>1} \phi(z, J u_{n,i}) \leq \phi(w, J x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \, n \geq 1. \end{cases} \quad (1.17)$$

Then, they proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, and $\{r_{n,\,i}\}$, the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T) \cap EP(f)} x_0$.

We note that the block iterative method is a method which often used by many authors to solve the convex feasibility problem (see, [37,38], etc.). In 2008, Plubtieng and Ungchittrakool [39] established strong convergence theorems of block iterative methods for a finite family of relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming. Chang et al. [30] proposed the modified block iterative algorithm for solving the convex feasibility problems for an infinite family of closed and uniformly quasi-φ-asymptotically nonexpansive mapping, and they obtained the strong convergence theorems in a Banach space.

In 2010, Saewan and Kumam [40] obtained the following result for the set of solutions of the generalized equilibrium problems and the set of common fixed points of an infinite family of closed and uniformly quasi-φ-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space E with Kadec-Klee property.

Theorem SK Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfing (A1)-(A4). Let B be a continuous monotone mapping of C into E^*. Let $\{S_i\}_{i=1}^\infty : C \to C$ be an infinite family of closed uniformly L_i-Lipschitz continuous and uniformly quasi-φ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $F := \cap_{i=1}^\infty F(S_i) \cap GEP(f, B)$ is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:

$$\begin{cases} y_n = J^{-1}(\beta_n J x_n + (1 - \beta_n) J z_n), \\ z_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^\infty \alpha_{n,i} J S_i^n x_n), \\ u_n \in C \text{ such that } f(u_n, \gamma) + \langle B y_n, \gamma - u_n \rangle + \dfrac{1}{r_n}\langle \gamma - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall \gamma \in C, \quad (1.18) \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where J is the duality mapping on E, $\theta_n = \sup_{q \in F} (k_n - 1)\varphi(q, x_n)$, $\{\alpha_{n,\,i}\}$, $\{\beta_n\}$ are sequences in [0, 1] and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\sum_{i=0}^\infty \alpha_{n,i} = 1$ for all $n \geq 0$ and $\liminf_{n \to \infty} \alpha_{n,\,0} \alpha_{n,\,i} > 0$ for all $i \geq 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Quite recently, Qin et al. [9] purposed the problem of approximating a common fixed point of two asymptotically quasi-φ-nonexpansive mappings based on hybrid projection methods. Strong convergence theorems are established in a real Banach space. Zegeye et al. [15] introduced an iterative process which converges strongly to a common element of set of common fixed points of countably infinite family of closed relatively quasi- nonexpansive mappings, the solution set of generalized equilibrium problem and the solution set of the variational inequality problem for an α-inverse-strongly monotone mapping in Banach spaces.

Motivated and inspired by the work of Chang et al. [30], Qin et al. [7], Takahashi and Zembayashi [33], Wattanawitoon and Kumam [14], Zegeye [41] and Saewan and Kumam [40], we introduce a new modified block hybrid projection algorithm for finding a common element of the set of the variational inequality for an α-inverse-strongly monotone operator, the set of solutions of the system of generalized mixed equilibrium problems and the set of common fixed points of an infinite family of closed and uniformly quasi-φ-asymptotically nonexpansive mappings in the framework Banach spaces. The results presented in this paper improve and generalize some well-known results in the literature.

2 Preliminaries

A Banach space E is said to be *strictly convex* if $\| \frac{x+y}{2} \| < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. Let $U = \{x \in E : ||x|| = 1\}$ be the unit sphere of E. Then a Banach space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}$$

exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. Let E be a Banach space. The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf \left\{ 1 - \| \frac{x+y}{2} \| : x, y \in E, \| x \| = \| y \| = 1, \| x - y \| \geq \varepsilon \right\}.$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$; see [42,43] for more details. Observe that every p-uniformly convex is uniformly convex. One should note that no a Banach space is p-uniformly convex for $1 < p < 2$. It is well known that a Hilbert space is *2-uniformly convex*, uniformly smooth. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Remark 2.1. The following basic properties can be found in Cioranescu [18].

(i) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.

(ii) If E is a smooth, strictly convex, and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E*}$ is single-valued, one-to-one, and onto.

(iii) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.

(iv) Each uniformly convex Banach space E has the *Kadec-Klee property*, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $||x_n|| \to ||x||$, then $x_n \to x$.

We also need the following lemmas for the proof of our main results.

Lemma 2.2. (*Beauzamy* [44] and *Xu* [45]). *If E be a 2-uniformly convex Banach space, then for all x, $y \in E$ we have*

$$\| x - y \| \leq \frac{2}{c^2} \| Jx - Jy \|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $\frac{1}{c}$ in lemma is called the *p-uniformly convex constant* of E.

Lemma 2.3. (*Beauzamy* [44] and *Zalinescu* [46]). *If E be a p-uniformly convex Banach space and let p be a given real number with $p \geq 2$, then for all x, $y \in E$, $j_x \in J_p$ (x) and $j_y \in J_p(y)$*

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2^{p-2}p} \| x - y \|^p,$$

where J_p is the generalized duality mapping of E and $\frac{1}{c}$ is the p-uniformly convexity constant of E.

Lemma 2.4. (*Kamimura and Takahashi* [19]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\varphi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.5. (*Alber* [16]). *Let C be a nonempty closed convex subset of a smooth Banach space and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.6. (*Alber* [[16], Lemma 2.4]). *Let E be a reflexive, strictly convex and smooth Banach space, and let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Let E be a reflexive, strictly convex, smooth Banach space and J is the duality mapping from E into E^*. Then J^{-1} is also single value, one-to-one, surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [16]:

$$V(x, x^*) = \| x \|^2 - 2\langle x, x^* \rangle + \| x^* \|^2, \tag{2.1}$$

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \varphi(x, J^{-1}(x^*))$.

Lemma 2.7. (*Alber* [16]). *Let E be a reflexive, strictly convex smooth Banach space and let V be as in (2.1). Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*),$$

for all $x \in E$ and x^*, $y^* \in E^*$.

A set valued mapping $U : E \rightrightarrows E^*$ with graph $G(U) = (x, x^*) : x^* \in Ux\}$, domain $D(U) = \{x \in E : Ux \neq \varnothing\}$, and range $R(U) = \cup\{Ux : x \in D(U)\}$. U is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*) \in G(U)$, $(y, y^*) \in G(U)$. We denote a set valued operator U from E to E^* by $U \subset E \times E^*$. A monotone U is said to be *maximal* if its graph is not property contained in the graph of any other monotone operator. If U is maximal monotone, then the solution set $U^{-1}0$ is closed and convex. Let E be a reflexive, strictly

convex and smooth Banach space, and it is known that U is a maximal monotone if and only if $R(J + rU) = E^*$ for all $r > 0$. Define the *resolvent* of U by $J_r x = x_r$. In other words, $Jr = (J + rU)^{-1}$ for all $r > 0$. J_r is a single-valued mapping from E to $D(U)$. Also, $U^{-1}(0) = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of all fixed points of Jr. Define, for $r > 0$, the *Yosida approximation* of U by $T_r x = (Jx - JJ_r x)/r$ for all $x \in C$: We know that $T_r x \in U (J_r x)$ for all $r > 0$ and $x \in E$.

