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Common fixed points of *R*-weakly commuting maps in generalized metric spaces

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Abstract

In this paper, using the setting of a generalized metric space, a unique common fixed point of four *R*-weakly commuting maps satisfying a generalized contractive condition is obtained. We also present example in support of our result.

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1 Introduction and preliminaries

The study of unique common fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity. Mustafa and Sims [1] generalized the concept of a metric, in which the real number is assigned to every triplet of an arbitrary set. Based on the notion of generalized metric spaces, Mustafa et al. [2-6] obtained some fixed point theorems for mappings satisfying different contractive conditions. Study of common fixed point theorems in generalized metric spaces was initiated by Abbas and Rhoades [7]. Abbas et al. [8] obtained some periodic point results in generalized metric spaces. While, Chugh et al. [9] obtained some fixed point results for maps satisfying property p in G-metric spaces. Saadati et al. [10] studied some fixed point results for contractive mappings in partially ordered G-metric spaces. Recently, Shatanawi [11] obtained fixed points of Φ-maps in G-metric spaces. Abbas et al. [12] gave some new results of coupled common fixed point results in two generalized metric spaces (see also [13]).

The aim of this paper is to initiate the study of unique common fixed point of four R-weakly commuting maps satisfying a generalized contractive condition in G-metric spaces.

Consistent with Mustafa and Sims [2], the following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to R^+$ satisfies:

$\mathbf{G}_1 : G(x, y, z) = 0$ if $x = y = z$;

$\mathbf{G}_2 : 0 < G(x, y, z)$ for all $x, y, z \in X$, with $x \neq y$;

$\mathbf{G}_3 : G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$;

$\mathbf{G}_4 : G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables); and

$\mathbf{G}_5 : G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a G-metric on X and (X, G) is called a G-metric space.

Definition 1.2. A sequence $\{x_n\}$ in a G-metric space X is:

(i) a *G-Cauchy* sequence if, for any $\varepsilon > 0$, there is an $n_0 \in N$ (the set of natural numbers) such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$,
(ii) a *G-convergent* sequence if, for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in N$, such that for all $n, m \geq n_0$, $G(x, x_n, x_m) < \varepsilon$.

A G-metric space on X is said to be G-complete if every G-Cauchy sequence in X is G-convergent in X. It is known that $\rightarrow 0$ as $n, m \rightarrow \infty$.

Proposition 1.3. Let X be a G-metric space. Then the following are equivalent:

(1) $\{x_n\}$ is G-convergent to x.
(2) $G(x_n, x_m, x) \rightarrow 0$ as $n, m \rightarrow \infty$.
(3) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.
(4) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 1.4. A G-metric on X is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 1.5. Every G-metric on X will define a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X. \tag{1.1}$$

For a symmetric G-metric,

$$d_G(x, y) = 2G(x, y, y), \quad \forall x, y \in X. \tag{1.2}$$

However, if G is non-symmetric, then the following inequality holds:

$$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \quad \forall x, y \in X. \tag{1.3}$$

It is also obvious that

$$G(x, x, y) \leq 2G(x, y, y).$$

Now, we give an example of a non-symmetric G-metric.

Example 1.6. Let $X = \{1, 2\}$ and a mapping $G : X \times X \times X \rightarrow R^+$ be defined as

(x, y, z)	$G(x, y, z)$
(1, 1, 1), (2, 2, 2)	0
(1, 1, 2), (1, 2, 1), (2, 1, 1)	0.5
(1, 2, 2), (2, 1, 2), (2, 2, 1)	1.

Note that G satisfies all the axioms of a generalized metric but $G(x, x, y) \neq G(x, y, y)$ for distinct x, y in X. Therefore, G is a non-symmetric G-metric on X.

In 1999, Pant [14] introduced the concept of weakly commuting maps in metric spaces. We shall study R-weakly commuting and compatible mappings in the frame work of G-metric spaces.

Definition 1.7. Let X be a G-metric space and f and g be two self-mappings of X. Then f and g are called R-weakly commuting if there exists a positive real number R such that $G(fgx, fgx, gfx) \leq RG(fx, fx, gx)$ holds for each $x \in X$.

Two maps f and g are said to be compatible if, whenever $\{x_n\}$ in X such that $\{fx_n\}$ and $\{gx_n\}$ are G-convergent to some $t \in X$, then $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n) = 0$.

