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Existence of positive solutions for nonlocal second-order boundary value problem with variable parameter in Banach spaces

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Abstract

By obtaining intervals of the parameter λ, this article investigates the existence of a positive solution for a class of nonlinear boundary value problems of second-order differential equations with integral boundary conditions in abstract spaces. The arguments are based upon a specially constructed cone and the fixed point theory in cone for a strict set contraction operator.
MSC: 34B15; 34B16.

Keywords: boundary value problem, positive solution, fixed point theorem, measure of noncompactness

1 Introduction

The existence of positive solutions for second-order boundary value problems has been studied by many authors using various methods (see [1-6]). Recently, the integral boundary value problems have been studied extensively. Zhang et al. [7] investigated the existence and multiplicity of symmetric positive solutions for a class of p-Laplacian fourth-order differential equations with integral boundary conditions. By using Mawhin's continuation theorem, some sufficient conditions for the existence of solution for a class of second-order differential equations with integral boundary conditions at resonance are established in [8]. Feng et al. [9] considered the boundary value problems with one-dimensional (1D) p-Laplacian and impulse effects subject to the integral boundary condition. This study in this article is motivated by Feng and Ge [1], who applied a fixed point theorem [10] in cone to the second-order differential equations.

$$\begin{cases} x''(t) + f(t, x(t)) = \theta, & 0 < t < 1, \\ x(0) = \int_0^1 g(t)x(t)dt, & x(1) = \theta. \end{cases}$$

Let E be a real Banach space with norm $||\cdot||$ and $P \subset E$ be a cone of E. The purpose of this article is to investigate the existence of positive solutions of the following second-order integral boundary value problem:

$$\begin{cases} -x''(t) + q(t)x'(t) = \lambda f(t, x), & 0 < t < 1, \\ x(0) = \int_0^1 g(t)x(t)dt, & x(1) = \theta, \end{cases} \tag{1.1}$$

where $q \in C[0, 1]$, $\lambda > 0$ is a parameter, $f(t, x) \in C([0, 1] \times P, P)$, and $g \in L^1[0, 1]$ is nonnegative, θ is the zero element of E.

The main features of this article are as follows. First, the author discusses the existence results in the case $q \in C[0, 1]$, not $q(t) = 0$ as in [1]. Second, comparing with [1], let us consider the existence results in the case $\lambda > 0$, not $\lambda = 1$ as in [1]. To our knowledge, no article has considered problem (1.1) in abstract spaces.

The organization of this article is as follows. In Section 2, the author provides some necessary background. In particular, the author states some properties of the Green function associated with problem (1.1). In Section 3, the main results will be stated and proved.

Basic facts about ordered Banach space E can be found in [10,11]. In this article, let me just recall a few of them. The cone P in E induces a partial order on E, i.e., $x \le y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \le x \le y$ implies $||x|| \le N||y||$. Without loss of generality, let us suppose that, in the present article, the normal constant $N = 1$.

Now let us consider problem (1.1) in $C[I, E]$, in which $I = [0, 1]$. Evidently, $(C[I, E], ||\cdot||_c)$ is a Banach space with norm $||x||_c = \max_{t \in I} ||x(t)||$ for $x \in C[I, E]$. In the following, $x \in C[I, E]$ is called a solution of (1.1) if it satisfies (1.1). x is a positive solution of (1.1) if, in addition, $x(t) > \theta$ for $t \in (0, 1)$.

In the following, the author denotes Kuratowski's measure of noncompactness by $\alpha(\cdot)$.

Lemma 1.1 [10]*Let K be a cone of Banach space E and $K_{r,R} = \{x \in K, r \le ||x|| \le R\}$, $R > r > 0$. Suppose that $A: K_{r,R} \to K$ is a strict set contraction such that one of the following two conditions is satisfied:*

$$(a)\ ||Ax|| \ge ||x||, \forall x \in K,\ ||x|| = r;\ ||Ax|| \le ||x||, \forall x \in K,\ ||x|| = R.$$

$$(b)\ ||Ax|| \le ||x||, \forall x \in K,\ ||x|| = r;\ ||Ax|| \ge ||x||, \forall x \in K,\ ||x|| = R.$$

Then, A has a fixed point $x \in K_{r,R}$ such that $r \le ||x|| \le R$.