Let A be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, and the mapping F of $[0, 1]$ into E^*, defined by $F(t) = A(tx + (1 - t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ *the normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \ \forall y \in C\}. \tag{2.2}$$

Lemma 2.8. (*Rockafellar* [47]). *Let C be a nonempty, closed convex subset of a Banach space E, and A is a monotone, hemicontinuous operator of C into E^*. Let $U \subset E \times E^*$ be an operator defined as follows:*

$$Uv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset & otherwise. \end{cases} \tag{2.3}$$

Then U is maximal monotone and $U^{-1}0 = VI(A, C)$.

Lemma 2.9. (*Chang et al.* [30]). *Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive number with $\sum_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \rightarrow [0, \infty)$ with $g(0) = 0$ such that, for any positive integer i, j with $i < j$,*

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \| x_n \|^2 - \lambda_i \lambda_j g(\| x_i - x_j \|). \tag{2.4}$$

Lemma 2.10. (*Chang et al.* [30]). *Let E be a real uniformly smooth and strictly convex Banach space, and C be a nonempty closed convex subset of E. Let $T : C \rightarrow C$ be a closed and quasi-φ-asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \rightarrow 1$. Then $F(T)$ is a closed convex subset of C:*

For solving the equilibrium problem for a bifunction $f : C \times C \rightarrow \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y);$$

(A4) for each $x \in C$, $y \alpha f(x, y)$ is convex and lower semicontinuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$f(x, y) = \langle Ax, y - x \rangle, \quad \forall x, y \in C.$$

Then, f satisfies (A1)-(A4). The following result is in Blum and Oettli [1].

Motivated by Combettes and Hirstoaga [2] in a Hilbert space and Takahashi and Zembayashi [33] in a Banach space, Zhang [48] obtained the following lemma.

Lemma 2.11. (*Zhang* [[48], *Lemma 1.5*]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that f be a bifunction from C × C to* $\mathbb{R}$ *satisfying (A1)-(A4), A : C → E* be a continuous and monotone mapping and* $\phi : C \to \mathbb{R}$ *be a semicontinuous and convex functional. For r >0 and let x ∈ E. Then, there exists z ∈ C such that*

$$Q(z, \gamma) + \frac{1}{r} \langle \gamma - z, Jz - Jx \rangle \geq 0, \quad \forall \gamma \in C,$$

where Q(z, y) = f(z, y) + ⟨Bz, y - z⟩ + ϕ(y) ϕ(z), x, y ∈ C. Furthermore, define a mapping $T_r : E \to C$ *as follows:*

$$T_r x = \left\{ z \in C : Q(z, \gamma) + \frac{1}{r} \langle \gamma - z, Jz - Jx \rangle \geq 0, \ \forall \gamma \in C \right\}.$$

Then the following hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., for all x, y ∈ E, ⟨T_rx - T_ry, JT_rx - JT_ry⟩ ≤ ⟨T_rx - T_ry, Jx -Jy⟩;*
3. *$F(T_r) = \widetilde{F(T_r)} = GMEP(f, B, \varphi)$;*
4. *GMEP(f, B, ϕ) is closed and convex;*
5. *$\varphi(p, T_r z) + \varphi(T_r z, z) \leq \varphi(p, z), \forall p \in F(T_r)$ and z ∈ E.*

3 Main results

In this section, we prove the new convergence theorems for finding the set of solutions of system of generalized mixed equilibrium problems, the common fixed point set of a family of closed and uniformly quasi-φ-asymptotically nonexpansive mappings, and the solution set of variational inequalities for an α-inverse strongly monotone mapping in a 2-uniformly convex and uniformly smooth Banach space.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. For each j = 1, 2, ..., m let f_j be a bifunction from C × C to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $B_j : C \to E^*$ be a continuous and monotone mapping and $\phi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into E* satisfying ||Ay|| ≤ ||Ay - Au||, ∀y ∈ C and u ∈ VI(A, C) ≠ ∅. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed uniformly L_i-Lipschitz continuous and uniformly quasi-φ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty), k_n \to 1$ such that $F := (\cap_{i=1}^{\infty} F(S_i)) \cap (\cap_{j=1}^{m} GMEP(f_j, B_j, \varphi_j))(\cap VI(A, C))$ is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n v_n), \\ \gamma_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n) Jz_n), \\ u_n = T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} \gamma_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\theta_n = \sup_{q \in F}(k_n -1)\varphi(q, x_n)$, for each i ≥ 0, $\{\alpha_{ni}\}$ and $\{\beta_n\}$ are sequences in [0, 1], $\{r_{j, n}\} \subset [d, \infty)$ for some d >0 and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$,

where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$, $\liminf_n \to \infty (1 - \beta_n) > 0$ and $\liminf_{n \to \infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Proof. We first show that C_{n+1} is closed and convex for each $n \geq 0$. Clearly, $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we know $\varphi(z, u_n) \leq \varphi(z, x_n) + \theta n$ is equivalent to $2\langle z, Jx_n - Ju_n \rangle \leq ||x_n||^2 - ||u_n||^2 + \theta_n$. So, C_{n+1} is closed and convex.

Next, we show that $F \subseteq C_n$ for all $n \geq 0$. Since $u_n = \Omega_n^m \gamma_n$, when $\Omega_n^j = T_{r_{j,n}}^{Q_j} T_{r_{j-1,n}}^{Q_{j-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1}$, $j = 1, 2, 3, ..., m$, $\Omega_n^0 = I$, by the convexity of $||\cdot||^2$, property of φ, Lemma 2.9 and by uniformly quasi-φ-asymptotically nonexpansive of S_n for each $q \in F \subseteq C_n$, we have

$$
\begin{aligned}
\phi(q, u_n) &= \phi(q, \Omega_n^m \gamma_n) \\
&\leq \phi(q, \gamma_n) \\
&= \phi(q, J^{-1}(\beta_n Jx_n + (1 - \beta_n)Jz_n)) \\
&= ||q||^2 - 2\langle q, \beta_n Jx_n + (1 - \beta_n)Jz_n \rangle + ||\beta_n Jx_n + (1 - \beta_n)Jz_n||^2 \\
&\leq ||q||^2 - 2\beta_n\langle q, Jx_n \rangle - 2(1 - \beta_n)\langle q, Jz_n \rangle + \beta_n ||x_n||^2 + (1 - \beta_n) ||z_n||^2 \\
&= \beta_n\phi(q, x_n) + (1 - \beta_n)\phi(q, z_n)
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
\phi(q, z_n) &= \phi(q, J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}JS_i^n v_n)) \\
&= ||q||^2 - 2\langle q, \alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}JS_i^n v_n \rangle + ||\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}JS_i^n v_n||^2 \\
&= ||q||^2 - 2\alpha_{n,0}\langle q, Jx_n \rangle - 2\sum_{i=1}^{\infty} \alpha_{n,i}\langle q, JS_i^n v_n \rangle + ||\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}JS_i^n v_n||^2 \\
&\leq ||q||^2 - 2\alpha_{n,0}\langle q, Jx_n \rangle - 2\sum_{i=1}^{\infty} \alpha_{n,i}\langle q, JS_i^n v_n \rangle + \alpha_{n,0} ||Jx_n||^2 + \sum_{i=1}^{\infty} \alpha_{n,i} ||JS_i^n v_n||^2 \\
&\quad - \alpha_{n,0}\alpha_{n,j}g ||Jv_n - JS_j^n v_n|| \\
&= ||q||^2 - 2\alpha_{n,0}\langle q, Jx_n \rangle + \alpha_{n,0} ||Jx_n||^2 - 2\sum_{i=1}^{\infty} \alpha_{n,i}\langle q, JS_i^n v_n \rangle \\
&\quad + \sum_{i=1}^{\infty} \alpha_{n,i} ||JS_i^n v_n||^2 - \alpha_{n,0}\alpha_{n,j}g ||Jv_n - JS_j^n v_n|| \\
&= \alpha_{n,0}\phi(q, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}\phi(q, S_i^n v_n) - \alpha_{n,0}\alpha_{n,j}g ||Jv_n - JS_j^n v_n|| \\
&\leq \alpha_{n,0}\phi(q, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}k_n\phi(q, v_n) - \alpha_{n,0}\alpha_{n,j}g ||Jv_n - JS_j^n v_n||.
\end{aligned}
\tag{3.3}
$$