Example 1.8. Let $X = [0, 2]$ with complete G-metric defined by

$$G(x, y, z) = \max\{|x-y|, |x-z|, |y-z|\}.$$

Let $f, g, S, T : X \to X$ defined by

$$fx = 1, x \geq 0,$$

$$gx = \left\{ \begin{array}{l} 1, \; x \in [0,1], \\ \frac{2-x}{2}, \; x \in (1,2], \end{array} \right| ,$$

$$Sx = \left\{ \begin{array}{l} 2-x, \; x \in [0,1], \\ x, \; x \in (1,2], \end{array} \right| ,$$

and

$$Tx = \left\{ \begin{array}{l} \frac{3-x}{2}, \; x \in [0,1], \\ \frac{x}{2}, \; x \in (1,2], \end{array} \right| .$$

Then note that the pairs $\{f, S\}$ and $\{g, T\}$ are R-weakly commuting as they commute at their coincidence points. The pair $\{f, S\}$ is continuous compatible while the pair $\{g, T\}$ is non-compatible. To see that g and T are non-compatible, consider a decreasing sequence $\{x_n\}$ in X such that $x_n \to 1$. Then $gx_n \to \frac{1}{2}$, $Tx_n \to \frac{1}{2}$. $gTx_n = \frac{4-x_n}{4} \to \frac{3}{4}$ and $Tgx_n = \frac{2-x_n}{4} \to \frac{1}{4}$. □

2 Common fixed point theorems

In this section, we obtain some unique common fixed point results for four mappings satisfying certain generalized contractive conditions in the framework of a generalized metric space. We start with the following result.

Theorem 2.1. Let X be a complete G-metric space. Suppose that $\{f, S\}$ and $\{g, T\}$ be pointwise R-weakly commuting pairs of self-mappings on X satisfying

$$G(fx, fx, gy) \leq h \max\{G(Sx, Sx, Ty), G(fx, fx, Sx), G(gy, gy, Ty), [G(fx, fx, Ty) + G(gy, gy, Sx)]/2\} \tag{2.1}$$

and

$$G(fx, gy, gy) \leq h \max\{G(Sx, Ty, Ty), G(fx, Sx, Sx), G(gy, Ty, Ty), [G(fx, Ty, Ty) + G(gy, Sx, Sx)]/2\} \tag{2.2}$$

for all $x, y \in X$, where $h \in [0, 1)$. Suppose that $fX \subseteq TX$, $gX \subseteq SX$, and one of the pair $\{f, S\}$ or $\{g, T\}$ is compatible. If the mappings in the compatible pair are continuous, then f, g, S and T have a unique common fixed point.

Proof. Suppose that f and g satisfy the conditions (2.1) and (2.2). If G is symmetric, then by adding these, we have

$$\begin{aligned} & d_G(fx, gy) \\ & \leq \frac{h}{2} \max\{d_G(Sx, Ty), d_G(fx, Sx), d_G(gy, Ty), [d_G(fx, Ty) + d_G(gy, Sx)]/2\} \\ & \quad + \frac{h}{2} \max\{d_G(Sx, Ty), d_G(fx, Sx), d_G(gy, Ty), [d_G(fx, Ty) + d_G(gy, Sx)]/2\} \\ & = h \max\{d_G(Sx, Ty), d_G(fx, Sx), d_G(gy, Ty), [d_G(fx, Ty) + d_G(gy, Sx)]/2\}, \end{aligned}$$

for all $x, y \in X$ with $0 \leq h < 1$, the existence and uniqueness of a common fixed point follows from [14]. However, if X is non-symmetric G-metric space, then by the definition of metric d_G on X and (1.3), we obtain

$$\begin{aligned}
&d_G(fx, gy) \\
&= G(fx, fx, gy) + G(fx, gy, gy) \\
&\leq \frac{2h}{3}\max\{d_G(Sx, Ty), d_G(fx, Sx), d_G(gy, Ty), [d_G(fx, Ty) + d_G(gy, Sx)]/2\} \\
&\quad + \frac{2h}{3}\max\{d_G(Sx, Ty), d_G(fx, Sx), d_G(gy, Ty), [d_G(fx, Ty) + d_G(gy, Sx)]/2\} \\
&= \frac{4h}{3}\max\{d_G(Sx, Ty), d_G(fx, Sx), d_G(gy, Ty), [d_G(fx, Ty) + d_G(gy, S_X)]/2\},
\end{aligned}$$