2 Preliminaries

To establish the existence and nonexistence of positive solutions in $C[I, P]$ of (1.1), let us list the following assumptions, which will hold throughout this article:

(H) $m(t) = \int_0^t q(s)ds$, $\int_0^1 e^{m(x)}dx = c \in R$, $\inf_{t \in I}\{m(t)\} = d > -\infty$, and for any $r > 0$, f is uniformly continuous on $I \times P_r$. $f(t, P_r)$ is relatively compact, and there exist $a, b \in L(I, R^+)$, *and* $w \in C(R^+, R^+)$, such that $||f(t, x)|| \le a(t) + b(t)w(||x||)$, *a.e.* $t \in I$, $x \in P$, where $P_r = P \cap T_r$.

In the case of main results of this study, let us make use of the following lemmas.

Lemma 2.1 *Assume that (H) holds, then x is a nonnegative solution of (1.1) if and only if x is a fixed point of the following integral operator:*

$$(Tx)(t) = \lambda \int_0^1 H(t,s)f(s,x(s))ds, \tag{2.1}$$

where

$$\begin{aligned}(Tx)'(t) &= \frac{\lambda}{c}e^{m(t)}\left[-\int_0^t e^{-m(s)}f(s,x(s))\int_0^s e^{m(x)}dxds + \int_t^1 e^{-m(s)}f(s,x(s))\int_s^1 e^{m(x)}dxds\right] \\ &\quad - \lambda\int_0^1 \frac{1}{c(1-\sigma)}e^{m(t)}\int_0^1 g(\tau)G(\tau,s)d\tau f(s,x(s))ds, \\ (Tx)''(t) &= m'(t)(Tx)'(t) - \lambda f(t,x(t)).\end{aligned}$$

Proof. By

$$\begin{aligned}
Tx(t) &= \lambda \int_0^1 H(t,s) f(s,x(s))ds \\
&= \lambda \int_0^1 G(t,s) f(s,x(s))ds + \lambda \int_0^1 \frac{1}{c(1-\sigma)} \int_t^1 e^{m(x)}dx \int_0^1 g(\tau) G(\tau,s) d\tau f(s,x(s))ds \\
&= \frac{\lambda}{c}\left[\int_0^t \frac{f(s,x(s))}{e^{m(s)}} \int_0^s e^{m(x)}dx \int_t^1 e^{m(x)}dxds + \int_t^1 \frac{f(s,x(s))}{e^{m(s)}} \int_0^t e^{m(x)}dx \int_s^1 e^{m(x)}dxds\right] \\
&\quad + \lambda \int_0^1 \frac{1}{c(1-\sigma)} \int_t^1 e^{m(x)}dx \int_0^1 g(\tau) G(\tau,s) d\tau f(s,x(s))ds,
\end{aligned}$$

we get

$$\begin{aligned}
(Tx)'(t) &= \frac{\lambda}{c} e^{m(t)}\left[-\int_0^t e^{-m(s)} f(s,x(s)) \int_0^s e^{m(x)}dxds + \int_t^1 e^{-m(s)} f(s,x(s)) \int_s^1 e^{m(x)}dxds\right] \\
&\quad - \lambda \int_0^1 \frac{1}{c(1-\sigma)} e^{m(t)} \int_0^1 g(\tau) G(\tau,s) d\tau f(s,x(s))ds, \\
(Tx)''(t) &= m'(t)(Tx)'(t) - \lambda f(t,x(t)).
\end{aligned}$$

Therefore,

$$-(Tx)''(t) + m'(t)(Tx)'(t) = \lambda f(t,x(t)), \quad t \in (0,1).$$

Moreover, by $G(0, s) = G(1, s) = 0$, it is easy to verify that $Tx(0) = \int_0^1 g(s)Tx(s)ds$, $Tx(1) = \theta$. The lemma is proved.