It follows from Lemma 2.7 that

$$
\begin{aligned}
\phi(q, v_n) &= \phi(q, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&\leq \phi(q, J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&= V(q, Jx_n - \lambda_n Ax_n) \\
&\leq V(q, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - q, \lambda_n Ax_n \rangle \\
&= V(q, Jx_n) - 2\lambda_n\langle J^{-1}(Jx_n - \lambda_n Ax_n) - q, Ax_n \rangle \\
&= \phi(q, x_n) - 2\lambda_n\langle x_n - q, Ax_n \rangle + 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \rangle.
\end{aligned}
\tag{3.4}
$$

Since $q \in VI(A, C)$ and A is an α-inverse-strongly monotone mapping, we have

$$
\begin{aligned}
-2\lambda_n\langle x_n - q, Ax_n \rangle &= -2\lambda_n\langle x_n - q, Ax_n - Aq \rangle - 2\lambda_n\langle x_n - q, Aq \rangle \\
&\leq -2\lambda_n\langle x_n - q, Ax_n - Aq \rangle \\
&\leq -2\alpha\lambda_n ||Ax_n - Aq||^2.
\end{aligned}
\tag{3.5}
$$

From Lemma 2.2 and $||Ax_n|| \leq ||Ax_n - Aq||$, $\forall q \in VI(A, C)$, we also have

$$
\begin{aligned}
2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \rangle &= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n), -\lambda_n Ax_n \rangle \\
&\leq 2 ||J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n)|| ||\lambda_n Ax_n|| \\
&\leq \frac{4}{c^2} ||JJ^{-1}(Jx_n - \lambda_n Ax_n) - JJ^{-1}(Jx_n)|| ||\lambda_n Ax_n|| \\
&= \frac{4}{c^2} ||Jx_n - \lambda_n Ax_n - Jx_n|| ||\lambda_n Ax_n|| \\
&= \frac{4}{c^2} ||\lambda_n Ax_n||^2 \\
&= \frac{4}{c^2}\lambda_n^2 ||Ax_n||^2 \\
&\leq \frac{4}{c^2}\lambda_n^2 ||Ax_n - Aq||^2.
\end{aligned}
\tag{3.6}
$$

Substituting (3.5) and (3.6) into (3.4), we obtain

$$
\begin{aligned}
\phi(q, v_n) &\leq \phi(q, x_n) - 2\alpha\lambda_n \parallel Ax_n - Aq \parallel^2 + \frac{4}{c^2}\lambda_n^2 \parallel Ax_n - Aq \parallel^2 \\
&= \phi(q, x_n) + 2\lambda_n(\frac{2}{c^2}\lambda_n - \alpha) \parallel Ax_n - Aq \parallel^2 \\
&\leq \phi(q, x_n).
\end{aligned}
$$

Substituting (3.7) into (3.3), we also have

$$
\begin{aligned}
\phi(q, z_n) &\leq \alpha_{n,0}\phi(q, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}k_n\phi(q, x_n) - \alpha_{n,0}\alpha_{n,j}g \parallel Jv_n - JS_j^n v_n \parallel \\
&\leq \alpha_{n,0}k_n\phi(q, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}k_n\phi(q, x_n) - \alpha_{n,0}\alpha_{n,j}g \parallel Jv_n - JS_j^n v_n \parallel \\
&= k_n\phi(q, x_n) - \alpha_{n,0}\alpha_{n,j}g \parallel Jv_n - JS_j^n v_n \parallel \\
&\leq \phi(q, x_n) + \sup_{q \in F}(k_n - 1)\phi(q, x_n) - \alpha_{n,0}\alpha_{n,j}g \parallel Jv_n - JS_j^n v_n \parallel \\
&= \phi(q, x_n) + \theta_n - \alpha_{n,0}\alpha_{n,j}g \parallel Jv_n - JS_j^n v_n \parallel \\
&\leq \phi(q, x_n) + \theta_n.
\end{aligned}
\tag{3.8}
$$

and substituting (3.8) into (3.2), we also have

$$
\phi(q, u_n) \leq \phi(q, x_n) + \theta_n.
\tag{3.9}
$$

This shows that $q \in C_{n+1}$ implies that $F \subset C_{n+1}$ and hence, $F \subset C_n$ for all $n \geq 0$. This implies that the sequence $\{x_n\}$ is well defined. From definition of C_{n+1} that $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0, \in C_{n+1} \subset C_n$, we have

$$
\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0.
\tag{3.10}
$$

By Lemma 2.6, we get

$$
\begin{aligned}
\phi(x_n, x_0) &= \phi(\Pi_{C_n}x_0, x_0) \\
&\leq \phi(q, x_0) - \phi(q, x_n) \\
&\leq \phi(q, x_0), \quad \forall q \in F.
\end{aligned}
\tag{3.11}
$$

From (3.10) and (3.11), then $\{\varphi(x_n, x_0)\}$ are nondecreasing and bounded. So, we obtain that $\lim_{n \to \infty} \phi(x_n, x_0)$ exists. In particular, by (1.9), the sequence $\{(\parallel x_n \parallel - \parallel x_0 \parallel)^2$ is bounded. This implies $\{x_n\}$ is also bounded. Denote

$$
M = \sup_{n \geq 0}\{\parallel x_n \parallel\} < \infty.
\tag{3.12}
$$

Moreover, by the definition of θ_n and (3.12), it follows that

$$
\theta_n \to 0 \quad \text{as } n \to \infty.
\tag{3.13}
$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Since $x_m = \Pi_{C_m}x_0 \in C_m \subset C_n$, for $m > n$, by Lemma 2.6, we have

$$
\begin{aligned}
\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n}x_0) \\
&\leq \phi(x_m, x_0) - \phi(\Pi_{C_n}x_0, x_0) \\
&= \phi(x_m, x_0) - \phi(x_n, x_0).
\end{aligned}
$$

Since $\lim_{n \to \infty} \varphi(x_n, x_0)$ exists and we take $m, n \to \infty$, we get $\varphi(x_m, x_n) \to 0$. From Lemma 2.4, we have $\lim_{n \to \infty} \parallel x_m - x_n \parallel = 0$. Thus, $\{x_n\}$ is a Cauchy sequence, and by the completeness of E, there exists a point $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Now, we claim that $\parallel Ju_n - Jx_n \parallel \to 0$, as $n \to \infty$. By definition of $x_n = \Pi_{C_n}x_0$, we have

$$
\begin{aligned}
\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n}x_0) \\
&\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n}x_0, x_0) \\
&= \phi(x_{n+1}, x_0) - \phi(x_n, x_0).
\end{aligned}
$$

