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Coupled coincidence points for monotone operators in partially ordered metric spaces

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Abstract

Using the notion of compatible mappings in the setting of a partially ordered metric space, we prove the existence and uniqueness of coupled coincidence points involving a (φ, ψ)-contractive condition for a mappings having the mixed g-monotone property. We illustrate our results with the help of an example.

Keywords: coupled coincidence point, partially ordered metric space, mixed g-monotone property

1 Introduction

The Banach contraction principle is the most celebrated fixed point theorem. Afterward many authors obtained many important extensions of this principle (cf. [1-16]). Recently Bhaskar and Lakshmikantham [5], Nieto and Lopez [12,13], Ran and Reurings [14] and Agarwal et al. [3] presented some new results for contractions in partially ordered metric spaces. Bhaskar and Lakshmikantham [5] noted that their theorem can be used to investigate a large class of problems and have discussed the existence and uniqueness of solution for a periodic boundary value problem.

Recently, Luong and Thuan [11] presented some coupled fixed point theorems for a mixed monotone mapping in a partially ordered metric space which are generalizations of the results of Bhaskar and Lakshmikantham [5]. In this paper, we establish the existence and uniqueness of coupled coincidence point involving a (φ,ψ)-contractive condition for mappings having the mixed g-monotone property. We also illustrate our results with the help of an example.

2 Preliminaries

A partial order is a binary relation $\preccurlyeq$ over a set X which is reflexive, antisymmetric, and transitive. Now, let us recall the definition of the monotonic function $f : X \to X$ in the partially order set $(X, \preccurlyeq)$. We say that f is non-decreasing if for $x, y \in X$, $x \preccurlyeq y$, we have $fx \preccurlyeq fy$. Similarly, we say that f is non-increasing if for $x, y \in X$, $x \preccurlyeq y$, we have $fx \succcurlyeq fy$. Any one could read on [9] for more details on fixed point theory.

Definition 2.1 [10]*(Mixed g-Monotone Property)*

Let $(X, \preccurlyeq)$ be a partially ordered set and $F : X \times X \to X$. We say that the mapping F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument. That is, for any $x, y \in X$,

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$$x_1, x_2 \in X, gx_1 \preccurlyeq gx_2 \Rightarrow F(x_1, y) \preccurlyeq F(x_2, y) \tag{1}$$

and

$$y_1, y_2 \in X, gy_1 \preccurlyeq gy_2 \Rightarrow F(x, y_1) \succcurlyeq F(x, y_2). \tag{2}$$

Definition 2.2 [10]*(Coupled Coincidence Point)*

Let $(x, y) \in X \times X$, $F : X \times X \to X$ *and* $g : X \to X$. *We say that* (x, y) *is a coupled coincidence point of F and g if* $F(x, y) = gx$ *and* $F(y, x) = gy$ *for* $x, y \in X$.

Definition 2.3 [10]*Let X be a non-empty set and let* $F : X \times X \to X$ *and* $g : X \to X$. *We say F and g are commutative if, for all* $x, y \in X$,

$$g(F(x, y)) = F(g(x), g(y)).$$

Definition 2.4 [6]*The mapping F and g where* $F : X \times X \to X$ *and* $g : X \to X$, *are said to be compatible if*

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0,$$

whenever $\{x_n\}$ *and* $\{y_n\}$ *are sequences in X, such that* $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ *and* $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$, *for all* $x, y \in X$ *are satisfied.*

3 Existence of coupled coincidence points

As in [11], let φ denote all functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfy

1. φ is continuous and non-decreasing,
2. $\varphi(t) = 0$ if and only if $t = 0$,
3. $\varphi(t + s) \le \varphi(t) + \varphi(s)$, $\forall t, s \in [0, \infty)$

and let ψ denote all the functions $\psi : [0, \infty) \to (0, \infty)$ which satisfy $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0^+} \psi(t) = 0$.