for all $x, y \in X$. Here, the contractivity factor $\frac{4h}{3}$ needs not be less than 1. Therefore, metric d_G gives no information. In this case, let x_0 be an arbitrary point in X. Choose x_1 and x_2 in X such that $gx_0 = Sx_1$ and $fx_1 = Tx_2$. This can be done, since the ranges of S and T contain those of g and f, respectively. Again choose x_3 and x_4 in X such that $gx_2 = Sx3$ and $fx_3 = Tx_4$. Continuing this process, having chosen x_n in X such that $gx_{2n} = Sx_{2n+1}$ and $fx_{2n+1} = Tx_{2n+2}$, $n = 0, 1, 2, \ldots$. Let

$$y_{2n} = Sx_{2n+1} = gx_{2n} \text{ and } y_{2n+1} = Tx_{2n+2} = fx_{2n+1} \text{ for all } n = 0, 1, 2, \ldots .$$

For a given $n \in \mathbf{N}$, if n is even, so $n = 2k$ for some $k \in \mathbf{N}$. Then from (2.1)

$$\begin{aligned}
&G(y_{n+1}, y_{n+1}, y_n) \\
&= G(y_{2k+1}, y_{2k+1}, y_{2k}) \\
&= G(fx_{2k+1}, fx_{2k+1}, gx_{2k}) \\
&\leq h\max\{G(Sx_{2k+1}, Sx_{2k+1}, Tx_{2k}), G(fx_{2k+1}, fx_{2k+1}, Sx_{2k+1}), \\
&\quad G(gx_{2k}, gx_{2k}, Tx_{2k}), [G(fx_{2k+1}, fx_{2k+1}, Tx_{2k}) + G(gx_{2k}, gx_{2k}, Sx_{2k+1})]/2\} \\
&= h\max\{G(y_{2k}, y_{2k}, y_{2k-1}), G(y_{2k+1}, y_{2k+1}, y_{2k}), \\
&G(y_{2k}, y_{2k}, y_{2k-1}), [G(y_{2k+1}, y_{2k+1}, y_{2k-1}) + G(y_{2k}, y_{2k}, y_{2k})]/2\} \\
&\leq h\max\{G(y_{2k}, y_{2k}, y_{2k-1}), G(y_{2k+1}, y_{2k+1}, y_{2k}), \\
&\quad [G(y_{2k+1}, y_{2k+1}, y_{2k}) + G(y_{2k}, y_{2k}, y_{2k-1})]/2\} \\
&= h\max\{G(y_n, y_n, y_{n-1}), G(y_{n+1}, y_{n+1}, y_n)\}.
\end{aligned}$$

This implies that

$$G(y_{n+1}, y_{n+1}, y_n) \leq hG(y_n, y_n, y_{n-1}).$$

If n is odd, then $n = 2k + 1$ for some $k \in \mathbf{N}$. In this case (2.1) gives

$$\begin{aligned}
&G(y_{n+1}, y_{n+1}, y_n) \\
&= G(y_{2k+2}, y_{2k+2}, y_{2k+1}) \\
&= G(fx_{2k+2}, fx_{2k+2} + gx_{2k+1}) \\
&\leq h\max\{G(Sx_{2k+2}, Sx_{2k+2}, Tx_{2k+1}), G(fx_{2k+2}, fx_{2k+2}, Sx_{2k+2}), \\
&\quad G(gx_{2k+1}, gx_{2k+1}, Tx_{2k+1}), [G(fx_{2k+2}, fx_{2k+2}, Tx_{2k+1}) + G(gx_{2k+1}, gx_{2k+1}, Sx_{2k+2})]/2\} \\
&= h\max\{G(y_{2k+1}, y_{2k+1}, y_{2k}), G(y_{2k+2}, y_{2k+2}, y_{2k+1}), \\
&\quad G(y_{2k+1}, y_{2k+1}, y_{2k}), [G(y_{2k+2}, y_{2k+2}, y_{2k}) + G(y_{2k+1}, y_{2k+1}, y_{2k+1})]/2\} \\
&\leq h\max\{G(y_{2k+1}, y_{2k+1}, y_{2k}), G(y_{2k+2}, y_{2k+2}, y_{2k+1}), \\
&\quad [G(y_{2k+2}, y_{2k+2}, y_{2k+1}) + G(y_{2k+1}, y_{2k+1}, y_{2k})]/2\} \\
&= h\max\{G(y_{2k+1}, y_{2k+1}, y_{2k}), G(y_{2k+2}, y_{2k+2}, y_{2k+1})\} \\
&= h\max\{G(y_n, y_n, y_{n-1}), G(y_{n+1}, y_{n+1}, y_n)\},
\end{aligned}$$