Since $\lim_{n\to\infty} \varphi(x_n, x_0)$ exists, we also have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.14}$$

Again from Lemma 2.4 that

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \tag{3.15}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \| Jx_{n+1} - Jx_n \| = 0. \tag{3.16}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and the definition of C_{n+1}, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \theta_n.$$

By (3.13) and (3.14) that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{3.17}$$

Again applying Lemma 2.4, we have

$$\lim_{n\to\infty} \| x_{n+1} - u_n \| = 0. \tag{3.18}$$

Since

$$\begin{aligned} \| u_n - x_n \| &= \| u_n - x_{n+1} + x_{n+1} - x_n \| \\ &\leq \| u_n - x_{n+1} \| + \| x_{n+1} - x_n \| . \end{aligned}$$

It follows from (3.15) and (3.18) that

$$\lim_{n\to\infty} \| u_n - x_n \| = 0. \tag{3.19}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we also have

$$\lim_{n\to\infty} \| Ju_n - Jx_n \| = 0. \tag{3.20}$$

Next, we will show that $p \in F := \cap_{j=1}^m \mathrm{GMEP}(f_j, B_j, \varphi_j) \cap (\cap_{i=1}^\infty F(S_i)) \cap \mathrm{VI}(A, C)$.

(a) We show that $p \in \cap_{i=1}^\infty F(S_i)$. Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, it follow from (3.8), we have

$$\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n) + \theta_n,$$

by (3.13) and (3.14), we get

$$\lim_{n\to\infty} \phi(x_{n+1}, z_n) = 0 \tag{3.21}$$

again from Lemma 2.4 that

$$\lim_{n\to\infty} \| x_{n+1} - z_n \| = 0. \tag{3.22}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \| Jx_{n+1} - Jz_n \| = 0. \tag{3.23}$$

From (3.50), we note that

$$
\begin{aligned}
\| Jx_{n+1} - Jz_n \| &= \| Jx_{n+1} - (\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}JS_i^n v_n) \| \\
&= \| \alpha_{n,0}Jx_{n+1} - \alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}Jx_{n+1} - \sum_{i=1}^{\infty} \alpha_{n,i}JS_i^n v_n \| \\
&= \| \alpha_{n,0}(Jx_{n+1} - Jx_n) + \sum_{i=1}^{\infty} \alpha_{n,i}(Jx_{n+1} - JS_i^n v_n) \| \\
&= \| \sum_{i=1}^{\infty} \alpha_{n,i}(Jx_{n+1} - JS_i^n v_n) - \alpha_{n,0}(Jx_n - Jx_{n+1}) \| \\
&\geq \sum_{i=1}^{\infty} \alpha_{n,i} \| Jx_{n+1} - JS_i^n v_n \| - \alpha_{n,0} \| Jx_n - Jx_{n+1} \|,
\end{aligned}
$$

and hence

$$
\| Jx_{n+1} - JS_i^n v_n \| \leq \frac{1}{\sum_{i=1}^{\infty} \alpha_{n,i}} (\| Jx_{n+1} - Jz_n \| + \alpha_{n,0} \| Jx_n - Jx_{n+1} \|). \tag{3.24}
$$

From (3.16), (3.23) and $\lim \inf_{n\to\infty} \sum_{i=1}^{\infty} \alpha_{n,i} > 0$, we obtain that

$$
\lim_{n\to\infty} ||Jx_{n+1} - JS_i^n v_n|| = 0. \tag{3.25}
$$

Since J^1 is uniformly norm-to-norm continuous on bounded sets, we have

$$
\lim_{n\to\infty} \| x_{n+1} - S_i^n v_n \| = 0. \tag{3.26}
$$

Using the triangle inequality that

$$
\begin{aligned}
\| x_n - S_i^n v_n \| &= \| x_n - x_{n+1} + x_{n+1} - S_i^n v_n \| \\
&\leq \| x_n - x_{n+1} \| + \| x_{n+1} - S_i^n v_n \|.
\end{aligned}
$$

From (3.15) and (3.26), we have

$$
\lim_{n\to\infty} \| x_n - S_i^n v_n \| = 0. \tag{3.27}
$$

On the other hand, we note that

$$
\begin{aligned}
\phi(q, x_n) - \phi(q, u_n) + \theta_n &= \| x_n \|^2 - \| u_n \|^2 - 2\langle q, Jx_n - Ju_n \rangle + \theta_n \\
&\leq \| x_n - u_n \| (\| x_n \| + \| u_n \|) + 2 \| q \| \| Jx_n - Ju_n \| + \theta_n.
\end{aligned}
$$

It follows from $\theta_n \to 0$, $||x_n - u_n|| \to 0$ and $||Jx_n - Ju_n|| \to 0$, that

$$
\phi(q, x_n) - \phi(q, u_n) + \theta_n \to 0 \quad \text{as } n \to \infty. \tag{3.28}
$$

From (3.2), (3.3) and (3.7) that

$$
\begin{aligned}
\phi(q, u_n) &\leq \phi(q, y_n) \\
&\leq \beta_n \phi(q, x_n) + (1 - \beta_n)\phi(q, z_n) \\
&\leq \beta_n \phi(q, x_n) + (1 - \beta_n)[\alpha_{n,0}\phi(q, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}k_n\phi(q, v_n) \\
&\quad - \alpha_{n,0}\alpha_{n,j}g \| Jv_n - JS_j^n v_n \|] \\
&= \beta_n \phi(q, x_n) + (1 - \beta_n)\alpha_{n,0}\phi(q, x_n) + (1 - \beta_n)\sum_{i=1}^{\infty} \alpha_{n,i}k_n\phi(q, v_n) \\
&\quad - (1 - \beta_n)\alpha_{n,0}\alpha_{n,j}g \| Jv_n - JS_j^n v_n \| \\
&\leq \beta_n \phi(q, x_n) + (1 - \beta_n)\alpha_{n,0}\phi(q, x_n) + (1 - \beta_n)\sum_{i=1}^{\infty} \alpha_{n,i}k_n\phi(q, v_n) \\
&\leq \beta_n \phi(q, x_n) + (1 - \beta_n)\alpha_{n,0}\phi(q, x_n) + (1 - \beta_n)\sum_{i=1}^{\infty} \alpha_{n,i}k_n[\phi(q, x_n) - 2\lambda_n(\alpha - \tfrac{2}{c^2}\lambda_n) \| Ax_n - Aq \|^2] \\
&\leq \beta_n \phi(q, x_n) + (1 - \beta_n)\alpha_{n,0}k_n\phi(q, x_n) + (1 - \beta_n)\sum_{i=1}^{\infty} \alpha_{n,i}k_n\phi(q, x_n) \\
&\quad - (1 - \beta_n)\sum_{i=1}^{\infty} \alpha_{n,i}k_n 2\lambda_n(\alpha - \tfrac{2}{c^2}\lambda_n) \| Ax_n - Aq \|^2 \\
&\leq \beta_n k_n\phi(q, x_n) + (1 - \beta_n)k_n\phi(q, x_n) - (1 - \beta_n)\sum_{i=1}^{\infty} \alpha_{n,i}k_n 2\lambda_n(\alpha - \tfrac{2}{c^2}\lambda_n) \| Ax_n - Aq \|^2 \\
&= k_n\phi(q, x_n) - (1 - \beta_n)\sum_{i=1}^{\infty} \alpha_{n,i}k_n 2\lambda_n(\alpha - \tfrac{2}{c^2}\lambda_n) \| Ax_n - Aq \|^2] \\
&\leq \phi(q, x_n) + \sup_{q\in F}(k_n - 1)\phi(q, x_n) - (1 - \beta_n)\sum_{i=1}^{\infty} \alpha_{n,i}k_n 2\lambda_n(\alpha - \tfrac{2}{c^2}\lambda_n) \| Ax_n - Aq \|^2 \\
&= \phi(q, x_n) + \theta_n - (1 - \beta_n)\sum_{i=1}^{\infty} \alpha_{n,i}k_n 2\lambda_n(\alpha - \tfrac{2}{c^2}\lambda_n) \| Ax_n - Aq \|^2,
\end{aligned}
$$