For example [11], functions $\varphi_1(t) = kt$ where $k > 0$, $\phi_2(t) = \frac{t}{t+1}$, $\varphi_3(t) = \ln(t + 1)$, and $\varphi_4(t) = \min\{t, 1\}$ are in Φ; $\psi_1(t) = kt$ where $k > 0$, $\psi_2(t) = \frac{\ln(2t+1)}{2}$, and

$$\psi_3(t) = \begin{cases} 1, & t = 0 \\ \frac{t}{t+1}, & 0 < t < 1 \\ 1, & t = 1 \\ \frac{1}{2}t, & t > 1 \end{cases}$$

are in Ψ,

Now, let us start proving our main results.

Theorem 3.1 *Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose there is a metric d on X such that* (X, d) *is a complete metric space. Let* $F : X \times X \to X$ *be a mapping having the mixed g-monotone property on X such that there exist two elements* $x_0, y_0 \in X$ *with*

$$gx_0 \preccurlyeq F(x_0, y_0) \quad \textit{and} \quad gy_0 \succcurlyeq F(y_0, x_0).$$

Suppose there exist $\varphi \in \Phi$ *and* $\psi \in \Psi$ *such that*

$$\phi(d(F(x,y), F(u,v))) \le \frac{1}{2}\phi(d(gx,gu) + d(gy,gv)) - \psi\left(\frac{d(gx,gu) + d(gy,gv))}{2}\right) \quad (3)$$

for all x, y, u, $v \in X$ *with* $gx \succcurlyeq gu$ *and* $gy \preccurlyeq gv$. *Suppose* $F(X \times X) \subseteq g(X)$, *g is continuous and compatible with F and also suppose either*

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, *then* $x_n \preccurlyeq x$, *for all n,*

(ii) if a non-increasing sequence $\{y_n\} \to y$, *then* $y \preccurlyeq y_n$, *for all n.*

Then there exists x, $y \in X$ *such that*

$$gx = F(x,y) \quad \textit{and} \quad gy = F(y,x),$$

i.e., F and g have a coupled coincidence point in X.

Proof. Let x_0, $y_0 \in X$ be such that $gx_0 \preccurlyeq F(x_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, x_0)$.

Using $F(X \times X) \subseteq g(X)$, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X as

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n) \ \text{ for all } n \ge 0. \quad (4)$$

We are going to prove that

$$gx_n \preccurlyeq gx_{n+1} \quad \text{for all } n \ge 0 \quad (5)$$

and

$$gy_n \succcurlyeq gy_{n+1} \quad \text{for all } n \ge 0. \quad (6)$$

To prove these, we are going to use the mathematical induction.

Let $n = 0$. Since $gx_0 \preccurlyeq F(x_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, x_0)$ and as $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, we have $gx_0 \preccurlyeq gx_1$ and $gy_0 \succcurlyeq gy_1$. Thus (5) and (6) hold for $n = 0$.

Suppose now that (5) and (6) hold for some fixed $n \ge 0$, Then, since $gx_n \preccurlyeq gx_{n+1}$ and $gy_n \succcurlyeq gy_{n+1}$, and by mixed g-monotone property of F, we have

$$gx_{n+2} = F(x_{n+1}, y_{n+1}) \succcurlyeq F(x_n, y_{n+1}) \succcurlyeq F(x_n, y_n) = gx_{n+1} \quad (7)$$

and

$$gy_{n+2} = F(y_{n+1}, x_{n+1}) \preccurlyeq F(y_n, x_{n+1}) \preccurlyeq F(y_n, x_n) = gy_{n+1}. \quad (8)$$

Using (7) and (8), we get

$$gx_{n+1} \preccurlyeq gx_{n+2} \quad \text{and} \quad gy_{n+1} \succcurlyeq gy_{n+2}.$$

Hence by the mathematical induction we conclude that (5) and (6) hold for all $n \ge 0$. Therefore,

$$gx_0 \preccurlyeq gx_1 \preccurlyeq gx_2 \preccurlyeq \cdots \preccurlyeq gx_n \preccurlyeq gx_{n+1} \preccurlyeq \cdots \quad (9)$$

and

$$gy_0 \succcurlyeq gy_1 \succcurlyeq gy_2 \succcurlyeq \cdots \succcurlyeq gy_n \succcurlyeq gy_{n+1} \succcurlyeq \cdots . \tag{10}$$