For convenience, let us define

$$\begin{aligned}
k &= \sup_{t\in(0,1)} \left\{\frac{1}{c} e^{-m(t)} \int_0^t e^{m(x)}dx \int_t^1 e^{m(x)}dx\right\}, \qquad e(t) = \frac{1}{c}\int_t^1 e^{m(x)}dx, \\
h(t) &= G(t,t) + \frac{1}{1-\sigma}\int_0^1 g(x)G(x,t)dx, \qquad k_0 = k + \frac{k}{1-\sigma}\int_0^1 g(x)dx.
\end{aligned}$$

For the Green's function $G(t, s)$, it is easy to prove that it has the following two properties.

Proposition 2.1 *For $t, s \in I$, we have* $0 \le H(t, s) \le h(s) \le k_0$.

Proposition 2.2 *For $t, w, s \in I$, we have* $H(t, t) \ge e(s)H(w, s)$.

To obtain a positive solution, let us construct a cone K by

$$K = \{x \in Q : x(t) \ge e(t)x(s), t, s \in I\} \tag{2.2}$$

where $Q = \{x \in C[I, E]: x(t) \ge \theta, t \in I\}$.

It is easy to see that K is a cone of $C[I, E]$ and $K_{r,R} = \{x \in K: r \le ||x|| \le R\} \subset K$, $K \subset Q$.

In the following, let $B_l = \{x \in C[I, E]: ||x||_c \le l\}$, $l > 0$.

Lemma 2.2 [10]*Let H be a countable set of strongly measurable function $x: J \to E$ such that there exists a $M \in L[I, R^+]$ such that $||x(t)|| \le M(t)$ a.e. $t \in I$ for all $x \in H$. Then $\alpha(H(t)) \in L[I, R^+]$ and*

$$\alpha\left(\left\{\int_J x(t)dt : x \in H\right\}\right) \le 2\int_J \alpha(H(t))dt.$$

Lemma 2.3 *Suppose that (H) holds. Then $T(K) \subset K$ and $T: K_{r,R} \to K$ is a strict set contraction.*

Proof. Observing $H(t, s) \in C(I \times I)$ and $f \in C(I \times P, P)$, we can get $Tu \in C(I, E)$. For any $u \in K$, we have

$Tu(t) = \lambda \int_0^1 H(t,s)f(s,u(s))ds \geq \lambda e(t) \int_0^1 H(w,s)f(s,u(s))ds = e(t)Tu(w), \quad t,w \in (0,1)$, thus, T: $K \to K$. Therefore, by (H), it is easily seen that $T \in C(K, K)$. On the other hand, let $V = \{u_n\}_{n=1}^{\infty}$, be a bounded sequence, $||u_n||_c \leq r$, let $M_r = \{w(v)\colon 0 \leq v \leq r\}$, be (H), then we have

$$f(t,u_n(t)) \leq a(t) + b(t)M_r, \quad u_n \in V, \quad a.e. t \in I.$$

Then

$$\begin{aligned} \alpha(Tu_n(t) : u_n \in V) &= \alpha\left(\lambda \int_0^1 H(t,s)f(s,u_n(s))ds : u_n \in V\right) \\ &\leq 2\lambda k_0 \int_0^1 \alpha(f(s,u_n(s)) : u_n \in V)ds = 0. \end{aligned}$$

Hence, T: $K_{r,\,R} \to K$ is a strict set contraction. The proof is complete.

3 Main results

Definition 3.1 Let P be a cone of real Banach space E. If $P^* = \{\phi \in E^* \mid \phi(x) \geq 0, x \in P\}$, then P^* is a dual cone of cone P. Write

$$f^{\beta} = \limsup_{\|x \to \beta\|} \max_{t \in I} \frac{\|f(t,x)\|}{\|x\|}, \qquad (\varphi f)_{\beta} = \liminf_{\|x \to \beta\|} \min_{t \in I} \frac{\varphi(f(t,x))}{\|x\|},$$
$$A = \max_{t \in I} \int_0^1 e(s)H(t,s)ds, \qquad B = \max_{t \in I} \int_0^1 H(t,s)ds,$$

where β denotes 0 or ∞, $\phi \in P^*$, and $||\phi|| = 1$.

In this section, let us apply Lemma 1.1 to establish the existence of a positive solution for problem (1.1).