and hence

$$2a(\alpha - \tfrac{2b}{c^2}) \parallel Ax_n - Aq \parallel^2 \le 2\lambda_n(\alpha - \tfrac{2}{c^2}\lambda_n) \parallel Ax_n - Aq \parallel^2$$
$$\le \frac{1}{(1-\beta_n)\sum_{i=1}^{\infty}\alpha_{n,i}k_n}(\phi(q,x_n) - \phi(q,u_n) + \theta_n). \tag{3.29}$$

From (3.28), $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\,\alpha/2$, $\lim\inf_{n\to\infty}(1 - \beta_n)$ >0 and $\lim\inf_{n\to\infty}\alpha_{n,0}\alpha_{n,i} > 0$, for $i \ge 0$ and $k_n \to 1$ as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \parallel Ax_n - Aq \parallel = 0. \tag{3.30}$$

From Lemmas 2.6, 2.7 and (3.6), we compute

$$\begin{aligned}
\phi(x_n, v_n) &= \phi(x_n, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&\le \phi(x_n, J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&= V(x_n, Jx_n - \lambda_n Ax_n) \\
&\le V(x_n, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n \rangle \\
&= \phi(x_n, x_n) + 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \rangle \\
&= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n \rangle \\
&\le \frac{4\lambda_n^2}{c^2} \parallel Ax_n - Aq \parallel^2 \\
&\le \frac{4b^2}{c^2} \parallel Ax_n - Aq \parallel^2.
\end{aligned}$$

Applying Lemma 2.4 and (3.30) that

$$\lim_{n\to\infty} \parallel x_n - v_n \parallel = 0 \tag{3.31}$$

and we also obtain

$$\lim_{n\to\infty} ||Jx_n - Jv_n|| = 0. \tag{3.32}$$

Since S_i^n is continuous, for any $i \ge 1$

$$\lim_{n\to\infty} \parallel S_i^n x_n - S_i^n v_n \parallel = 0. \tag{3.33}$$

Again by the triangle inequality, we get

$$\parallel x_n - S_i^n x_n \parallel \le \parallel x_n - S_i^n v_n \parallel + \parallel S_i^n v_n - S_i^n x_n \parallel.$$

From (3.27) and (3.33), we have

$$\lim_{n\to\infty} \parallel x_n - S_i^n x_n \parallel = 0, \quad \forall i \ge 1. \tag{3.34}$$

By using triangle inequality, we get

$$\parallel S_i^n x_n - p \parallel \le \parallel S_i^n x_n - x_n \parallel + \parallel x_n - p \parallel, \quad \forall i \ge 1.$$

We know that $x_n \to p$ as $n \to \infty$ and from (3.34)

$$S_i^n x_n \to p \quad \text{for each } i \ge 1.$$

Moreover, by the assumption that $\forall i \ge 1$, S_i is uniformly L_i-Lipschitz continuous, and hence we have.

$$\begin{aligned}
\parallel S_i^{n+1}x_n - S_i^n x_n \parallel &\le \parallel S_i^{n+1}x_n - S_i^{n+1}x_{n+1} \parallel + \parallel S_i^{n+1}x_{n+1} - x_{n+1} \parallel + \parallel x_{n+1} - x_n \parallel + \parallel x_n - S_i^n x_n \parallel \\
&\le (L_i + 1) \parallel x_{n+1} - x_n \parallel + \parallel S_i^{n+1}x_{n+1} - x_{n+1} \parallel + \parallel x_n - S_i^n x_n \parallel.
\end{aligned} \tag{3.35}$$

By (3.15) and (3.34), it yields that $\parallel S_i^{n+1}x_n - S_i^n x_n \parallel \to 0$. From $S_i^n x_n \to p$, we have $S_i^{n+1}x_n \to p$, that is $S_i S_i^n x_n \to p$. In view of closeness of S_i, we have $S_i p = p$, for all $i \ge 1$. This implies that $p \in \cap_{i=1}^{\infty} F(S_i)$.

(b) We show that $p \in \cap_{j=1}^{m} \mathrm{GMEP}(f_j, B_j, \varphi_j)$.

Let $u_n = \Omega_n^m y_n$, when $\Omega_n^j = T_{r_{j,n}}^{Q_j} T_{r_{j-1,n}}^{Q_{j-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1}$, $j = 1,2,3, ..., m$ and $\Omega_n^0 = I$, we obtain

$$
\begin{aligned}
\phi(q, u_n) &= \phi(q, \Omega_n^m y_n) \\
&\leq \phi(q, \Omega_n^{m-1} y_n) \\
&\leq \phi(q, \Omega_n^{m-2} y_n) \\
&\vdots \\
&\leq \phi(q, \Omega_n^j y_n).
\end{aligned}
\tag{3.36}
$$

By Lemma (2.11)(5), we have for $j = 1, 2, 3, ..., m$

$$
\begin{aligned}
\phi(\Omega_n^j y_n, y_n) + \theta_n &\leq \phi(q, y_n) - \phi(q, \Omega_n^j y_n) + \theta_n \\
&\leq \phi(q, x_n) - \phi(q, \Omega_n^j y_n) + \theta_n \\
&\leq \phi(q, x_n) - \phi(q, u_n) + \theta_n.
\end{aligned}
\tag{3.37}
$$

From (3.13) and (3.28), we get $\phi(\Omega_n^j y_n, y_n) \to 0$ as $n \to \infty$, for $j = 1, 2, 3, ..., m$ and Lemma 2.4 implies that

$$
\lim_{n \to \infty} \| \Omega_n^j y_n - y_n \| = 0, \forall j = 1, 2, 3, \ldots, m.
\tag{3.38}
$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, it follows from (3.2) and (3.8) that

$$
\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + \theta_n.
$$

By (3.13) and (3.14), we have

$$
\lim_{n \to \infty} \phi(x_{n+1}, y_n) = 0.
$$

Applying Lemma 2.4 that

$$
\lim_{n \to \infty} \| x_{n+1} - y_n \| = 0.
\tag{3.39}
$$

Using the triangle inequality, we obtain

$$
\| x_n - y_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - y_n \|.
$$

From (3.15) and (3.39), we get

$$
\lim_{n \to \infty} \| x_n - y_n \| = 0.
$$

Since $x_n \to p$ and $\|x_n - y_n\| \to 0$, we have $y_n \to p$ *as* $n \to \infty$.
Again by using the triangle inequality, we have for $j = 1, 2, 3, ..., m$

$$
\| p - \Omega_n^j y_n \| \leq \| p - y_n \| + \| y_n - \Omega_n^j y_n \|.
$$

From (3.38) and $y_n \to p$ *as* $n \to \infty$, we get

$$
\lim_{n \to \infty} \| p - \Omega_n^j y_n \| = 0, \forall j = 1, 2, 3, \ldots, m.
\tag{3.41}
$$