Since $gx_n \succcurlyeq gx_{n-1}$ and $gy_n \preccurlyeq gy_{n-1}$, using (3) and (4), we have

$$\begin{aligned}\phi(d(gx_{n+1}, gx_n)) &= \phi(d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))) \\ &\leq \frac{1}{2}\phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})) \\ &- \psi\left(\frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})}{2}\right).\end{aligned} \tag{11}$$

Similarly, since $gy_{n-1} \succcurlyeq gy_n$ and $gx_{n-1} \preccurlyeq gx_n$, using (3) and (4), we also have

$$\begin{aligned}\phi(d(gy_n, gy_{n+1})) &= \phi(d(F(y_{n-1}, x_{n-1}), F(y_n, x_n))) \\ &\leq \frac{1}{2}\phi(d(gy_{n-1}, gy_n) + d(gx_{n-1}, gx_n)) \\ &- \psi\left(\frac{d((gy_{n-1}, gy_n) + d(gx_{n-1}, gx_n)}{2}\right).\end{aligned} \tag{12}$$

Using (11) and (12), we have

$$\begin{aligned}\phi(d(gx_{n+1}, gx_n)) + \phi(d(gy_{n+1}, gy_n)) &\leq \phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})) \\ &- 2\psi\left(\frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})}{2}\right).\end{aligned} \tag{13}$$

By property (iii) of φ, we have

$$\phi(d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)) \leq \phi(d(gx_{n+1}, gx_n)) + \phi(d(gy_{n+1}, gy_n)). \tag{14}$$

Using (13) and (14), we have

$$\begin{aligned}\phi(d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)) &\leq \phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})) \\ &- 2\psi\left(\frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})}{2}\right)\end{aligned} \tag{15}$$

which implies, since ψ is a non-negative function,

$$\phi(d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)) \leq \phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})).$$

Using the fact that φ is non-decreasing, we get

$$d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \leq d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}).$$

Set

$$\delta_n = d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n).$$

Now we would like to show that $\delta_n \to 0$ as $n \to \infty$. It is clear that the sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)] = \delta. \tag{16}$$

We shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then taking the limit as $n \to \infty$ (equivalently, $\delta_n \to \delta$) of both sides of (15) and remembering $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and φ is continuous, we have

$$\begin{aligned}\phi(\delta) &= \lim_{n\to\infty}\phi(\delta_n) \le \lim_{n\to\infty}\left[\phi(\delta_{n-1}) - 2\psi\left(\frac{\delta_{n-1}}{2}\right)\right]\\ &= \phi(\delta) - 2\lim_{\delta_{n-1}\to\delta}\psi\left(\frac{\delta_{n-1}}{2}\right) < \phi(\delta)\end{aligned}$$

a contradiction. Thus $\delta = 0$, that is

$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}[d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)] = 0. \tag{17}$$

Now, we will prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{gx_n\}$ or $\{gy_n\}$ is not Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}$, $\{gx_{m(k)}\}$ of $\{gx_n\}$ and $\{gy_{n(k)}\}$, $\{gy_{m(k)}\}$ of $\{gy_n\}$ with $n(k) > m(k) \ge k$ such that

$$d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) \ge \varepsilon. \tag{18}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (18). Then

$$d(gx_{n(k)-1}, gx_{m(k)}) + d(gy_{n(k)-1}, gy_{m(k)}) < \varepsilon. \tag{19}$$

Using (18), (19) and the triangle inequality, we have

$$\begin{aligned}\varepsilon \le r_k &:= d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})\\ &\le d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)}) + d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)})\\ &\le d(gx_{n(k)}, gx_{n(k)-1}) + d(gy_{n(k)}, gy_{n(k)-1}) + \varepsilon.\end{aligned}$$

Letting $k \to \infty$ and using (17), we get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty}[d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}] = \varepsilon. \tag{20}$$

By the triangle inequality

$$\begin{aligned}r_k &= d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})\\ &\le d(gx_{n(k)}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)})\\ &+ d(gy_{n(k)}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) + d(gy_{m(k)+1}, gy_{m(k)})\\ &= \delta_{n(k)} + \delta_{m(k)} + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}).\end{aligned}$$