that is,

$$G(y_{n+1}, y_{n+1}, y_n) \leq hG(y_n, y_n, y_{n-1}).$$

Continuing the above process, we have

$$G(y_{n+1}, y_{n+1}, y_n) \leq h^n G(y_1, y_1, y_0).$$

Thus, if $y_0 = y_1$, we get $G(y_n, y_{n+1}, y_{n+1}) = 0$ for each $n \in \mathbf{N}$. Hence, $y_n = y_{n+1}$ for each $n \in \mathbf{N}$. Therefore, $\{y_n\}$ is G-Cauchy. So we may assume that $y_0 \neq y_1$.

Let $n, m \in \mathbf{N}$ with $m > n$,

$$\begin{aligned} G(y_n, y_m, y_m) & \\ &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \cdots + G(y_{m-1}, y_m, y_m) \\ &\leq h^n G(y_0, y_1, y_1) + h^{n+1} G(y_0, y_1, y_1) + \cdots + h^{m-1} G(y_0, y_1, y_1) \\ &= h^n G(y_0, y_1, y_1) \sum_{i=0}^{m-n-1} h^i \\ &\leq \frac{h^n}{1-h} G(y_0, y_1, y_1), \end{aligned}$$

and so $G(y_n, y_m, y_m) \to 0$ as $m, n \to \infty$. Hence $\{y_n\}$ is a Cauchy sequence in X. Since X is G-complete, there exists a point $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

Consequently

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} gx_{2n} = z$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Tx_{2n+2} = \lim_{n\to\infty} fx_{2n+1} = z.$$

Let f and S be continuous compatible mappings. Compatibility of f and S implies that $\lim_{n\to\infty} G(fSx_{2n+1}, fSx_{2n+1}, Sfx_{2n+1}) = 0$, that is $G(fz, fz, Sz) = 0$ which implies that $fz = Sz$. Since $fX \subset TX$, there exists some $u \in X$ such that $fz = Tu$. Now from (2.1), we have

$$\begin{aligned} G(fz, fz, gu) \leq h \max\{&G(Sz, Sz, Tu), G(fz, fz, Sz), G(gu, gu, Tu), \\ &[G(fz, fz, Tu) + G(gu, gu, Sz)]/2\} \\ = h \max\{&G(fz, fz, fz), G(fz, fz, fz), G(gu, gu, fz), \\ &[G(fz, fz, fz) + G(gu, gu, fz)]/2\} \\ = hG(&fz, gu, gu). \end{aligned} \tag{2.3}$$

Also, from (2.2)

$$\begin{aligned} G(fz, gu, gu) \leq h \max\{&G(Sz, Tu, Tu), G(fz, Sz, Sz), G(gu, Tu, Tu), \\ &[G(fz, Tu, Tu) + G(gu, Sz, Sz)]/2\} \\ = h \max\{&G(fz, fz, fz), G(fz, fz, fz), G(gu, fz, fz), \\ &[G(fz, fz, fz) + G(gu, fz, fz)]/2\} \\ = hG(&fz, fz, gu). \end{aligned} \tag{2.4}$$

Combining above two inequalities, we get

$$G(fz, fz, gu) \leq h^2 G(fz, fz, gu).$$

Since $h < 1$, so that $fz = gu$. Hence, $fz = Sz = gu = Tu$. As the pair $\{g, T\}$ is R-weakly commuting, there exists $R > 0$ such that

$$G(gTu, gTu, Tgu) \leq RG(gu, gu, Tu) = 0,$$

that is, $gTu = Tgu$. Moreover, $ggu = gTu = Tgu = TTu$. Similarly, the pair $\{f, S\}$ is R-weakly commuting, there exists some $R > 0$ such that

$$G(fSz, fSz, Sfz) \leq RG(fz, fz, Sz) = 0,$$

so that $fSz = Sfz$ and $ffz = fSz = Sfz = SSz$.