Theorem 3.1 *Assume that (H) holds, P is normal and for any* $x \in P$, $A(\phi f)_{\infty} > Bf^0$. *Then problem (1.1) has at least one positive solution in K provided*

$$\frac{1}{A(\varphi f)_{\infty}} < \lambda < \frac{1}{Bf^0}.$$

Proof. Let T be a cone preserving, strict set contraction that was defined by (2.1).

According to (3.1), there exists $\varepsilon > 0$ such that

$$\frac{1}{A[(\varphi f)_{\infty} - \varepsilon]} < \lambda < \frac{1}{B(f^0 + \varepsilon)}.$$

Considering $f^0 < \infty$, there exists $r_1 > 0$ such that $||f(t, x)|| \leq (f^0 + \varepsilon)||x||$, for $||x|| \leq r_1$, $x \in P$, and $t \in I$.

Therefore, for $t \in I$, $x \in K$, $||x||_c = r_1$, we have

$$\begin{aligned} \|Tx(t)\| &= \lambda \left\| \int_0^1 H(t,s)f(s,x(s))ds \right\| \\ &\leq \lambda(f^0 + \varepsilon) \int_0^1 H(t,s)\|x(s)\|ds \\ &\leq \lambda(f^0 + \varepsilon)\|x\|_c \int_0^1 H(t,s)ds \\ &\leq \lambda(f^0 + \varepsilon)\|x\|_c B \\ &\leq \|x\|_c. \end{aligned}$$

Therefore,

$$\|Tx\|_c \leq \|x\|_c, \quad t \in I, x \in K, \quad \|x\|_c = r_1.$$

Next, turning to $(\phi f)_\infty > 0$, there exists $r_2 > r_1$, such that $\phi(f(t, x(t))) \geq [(\phi f)_\infty - \varepsilon] \, ||x||$, for $||x|| \geq r_2$, $x \in P$, $t \in I$. Then, for $t \in I$, $x \in K$, $||x||_c = r_2$, we have by Proposition 2.2 and (2.8),

$$\begin{aligned}
\|Tx(t)\| &\geq \varphi((Tu)(t)) = \lambda \int_0^1 H(t,s)\varphi(f(s,x(s)))ds \\
&\geq \lambda \int_0^1 H(t,s)((\varphi f)_\infty - \varepsilon)\|x(s)\|ds \\
&\geq \lambda((\varphi f)_\infty - \varepsilon) \int_0^1 H(t,s)e(s)\|x\|_c ds \\
&\geq \lambda((\varphi f)_\infty - \varepsilon)\|x\|_c A \\
&\geq \|x\|_c.
\end{aligned}$$

Therefore,

$$||Tx||_c \;\geq\; ||x||_c, \quad t \in I, x \in K, \quad ||x||_c \;= r_2.$$

Applying (b) of Lemma 1.1 to (3.3) and (3.4) yields that T has a fixed point $x^* \in K_{r_1,r_2}, r_1 \leq \|x^*\|_c \leq r_2$ and $x^*(t) \leq e(t)x^*(s) > \theta$, $t \in I$, $s \in I$.

The proof is complete.

Similar to the proof of Theorem 3.1, we can prove the following results.

Theorem 3.2 *Assume that (H) holds, P is normal and for any* $x \in P$, $A(\phi f)_0 > Bf^\infty$. *Then problem (1.1) has at least one positive solution in K provided*

$$\frac{1}{A(\varphi f)_0} < \lambda < \frac{1}{Bf^\infty}.$$

Proof. Considering $(\phi f)_0 > 0$, there exists $r_3 > 0$ such that $\phi(f(t, x)) \geq [(\phi f)_0 - \varepsilon]||x||$, for $||x|| \leq r_3$, $x \in P$, $t \in I$.

Therefore, for $t \in I$, $x \in K$, $||x||_c = r_3$, similar to (3.3), we have

$$\|Tx(t)\| \geq \varphi((Tu)(t)) \geq \lambda[(\varphi f)_0 - \varepsilon]\|x\|_c A \geq \|x\|_c.$$

Therefore,

$$\|Tx\|_c \geq \|x\|_c, \quad t \in I, x \in K, \quad \|x\|_c = r_3.$$

Using a similar method, we can get $r_4 > r_3$, such that

$$||Tx||_c \;\leq\; ||x||_c, \quad t \in I, x \in K, \quad ||x||_c \;= r_4.$$

Applying (a) of Lemma 1.1 to (3.3) and (3.4) yields that T has a fixed point $x^* \in K_{r_3,r_4}, r_3 \leq \|x^*\|_c \leq r_4$ and $x^*(t) \leq e(t)x^*(s) > \theta$, $t \in I$, $s \in I$.