By using the triangle inequality, we obtain

$$
\| \Omega_n^j y_n - \Omega_n^{j-1} y_n \| \leq \| \Omega_n^j y_n - p \| + \| p - \Omega_n^{j-1} y_n \|.
$$

From (3.41), we have

$$
\lim_{n \to \infty} \| \Omega_n^j y_n - \Omega_n^{j-1} y_n \| = 0, \forall j = 1, 2, 3, \ldots, m.
\tag{3.42}
$$

Since $\{r_{j,\,n}\} \subset [d, \infty)$ and J is uniformly continuous on any bounded subset of E,

$$\lim_{n \to \infty} \frac{\| J\Omega_n^j \gamma_n - J\Omega_n^{j-1} \gamma_n \|}{r_{j,n}} = 0, \ \forall j = 1, 2, 3, \ldots, m. \tag{3.43}$$

From Lemma 2.11, we get for $j = 1, 2, 3, \ldots, m$

$$Q_j(\Omega_n^j \gamma_n, \gamma) + \frac{1}{r_{j,n}} \langle \gamma - \Omega_n^j \gamma_n, J\Omega_n^j \gamma_n - J\Omega_n^{j-1} \gamma_n \rangle \geq 0, \quad \forall \gamma \in C.$$

From (A2),

$$\frac{1}{r_{j,n}} \langle \gamma - \Omega_n^j \gamma_n, J\Omega_n^j \gamma_n - J\Omega_n^{j-1} \gamma_n \rangle \geq Q_j(\gamma, \Omega_n^j \gamma_n), \quad \forall \gamma \in C, \ \forall j = 1, 2, 3, \ldots, m.$$

From (3.41) and (3.43), we have

$$0 \geq Q_j(\gamma, p), \quad \forall \gamma \in C, \ \forall j = 1, 2, 3, \ldots, m. \tag{3.44}$$

For t with $0 < t \leq 1$ and $y \in C$; let $y_t = ty + (1 - t)p$. Then, we get that $y_t \in C$. From (3.44), and it follows that

$$Q_j(\gamma_t, p) \leq 0, \quad \forall \gamma \in C, \ \forall j = 1, 2, 3, \ldots, m. \tag{3.45}$$

By the conditions (A1) and (A4), we have for $j = 1, 2, 3, \ldots, m$

$$\begin{aligned} 0 &= Q_j(\gamma_t, \gamma_t) \\ &\leq tQ_j(\gamma_t, \gamma) + (1 - t)Q_j(\gamma_t, p) \\ &\leq tQ_j(\gamma_t, \gamma) \\ &= Q_j(\gamma_t, \gamma). \end{aligned} \tag{3.46}$$

From (A3) and letting $t \to 0$, This implies that $p \in \mathrm{GMEP}(f_j, B_j, \phi_j)$, $\forall j = 1, 2, 3, \ldots, m$. Therefore $p \in \cap_{j=1}^m \mathrm{GMEP}(f_j, B_j, \varphi_j)$

(c) We show that $p \in \mathrm{VI}(A, C)$. Indeed, define $U \subset E \times E^*$ by

$$Uv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{3.47}$$

By Lemma 2.8, U is maximal monotone and $U^{-1}0 = \mathrm{VI}(A, C)$. Let $(v, w) \in G(U)$. Since $w \in Uv = Av + N_C(v)$, we get $w - Av \in N_C(v)$.

From $v_n \in C$, we have

$$\langle v - v_n, w - Av \rangle \geq 0. \tag{3.48}$$

On the other hand, since $v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$. Then, by Lemma 2.5, we have

$$\langle v - v_n, Jv_n - (Jx_n - \lambda_n Ax_n) \rangle \geq 0,$$

and thus

$$\left\langle v - v_n, \frac{Jx_n - Jv_n}{\lambda_n} - Ax_n \right\rangle \leq 0. \tag{3.49}$$

It follows from (3.48), (3.49) and A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous that

$$
\begin{aligned}
\langle v - v_n, w \rangle &\geq \langle v - v_n, Av \rangle \\
&\geq \langle v - v_n, Av \rangle + \left\langle v - v_n, \frac{Jx_n - Jv_n}{\lambda_n} - Ax_n \right\rangle \\
&= \langle v - v_n, Av - Ax_n \rangle + \left\langle v - zv_n, \frac{Jx_n - Jv_n}{\lambda_n} \right\rangle \\
&= \langle v - v_n, Av - Av_n \rangle + \langle v - v_n, Av_n - Ax_n \rangle + \left\langle v - v_n, \frac{Jx_n - Jv_n}{\lambda_n} \right\rangle \\
&\geq - \| v - v_n \| \frac{\| v_n - x_n \|}{\alpha} - \| v - v_n \| \frac{\| Jx_n - Jv_n \|}{a} \\
&\geq -H \left(\frac{\| v_n - x_n \|}{\alpha} + \frac{\| Jx_n - Jv_n \|}{a} \right),
\end{aligned}
$$

where $H = \sup_{n \geq 1} \|v - v_n\|$. Take the limit as $n \ i\to \infty$, (3.31) and (3.32), we obtain $\langle v - p, w \rangle \geq 0$. By the maximality of B we have $p \in B^{-1}0$, that is $p \in \mathrm{VI}(A, C)$. Hence, from (a), (b) and (c), we obtain $p \in F$.

Finally, we show that $p = \Pi_F x_0$. From $x_n = \Pi_{C_n} x_0$, we have $\langle Jx_0 - Jx_n, x_n - z \rangle \geq 0$, $\forall z \in C_n$. Since $F \subset C_n$, we also have

$$
\langle Jx_0 - Jx_n, x_n - \gamma \rangle \geq 0, \quad \forall \gamma \in F.
$$

Taking limit $n \to \infty$, we obtain

$$
\langle Jx_0 - Jp, p - \gamma \rangle \geq 0, \quad \forall \gamma \in F.
$$

By Lemma 2.5, we can conclude that $p = \Pi_F x_0$ and $x_n \to p$ as $n \to \infty$. This completes the proof. □

If $S_i = S$ for each $i \in \mathbb{N}$, then Theorem 3.1 is reduced to the following corollary.

Corollary 3.2. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. For each $j = 1, 2, ..., m$ let f_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $B_j : C \to E^*$ be a continuous and monotone mapping and $\phi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, $\forall y \in C$ and $u \in \mathrm{VI}(A, C) \neq \varnothing$. Let $S : C \to C$ be a closed L-Lipschitz continuous and quasi-φ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $F := F(S) \cap (\cap_{j=1}^m \mathrm{GMEP}(f_j, B_j, \varphi_j)) \cap \mathrm{VI}(A, C)$ is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$
\begin{cases}
v_n = \Pi_C J^{-1} (Jx_n - \lambda_n Ax_n), \\
z_n = J^{-1} (\alpha_n Jx_n + (1 - \alpha_n) JS^n v_n), \\
y_n = J^{-1} (\beta_n Jx_n + (1 - \beta_n) Jz_n), \\
u_n = T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{cases}
\tag{3.50}
$$

where $\theta_n = \sup_{q \in F} (k_n - 1) \varphi(q, x_n)$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$, $\{r_{j, n}\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}c$ is the 2-uniformly convexity constant of E. If $\lim \inf_{n \to \infty} (1 - \beta_n) > 0$ and $\lim \inf_{n \to \infty} (1 - \alpha_n) > 0$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

For a special case that $i = 1, 2$, we can obtain the following results on a pair of quasi-φ-asymptotically nonexpansive mappings immediately from Theorem 3.1.