Now by (2.1)

$$\begin{aligned} G(ffz, ffz, fz) &= G(ffz, ffz, gu) \\ &\leq h\max\{G(Sfz, Sfz, Tu), G(ffz, ffz, Sfz), G(gu, gu, Tu), \\ &\quad [G(ffz, ffz, Tu) + G(gu, gu, Sfz)]/2\} \\ &= h\max\{G(ffz, ffz, gu), G(ffz, ffz, ffz), G(gu, gu, gu), \\ &\quad [G(ffz, ffz, gu) + G(gu, gu, ffz)]/2\} \\ &= h\max\{G(ffz, ffz, fz), [G(ffz, ffz, fz) + G(fz, fz, ffz)]/2\} \\ &= \frac{h}{2}[G(ffz, ffz, fz) + G(fz, fz, ffz)], \end{aligned}$$

so that

$$G(ffz, ffz, fz) \leq hG(fz, fz, ffz). \tag{2.5}$$

Again from (2.2), we have

$$\begin{aligned} G(ffz, fz, fz) &= G(ffz, gu, gu) \\ &\leq h\max\{G(Sfz, Tu, Tu), G(ffz, Sfz, Sfz), G(gu, Tu, Tu), \\ &\quad [G(ffZ, Tu, Tu) + G(gu, Sfz, Sfz)]/2\} \\ &= h\max\{G(Sfz, gu, gu), G(ffz, ffz, ffz), G(gu, gu, gu), \\ &\quad [G(ffz, gu, gu) + G(gu, ffz, ffz)]/2\} \\ &= h\max\{G(ffz, fz, fz), [G(ffz, fz, fz) + G(fz, ffz, ffz)]/2\} \\ &= \frac{h}{2}[G(ffz, fz, fz) + G(ffz, ffz, fz)], \end{aligned}$$

which implies

$$G(ffz, fz, fz) \leq hG(ffz, ffz, fz). \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$G(ffz, ffz, fz) \leq h^2 G(ffz, ffz, fz),$$

and since $h^2 < 1$ so that $ffz = fz$. Hence, $ffz = Sfz = fz$, and fz is the common fixed point of f and S. Since $gu = fz$, following arguments similar to those given above we conclude that fz is a common fixed point of g and T as well. Now we show the uniqueness of fixed point. For this, assume that there exists another point w in X which is the common fixed point of f, g, S and T. From (2.1), we obtain

$$\begin{aligned} G(fz, fz, w) &= G(ffz, ffz, gw) \\ &\le h\max\{G(Sfz, Sfz, Tw), G(ffz, ffz, Sfz), G(gw, gw, Tw), \\ &\quad [G(ffz, ffz, Tw) + G(gw, gw, Sfz)]/2\} \\ &= h\max\{G(fz, fz, w), G(fz, fz, fz), G(w, w, w), \\ &\quad [G(fz, fz, w) + G(w, w, fz)]/2\} \\ &= \frac{h}{2}[G(fz, fz, w) + G(w, w, fz)], \end{aligned}$$

which implies that

$$G(fz, fz, w) \le hG(w, w, fz). \tag{2.7}$$

From (2.2), we get

$$\begin{aligned} G(fz, w, w) &= G(ffz, gw, gw) \\ &\le h\max\{G(Sfz, Tw, Tw), G(ffz, Sfz, Sfz), G(gw, Tw, Tw), \\ &\quad [G(ffz, Tw, Tw) + G(gw, Sfz, Sfz)]/2\} \\ &= h\max\{G(fz, w, w), G(fz, fz, fz), G(w, w, w), \\ &\quad [G(fz, w, w) + G(w, fz, fz)]/2\} \\ &= \frac{h}{2}[G(fz, w, w) + G(w, fz, fz)], \end{aligned}$$

which implies

$$G(fz, w, w) \le hG(fz, fz, w). \tag{2.8}$$

Now (2.7) and (2.8) give

$$G(fz, fz, w) \le h^2 G(fz, fz, w),$$

and $fz = w$. This completes the proof.

Example 2.2. Let $X = \{0, 1, 2\}$ with G-metric defined by

(x, y, z)	$G(x, y, z)$
$(0, 0, 0), (1, 1, 1), (2, 2, 2),$	0
$(0, 0, 1), (0, 1, 0), (1, 0, 0),$	
$(0, 0, 2), (0, 2, 0), (2, 0, 0),$	1
$(0, 2, 2), (2, 0, 2), (2, 2, 0),$	
$(0, 1, 1), (1, 0, 1), (1, 1, 0),$	
$(1, 1, 2), (1, 2, 1), (2, 1, 1),$	2
$(1, 2, 2), (2, 1, 2), (2, 2, 1),$	
$(0, 1, 2), (0, 2, 1), (1, 0, 2),$	2
$(1, 2, 0), (2, 0, 1), (2, 1, 0),$	

is a non-symmetric G-metric on X because $G(0, 0, 1) \neq G(0, 1, 1)$.