The proof is complete.

Theorem 3.3 *Assume that (H) holds, P is normal and for any* $||f(t, x)|| \leq ||x||$, $||x|| > 0$. *Then problem (1.1) has no positive solution in K provided* $\lambda B < 1$.

Proof. Assume to the contrary that $x(t)$ is a positive solution of the problem (1.1). Then $x \in K$, $||x||_c > 0$ for $t \in I$, and

$$\begin{aligned}\|x(t)\| &= \|\lambda \int_0^1 H(t,s)f(s,x(s))ds\| \le \lambda \int_0^1 H(t,s)\|x(s)\|ds \\ &\le \lambda\|x\|_c \int_0^1 H(t,s)ds \le \lambda B\|x\|_c \le \|x\|_c,\end{aligned}$$

which is a contradiction, and completes the proof.

Similarly, we have the following results.

Theorem 3.4 *Assume that (H) holds, P is normal and for any* $||f(t, x)|| \ge ||x||$, $||x|| > 0$ *Then problem (1.1) has no positive solution in K provided* $\lambda A > 1$.

Remark 3.1 *When* $q(t) \equiv 0$, $\lambda = 1$, *the problem (1.1) reduces to the problem studied in* [1], *and so our results generalize and include some results in* [1].

Competing interests

The author declare that they have no competing interests.

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References

1. Feng, M, Ji, D, Ge, W: Positive solutions for a class of boundary-value problem with integral boundary conditions in Banach spaces. J Comput Appl Math. **222**(2), 351–363 (2008). doi:10.1016/j.cam.2007.11.003
2. Li, F, Sun, J, Jia, M: Monotone iterative method for the second-order three-point boundary value problem with upper and lower solutions in the reversed order. Appl Math Comput. **217**(9), 4840–4847 (2011). doi:10.1016/j.amc.2010.11.003
3. Sun, Y, Liu, L, Zhang, J, Agarwal, R: Positive solutions of singular three-point boundary value problems for second-order differential equations. J Comput Appl Math. **230**(2), 738–750 (2009). doi:10.1016/j.cam.2009.01.003
4. Li, F, Jia, M, Liu, X, Li, C, Li, G: Existence and uniqueness of solutions of second-order three-point boundary value problems with upper and lower solutions in the reversed order. Nonlinear Anal. Theory Methods Appl. **68**(8), 2381–2388 (2008). doi:10.1016/j.na.2007.01.065
5. Lee, Y, Liu, X: Study of singular boundary value problems for second order impulsive differential equations. J Math Anal Appl. **331**(1), 159–176 (2007). doi:10.1016/j.jmaa.2006.07.106
6. Zhang, G, Sun, J: Multiple positive solutions of singular second-order m-point boundary value problems. J Math Anal Appl. **317**(2), 442–447 (2006). doi:10.1016/j.jmaa.2005.08.020
7. Zhang, X, Feng, M, Ge, W: Symmetric positive solutions for p-Laplacian fourth-order differential equations with integral boundary conditions. J Comput Appl Math. **222**(2), 561–573 (2008). doi:10.1016/j.cam.2007.12.002
8. Zhang, X, Feng, M, Ge, W: Existence result of second-order differential equations with integral boundary conditions at resonance. J Math Anal Appl. **353**(1), 311–319 (2009). doi:10.1016/j.jmaa.2008.11.082
9. Feng, M, Du, B, Ge, W: Impulsive boundary value problems with integral boundary conditions and one-dimensional p-Laplacian. Nonlinear Anal. **70**(9), 3119–3126 (2009). doi:10.1016/j.na.2008.04.015
10. Guo, D, Lakshmikantham, V, Liu, X: Nonlinear Integral Equations in Abstract Spaces. Kluwer Academic Publishers, Dordrecht (1996)
11. Guo, D, Lakskmikantham, V: Nonlinear Problems in Abstract Cones. Academic Press, New York (1988)

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