Corollary 3.3. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. For each $j = 1, 2, ..., m$ let f_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $B_j : C \to E^*$ be a continuous and monotone mapping and $\phi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let A be an α-inversestrongly monotone mapping of C into E^* satisfying $||Ay|| \leq ||Ay - Au||, \forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $S, T : C \to C$ be two closed quasi-φ-asymptotically nonexpansive mappings and L_S, L_T -Lipschitz continuous, respectively with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $F := F(S) \cap F(T) \cap (\cap_{j=1}^m GMEP(f_j, B_j, \varphi_j)) \cap VI(A, C)$ is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} v_n = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \\ z_n = J^{-1}(\alpha_n J x_n + \beta_n J S^n v_n + \gamma_n J T^n v_n), \\ y_n = J^{-1}(\delta_n J x_n + (1 - \delta_n) J z_n), \\ u_n = T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{3.51}$$

where $\theta_n = \sup_{q \in F} (k_n - 1)\varphi(q, x_n)$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$, $\{r_{j, n}\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2 \alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 0$ and $\lim \inf_{n \to \infty} \alpha_n \beta_n > 0$, $\lim \inf_{n \to \infty} \alpha_n \gamma_n > 0$, $\lim \inf_{n \to \infty} \beta_n \gamma_n > 0$ and $\lim \inf_{n \to \infty} (1 - \delta_n) > 0$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Corollary 3.4. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. For each $j = 1, 2, ..., m$ let f_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $B_j : C \to E^*$ be a continuous and monotone mapping and $\phi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $||Ay|| \leq ||Ay - Au||, \forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $\{S_i\}_{i=1}^\infty : C \to C$ be an infinite family of closed quasi-φ-nonexpansive mappings such that $F := \cap_{i=1}^\infty F(S_i) \cap (\cap_{j=1}^m GMEP(f_j, B_j, \varphi_j)) \cap VI(A, C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} v_n = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \\ z_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^\infty \alpha_{n,i} J S_i v_n), \\ y_n = J^{-1}(\beta_n J x_n + (1 - \beta_n) J z_n), \\ u_n = T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n), \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{3.52}$$

where $\{\alpha_{n, i}\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, $\{r_{j, n}\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2 \alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\sum_{i=0}^\infty \alpha_{n,i} = 1$ for all $n \geq 0$, $\lim \inf_{n \to \infty} (1 - \beta_n) > 0$ and $\lim \inf_{n \to \infty} \alpha_{n, 0} \alpha_{n, i} > 0$ for all $i \geq 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Proof. Since $\{S_i\}_{i=1}^\infty : C \to C$ is an infinite family of closed quasi-φ-nonexpansive mappings, it is an infinite family of closed and uniformly quasi-φ-asymptotically

nonexpansive mappings with sequence $k_n = 1$. Hence, the conditions appearing in Theorem 3.1 F is a bounded subset in C and for each $i \geq 1$, S_i is uniformly L_i-Lipschitz continuous are of no use here. By virtue of the closeness of mapping S_i for each $i \geq 1$, it yields that $p \in F(S_i)$ for each $i \geq 1$, that is, $p \in \cap_{i=1}^{\infty} F(S_i)$. Therefore, all conditions in Theorem 3.1 are satisfied. The conclusion of Corollary 3.4 is obtained from Theorem 3.1 immediately. □

4 Deduced theorems

Corollary 4.1. [[41], Theorem 3.2] *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and $\phi : C \to \mathbb{R}$ is convex and lower semicontinuous. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $||Ay|| \leq ||Ay - Au||$, $\forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $\{S_i\}_{i=1}^{N} : C \to C$ be a finite family of closed quasi-φ-nonexpansive mappings such that $F := \cap_{i=1}^{N} F(S_i) \cap \text{GMEP}(f, B, \varphi) \cap VI(A, C) \neq \emptyset$. For an initial point $x_0 \in E$ with $\frac{1}{c}$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$
\begin{cases}
z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\
y_n = J^{-1}(\alpha_0 Jx_n + \sum_{i=1}^{N} \alpha_i JS_i z_n), \\
f(u_n, y) + \langle Bu_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, (4.1) \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n), \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0,
\end{cases}
$$

where $\{\alpha_i\}$ is sequence in $[0, 1]$, $\{r_n\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\alpha_i \in (0, 1)$ such that $\sum_{i=0}^{N} \alpha_i = 1$ then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Remark 4.2. Theorems 3.1, Corollaries 3.4 and 4.1 improve and extend the corresponding results of Wattanawitoon and Kumam [14] and Zegeye [41] in the following senses:

- from a solution of the classical equilibrium problem to the generalized mixed equilibrium problem with an infinite family of quasi-φ-asymptotically mappings;
- for the mappings, we extend the mappings from nonexpansive mappings, relatively quasi-nonexpansive mappings or quasi-φ-nonexpansive mappings and a finite family of closed relatively quasi-nonexpansive mappings to an infinite family of quasi-φ-asymptotically nonex-pansive mappings.

Corollary 4.3. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and $\phi : C \to \mathbb{R}$ is convex and lower semicontinuous. Let B be a continuous monotone mapping of C into E^*. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed and uniformly quasi-φ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ and uniformly L_i-Lipschitz continuous such that $F := \cap_{i=1}^{\infty} F(S_i) \cap \text{GMEP}(f, B, \varphi)$ is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} y_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n), \\ f(u_n, y) + \langle Bu_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 0, \end{cases} \quad (4.2)$$

where $\theta_n = \sup_{q \in F}(k_n - 1)\varphi(q, x_n)$, $\{\alpha_{n,i}\}$ *is sequence in* [0, 1], $\{r_n\} \subset [a, \infty)$ *for some a* >0. *If* $\sum_{i=0}^{\infty}\alpha_{n,i} = 1$*for all* $n \geq 0$ *and* $\liminf_{n \to \infty}\alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$, *then* $\{x_n\}$ *converges strongly to* $p \in F$, *where* $p = \Pi_F x_0$.

Proof. Put $A \equiv 0$ in Theorem 3.1 Then, we get that $z_n = x_n$. Thus, the method of proof of Theorem 3.1 gives the required assertion without the requirement that E be 2-uniformly convex. □

If setting $B \equiv 0$ and $\phi \equiv 0$ in Corollary 4.3, then we have the following corollary.

Corollary 4.4. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from* C × C *to* ℝ *satisfying (A1)-(A4) and* $\phi : C \to$ ℝ *is convex and lower semicontinuous. Let* $\{S_i\}_{i=1}^{\infty} : C \to C$*be an infinite family of closed and uniformly quasi-φ-asymptotically nonexpansive mappings with a sequence* $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ *and uniformly* L_i-*Lipschitz continuous such that* $F := \cap_{i=1}^{\infty}F(S_i) \cap EP(f)$ *is a nonempty and bounded subset in C. For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1}x_0$*and* $C_1 = C$, *we define the sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} y_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n x_n), \\ f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 0, \end{cases} \quad (4.3)$$

where $\theta_n = \sup_{q \in F}(k_n - 1)\varphi(q, x_n)$, $\{\alpha_{n,i}\}$ *is sequence in* [0, 1], $\{r_n\} \subset [a, \infty)$ *for some a* >0. *If* $\sum_{i=0}^{\infty}\alpha_{n,i} = 1$*for all* $n \geq 0$ *and* $\liminf_{n \to \infty}\alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$, *then* $\{x_n\}$ *converges strongly to* $p \in F$, *where* $p = \Pi_F x_0$.