Let $f, g, S, T : X \to X$ defined by

x	$f(x)$	$g(x)$	$S(x)$	$T(x)$
0	0	0	0	0
1	0	2	2	1
2	0	0	1	1

Then $fX \subseteq TX$ and $gX \subseteq SX$, with the pairs $\{f, S\}$ and $\{g, T\}$ are R-weakly commuting as they commute at their coincidence points.

Now to get (2.1) and (2.2) satisfied, we have the following nine cases: (I) x, $y = 0$, (II) $x = 0$, $y = 2$, (III) $x = 1$, $y = 0$, (IV) $x = 1$, $y = 2$, (V) $x = 2$, $y = 0$, (VI) $x = 2$, $y = 2$. For all these cases, $f(x) = g(y) = 0$ implies $G(fx, fx, gy) = 0$ and (2.1) and (2.2) hold.

(VII) For $x = 0$, $y = 1$, then $fx = 0$, $gy = 2$, $Sx = 0$, $Ty = 1$.

$$\begin{aligned} &G(fx, fx, gy) \\ &\quad = G(0, 0, 2) = 1 \\ &\quad \leq h \max\{1, 0, 2, 1\} \\ &\quad = h \max\{G(0, 0, 1), G(0, 0, 0), G(2, 2, 1), [G(0, 0, 1) + G(2, 2, 0)]/2\} \\ &\quad = h \max\{G(Sx, Sx, Ty), G(fx, fx, Sx), G(gy, gy, Ty), \\ &\qquad [G(fx, fx, Ty) + G(gy, gy, Sx)]/2\}. \end{aligned}$$

Thus, (2.1) is satisfied where $h = \frac{4}{5}$.

Also

$$\begin{aligned} &G(fx, gy, gy) \\ &\quad = G(0, 2, 2) = 1 \\ &\quad \leq h \max\{2, 0, 2, 1.5\} \\ &\quad = h \max\{G(0, 1, 1), G(0, 0, 0), G(2, 1, 1), [G(0, 1, 1) + G(2, 0, 0)]/2\} \\ &\quad = h \max\{G(Sx, Ty, Ty), G(fx, Sx, Sx), G(gy, Ty, Ty), \\ &\qquad [G(fx, Ty, Ty) + G(gy, Sx, Sx)]/2\}. \end{aligned}$$

Thus, (2.2) is satisfied where $h = \frac{4}{5}$.

(VIII) Now when $x = 1$, $y = 1$, then $fx = 0$, $gy = 2$, $Sx = 2$, $Ty = 1$.

$$\begin{aligned} &G(fx, fx, gy) \\ &\quad = G(0, 0, 2) = 1 \\ &\quad \leq h \max\{2, 1, 2, 0.5\} \\ &\quad = h \max\{G(2, 2, 1), G(0, 0, 2), G(2, 2, 1), [G(0, 0, 1) + G(2, 2, 2)]/2\} \\ &\quad = h \max\{G(Sx, Sx, Ty), G(fx, fx, Sx), G(gy, gy, Ty), \\ &\qquad [G(fx, fx, Ty) + G(gy, gy, Sx)]/2\}. \end{aligned}$$

Thus, (2.1) is satisfied where $h = \frac{4}{5}$.

And

$$\begin{aligned} &G(fx, gy, gy) \\ &\quad = G(0, 2, 2) = 1 \\ &\quad \leq h \max\{2, 1, 2, 1\} \\ &\quad = h \max\{G(2, 1, 1), G(0, 2, 2), G(2, 1, 1), [G(0, 1, 1) + G(2, 2, 2)]/2\} \\ &\quad = h \max\{G(Sx, Ty, Ty), G(fx, Sx, Sx), G(gy, Ty, Ty), \\ &\qquad [G(fx, Ty, Ty) + G(gy, Sx, Sx)]/2\}. \end{aligned}$$

Thus, (2.2) is satisfied where $h = \frac{4}{5}$.

(IX) If $x = 2$, $y = 1$, then $fx = 0$, $gy = 2$, $Sx = 1$, $Ty = 1$ and

$$\begin{aligned} &G(fx, fx, gy) \\ &\quad = G(0, 0, 2) = 1 \\ &\quad \leq h \max\{0, 1, 2, 1.5\} \\ &\quad = h \max\{G(1, 1, 1), G(0, 0, 1), G(2, 2, 1), [G(0, 0, 1) + G(2, 2, 1)]/2\} \\ &\quad = h \max\{G(Sx, Sx, Ty), G(fx, fx, Sx), G(gy, gy, Ty), \\ &\qquad [G(fx, fx, Ty) + G(gy, gy, Sx)]/2\}. \end{aligned}$$

Thus, (2.1) is satisfied where $h = \frac{4}{5}$.

Also

$$\begin{aligned} &G(fx,\ gy,\ gy)\\ &\quad = G(0,2,2) = 1\\ &\quad \le h\max\{0,2,2,2\}\\ &\quad = h\max\{G(1,1,1),\ G(0,\ 1,1),\ G(2,1,1),\ [G(0,\ 1,1)+G(2,1,1)]/2\}\\ &\quad = h\max\{G(Sx,\ Ty,\ Ty),\ G(fx,\ Sx,\ Sx),\ G(gy,\ Ty,\ Ty),\\ &\qquad [G(fx,\ Ty,\ Ty)+G(gy,\ Sx,\ Sx)]/2\}. \end{aligned}$$

Thus, (2.2) is satisfied where $h = \frac{4}{5}$.

Hence, for all $x, y \in X$, (2.1) and (2.2) are satisfied for $h = \frac{4}{5} < 1$ so that all the conditions of Theorem 2.1 are satisfied. Moreover, 0 is the unique common fixed point for all of the mappings f, g, S and T.

In *Theorem* 2.1, if we take $f = g$, then we have the following corollary.

Corollary 2.3. Let X be a complete G-metric space. Suppose that $\{f, S\}$ and $\{f, T\}$ be pointwise R-weakly commuting pairs of self-mappings on X satisfying

$$\begin{aligned} G(fx,\ fx,\ fy) \le h\max\{&G(Sx,\ Sx,\ Ty),\ G(fx,\ fx,\ Sx),\ G(fy,\ fy,\ Ty),\\ &[G(fx,\ fx,\ Ty)+G(fy,\ fy,\ Sx)]/2\} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} G(fx,\ fy,\ fy) \le h\max\{&G(Sx, Ty,\ Ty),\ G(fx, Sx,\ Sx),\ G(fy, Ty,\ Ty)\}\\ &[G(fx,\ Ty,\ Ty)+G(fy,\ Sx,\ Sx)]/2\} \end{aligned} \tag{2.10}$$

for all $x, y \in X$, where $h \in [0, 1)$. Suppose that $fX \subseteq SX \cup TX$, and one of the pairs $\{f, S\}$ or $\{f, T\}$ is compatible. If the mappings in the compatible pair are continuous, then f, S and T have a unique common fixed point.

Also, if we take $S = T$ in *Theorem 2.1*, then we get the following.

Corollary 2.4. Let X be a complete G-metric space. Suppose that $\{f, S\}$ and $\{g, S\}$ are pointwise R-weakly commuting pairs of self-maps on X and

$$\begin{aligned} G(fx,\ fx,\ gy) \le h\max\{&G(Sx, Sx,\ Sy),\ G(fx, fx,\ Sx),\ G(gy, gy,\ Sy),\\ &[G(fx,\ fx,\ Sy)+G(gy,\ gy,\ Sx)]/2\} \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} G(fx,\ gy,\ gy) \le h\max\{&G(Sx, Sy,\ Sy),\ G(fx, Sx,\ Sx),\ G(gy, Sy,\ Sy),\\ &[G(fx,\ Sy,\ Sy)+G(gy,\ Sx,\ Sx)]/2\} \end{aligned} \tag{2.12}$$

hold for all $x, y \in X$, where $h \in [0, 1)$. Suppose that $fX \cup gX \subseteq SX$ and one of the pairs $\{f, S\}$ or $\{g, S\}$ is compatible. If the mappings in the compatible pair are continuous, then f, g and S have a unique common fixed point.