Remark 4.5. Corollaries 4.3 and 4.4 improve and extend the corresponding results of Zegeye [41] and Wattanawitoon and Kumam [14] in the sense from a finite family of closed relatively quasi-nonexpansive mappings and closed relatively quasi-nonexpansive mappings to more general than an infinite family of closed and uniformly quasi-φ-asymptotically nonexpansive mappings.

Remark 4.6. Moreover, Our theorems improve, generalize, unify and extend Qin et al. [9], Zeg-eye et al. [15], Zegeye [41] and Wattanawitoon and Kumam [14,49] and several results recently announced.

5 Applications
5.1 Application to complementarity problems
Let K be a nonempty, closed convex cone in E. We define the *polar* K^* of K as follows:

$$K^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0, \forall x \in K\}. \quad (5.1)$$

If $A : K \to E^*$ is an operator, then an element $u \in K$ is called a solution of the *complementarity problem* [20] if

$$Au \in K^* \quad \text{and} \quad \langle u, Au \rangle = 0. \quad (5.2)$$

The set of solutions of the complementarity problem is denoted by $CP(A, K)$.

Theorem 5.1. *Let K be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. For each j = 1, 2, ..., m let f_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $B_j : C \to E^*$ be a continuous and monotone mapping and $\phi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of K into E^* satisfying $||Ay|| \leq ||Ay - Au||$, $\forall y \in K$ and $u \in CP(A, K) \neq \varnothing$. Let $\{S_i\}_{i=1}^{\infty} : K \to K$ be an infinite family of closed uniformly L_i-Lipschitz continuous and uniformly quasi-φ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $F := \cap_{i=1}^{\infty} F(S_i) \cap (\cap_{j=1}^{m} GMEP(f_j, B_j, \varphi_j)) \cap CP(A, K)$ is a nonempty and bounded subset in K. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}x_0$ and $K_1 = K$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} v_n = \Pi_K J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}JS_i^n v_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)Jz_n), \\ u_n = T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\ K_{n+1} = \{z \in K_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{K_{n+1}}x_0, \quad \forall n \geq 0, \end{cases} \quad (5.3)$$

where J is the duality mapping on E, $\theta_n = \sup_{q \in F} (k_n - 1)\varphi(q, x_n)$, for each $i \geq 0$, $\{\alpha_{n,i}\}$ and $\{\beta_n\}$ are sequences in [0, 1], $\{r_{j,n}\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$, $\lim \inf_{n \to \infty}(1 - \beta_n) > 0$ and $\lim \inf_{n \to \infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Proof. As in the proof of Takahashi in [[20], Lemma 7.11], we get that $VI(A, K) = CP(A, K)$. So, we obtain the result. □

5.2 Application to zero points

Next, we consider the problem of finding a zero point of an inverse-strongly monotone operator of E into E^*. Assume that A satisfies the conditions:

(C1) A is α-inverse-strongly monotone,

(C2) $A^{-1}0 = \{u \in E : Au = 0\} \neq \varnothing$.

Theorem 5.2. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. For each j = 1, 2, ..., m let f_j be a bifunction from $C \times C$ to R which satisfies conditions (A1)-(A4), $B_j : C \to E^*$ be a continuous and monotone mapping and $\phi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let A be an operator of E into E^* satisfying (C1) and (C2). Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed uniformly L_i- Lipschitz continuous and uniformly quasi-φ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that*

$$F := \cap_{i=1}^{\infty} F(S_i) \cap (\cap_{j=1}^{m} GMEP(f_j, B_j, \varphi_j)) \cap A^{-1}0$$

is a nonempty and bounded subset in C: For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:

$$\begin{cases} v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}JS_i^n v_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)Jz_n), \\ u_n = T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 0, \end{cases} \quad (5.4)$$

where J is the duality mapping on E, $\theta_n = \sup_{q \in F} (kn - 1)\varphi(q, x_n)$, for each $i \geq 0$, $\{\alpha_{n, i}\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, $\{r_{j, n}\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$, $\liminf_{n \to \infty}(1 - \beta_n) > 0$ and $\liminf_{n \to \infty} \alpha_{n, 0}\alpha_{n, i} > 0$ for all $i \geq 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Proof. Setting $C = E$ in Corollary 3.4, we also get $\Pi_E = I$. We also have $VI(A, C) = VI(A, E)\ \{x \in E : Ax = 0\} \neq \varnothing$ and then the condition $||Ay|| \leq ||Ay - Au||$ holds for all $y \in E$ and $u \in A^{-1}0$. So, we obtain the result. □

5.3 Application to Hilbert spaces

If $E = H$, a Hilbert space, then E is 2-uniformly convex (we can choose $c = 1$) and uniformly smooth real Banach space and closed relatively quasi-nonexpansive map reduces to closed quasi-nonexpansive map. Moreover, $J = I$, identity operator on H and $\Pi_C = P_C$, projection mapping from H into C: Thus, the following corollaries hold.

Theorem 5.3. *Let C be a nonempty closed and convex subset of a Hilbert space H. For each $j = 1, 2, ..., m$ let f_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $B_j : C \to E^*$ be a continuous and monotone mapping and $\phi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H satisfying $||Ay|| \leq ||Ay - Au||$, $\forall y \in C$ and $u \in VI(A, C) \neq \varnothing$. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ and uniformly L_i-Lipschitz continuous such that $F := \cap_{i=1}^{\infty} F(S_i) \cap (\cap_{j=1}^{m} GMEP(f_j, B_j, \varphi_j)) \cap VI(A, C)$ is a nonempty and bounded subset in C. For an initial point $x_0 \in H$ with $x_1 = P_{C_1}x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = P_C(x_n - \lambda_n Ax_n), \\ y_n = \alpha_{n,0}x_n + \sum_{i=1}^{\infty} \alpha_{n,i}S_i^n z_n, \\ u_n = T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\ C_{n+1} = \{z \in C_n : \| z - u_n \| \leq \| z - x_n \| + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \geq 0, \end{cases} \tag{5.5}$$

where $\theta_n = \sup_{q \in F} (k_n - 1)|||q - x_n||$, $\{\alpha_{n, i}\}$ is sequence in $[0, 1]$, $\{r_{j, n}\} \subset [a, \infty)$ for some $a > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a; b with $0 < a < b < \alpha/2$. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$ and $\liminf_{n \to \infty} \alpha_{n,0}\alpha_{n, i} > 0$ for all $i \geq 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Remark 5.4. Theorem 5.3 improves and extends the Corollary 3.7 in Zegeye [41] in the aspect for the mappings, and we extend the mappings from a finite family of closed relatively quasi-nonexpansive mappings to a more general infinite family of closed and uniformly quasi-φ-asymptotically nonexpansive mappings.

6 Competing interests

The authors declare that they have no competing interests.

7 Authors' contributions

All authors contribute equally and significantly in this research work. All authors read and approved the final manuscript.

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