Corollary 2.5. Let X be a complete G-metric space. Suppose that f and g are two self-mappings on X satisfying

$$\begin{aligned} G(fx,\ fx,\ gy)\ \le\ h\max\{&G(x,\ x,\ y),\ G(fx,\ fx,\ x),\ G(gy,\ gy,\ y),\\ &[G(fx,\ fx,\ y)+G(gy,\ gy,\ x)]/2\} \end{aligned} \tag{2.13}$$

and

$$G(fx, gy, gy) \le h \max\{G(x, y, y), G(fx, x, x), G(gy, y, y), [G(fx, y, y) + G(gy, x, x)]/2\} \quad (2.14)$$

for all $x, y \in X$, where $h \in [0, 1)$. Suppose that one of f or g is continuous, then f and g have a unique common fixed point.

Proof. Taking S and T as identity maps on X, the result follows from *Theorem 2.1*.

Corollary 2.6. Let X be a complete G-metric space and f be a self-map on X such that

$$G(fx, fx, fy) \le h \max\{G(x, x, y), G(fx, fx, x), G(fy, fy, y), [G(fx, fx, y) + G(fy, fy, x)]/2\} \quad (2.15)$$

and

$$G(fx, fy, fy) \le h \max\{G(x, y, y), G(fx, x, x), G(fy, y, y), [G(fx, y, y) + G(fy, x, x)]/2\} \quad (2.16)$$

hold for all $x, y \in X$, where $h \in [0, 1)$. Then f has a unique fixed point.

Proof. If we take $f = g$, and S and T as identity maps on X, then from f has a unique fixed point by *Theorem 2.1*.

3 Application

Let $\Omega = [0, 1]$ be bounded open set in $\mathbb{R}$, $L^2(\Omega)$, the set of functions on Ω whose square is integrable on Ω. Consider an integral equation

$$p(t, x(t)) = \int_\Omega q(t, s, x(s))ds \quad (3.1)$$

where $p : \Omega \times \mathbb{R} \to \mathbb{R}$ and $q : \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ be two mappings. Define $G : X \times X \times X \to \mathbb{R}_+$ by

$$G(x, y, z) = \sup_{t\in\Omega} |x(t) - y(t)| + \sup_{t\in\Omega} |y(t) - z(t)| + \sup_{t\in\Omega} |z(t) - x(t)|.$$

Then X is a G-complete metric space. We assume the following that is there exists a function $G : \Omega \times \mathbb{R} \to \mathbb{R}^+$:

(i) $p(s, v(t)) \ge \int_\Omega q(t, s, u(s))\, ds \ge G(s, v(t))$ for each $s, t \in \Omega$..
(ii) $p(s, v(t))$ - $G(s, v(t)) \le h\, |p(s, v(t))$ - $v(t)|$.

Then integral equation (3.1) has a solution in $L^2(\Omega)$.

Proof. Define $(fx)(t) = p(t, x(t))$ and $(gx)(t) = \int_\Omega q(t, s, x(s))\, ds$. Now

$$\begin{aligned} G(fx, fx, gy) &= 2 \sup_{t\in\Omega} |(fx)(t) - (gy)(t)| \\ &= 2 \sup_{t\in\Omega} \left| p(t, x(t)) - \int_\Omega q(t, s, y(t))dt \right| \\ &\le 2 \sup_{t\in\Omega} |p(t, x(t)) - G(t, x(t))| \\ &\le 2h \sup_{t\in\Omega} |p(t, x(t)) - x(t)| \\ &= hG(fx, fx, x). \end{aligned}$$

Thus

$$G(fx, fx, gy) \le h \max\{G(x, x, y), G(fx, fx, x), G(gy, gy, y), [G(fx, fx, y) + G(gy, gy, x)]/2\}$$

is satisfied. Similarly (2.14) is satisfied. Now we can apply Corollary 2.5 to obtain the solution of integral equation (3.1) in $L^2(\Omega)$.

Remark 1. Theorems 2.8-2.9 in [3] and Corollaries 2.6-2.8 in [4] are special cases of our results Theorem 2.1 and Corollaries 2.3-2.6.

Remark 2. A G-metric naturally induces a metric d_G given by $d_G(x, y) = G(x, y, y) + G(x, x, y)$. If the G-metric is not symmetric, the inequalities (2.1) and (2.2) do not reduce to any metric inequality with the metric d_G. Hence, our theorems do not reduce to fixed point problems in the corresponding metric space (X, d_G).

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