Using the property of φ, we have

$$\begin{aligned}\phi(r_k) &= \phi(\delta_{n(k)} + \delta_{m(k)} + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}))\\ &\le \phi(\delta_{n(k)} + \delta_{m(k)}) + \phi(d(gx_{n(k)+1}, gx_{m(k)+1}))\\ &+ \phi(d(gy_{n(k)+1}, gy_{m(k)+1})).\end{aligned} \tag{21}$$

Since $n(k) > m(k)$, hence $gx_{n(k)} \succcurlyeq gx_{m(k)}$ and $gy_{n(k)} \succcurlyeq gy_{m(k)}$. Using (3) and (4), we get

$$\begin{aligned}\phi(d(gx_{n(k)+1}, gx_{m(k)+1})) &= \phi(d(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)})))\\ &\le \frac{1}{2}\phi(d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}))\\ &- \psi\left(\frac{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})}{2}\right)\\ &= \frac{1}{2}\phi(r_k) - \psi\left(\frac{r_k}{2}\right).\end{aligned} \tag{22}$$

By the same way, we also have

$$\begin{aligned}\phi(d(gy_{m(k)+1}, gy_{n(k)+1})) &= \phi(d(F(y_{m(k)}, x_{m(k)}), F(y_{n(k)}, x_{n(k)})))\backslash \\ &\le \frac{1}{2}\phi(d(gy_{m(k)}, gy_{n(k)}) + d(gx_{m(k)}, gx_{n(k)})) \\ &\quad - \psi\left(\frac{d(gy_{m(k)}, gy_{n(k)}) + d(gx_{m(k)}, gx_{n(k)})}{2}\right) \\ &= \frac{1}{2}\phi(r_k) - \psi\left(\frac{r_k}{2}\right).\end{aligned} \tag{23}$$

Inserting (22) and (23) in (21), we have

$$\phi(r_k) \le \phi(\delta_{n(k)} + \delta_{m(k)}) + \phi(r_k) - 2\psi\left(\frac{r_k}{2}\right).$$

Letting $k \to \infty$ and using (17) and (20), we get

$$\phi(\varepsilon) \le \phi(0) + \phi(\varepsilon) - 2\lim_{k\to\infty}\psi\left(\frac{r_k}{2}\right) = \phi(\varepsilon) - 2\lim_{r_k\to\varepsilon}\psi\left(\frac{r_k}{2}\right) < \phi(\varepsilon)$$

a contradiction. This shows that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

Since X is a complete metric space, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x \quad \textit{and} \quad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y. \tag{24}$$

Since F and g are compatible mappings, we have

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0 \tag{25}$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0 \tag{26}$$

We now show that $gx = F(x, y)$ and $gy = F(y, x)$. Suppose that the assumption (a) holds. For all $n \ge 0$, we have,

$$d(gx, F(gx_n, gy_n)) \le d(gx, g(F(x_n, y_n))) + d(g(F(x_n, y_n)), F(gx_n, gy_n).$$

Taking the limit as $n \to \infty$, using (4), (24), (25) and the fact that F and g are continuous, we have $d(gx, F(x, y)) = 0$.

Similarly, using (4), (24), (26) and the fact that F and g are continuous, we have $d(gy, F(y, x)) = 0$.

Combining the above two results we get

$$gx = F(x, y) \quad \textit{and} \quad gy = F(y, x).$$

Finally, suppose that (b) holds. By (5), (6) and (24), we have $\{gx_n\}$ is a non-decreasing sequence, $gx_n \to x$ and $\{gy_n\}$ is a non-increasing sequence, $gy_n \to y$ as $n \to \infty$. Hence, by assumption (b), we have for all $n \ge 0$,

$$gx_n \preccurlyeq x \quad \textit{and} \quad gy_n \preccurlyeq y. \tag{27}$$

Since F and g are compatible mappings and g is continuous, by (25) and (26) we have

$$\lim_{n\to\infty} g(gx_n) = gx = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(gx_n, gy_n) \tag{28}$$

and,

$$\lim_{n\to\infty} g(gy_n) = gy = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(gy_n, gx_n). \tag{29}$$

Now we have

$$d(gx, F(x, y)) \leq d(gx, g(gx_{n+1})) + d(g(gx_{n+1}), F(x, y)).$$

Taking $n \to \infty$ in the above inequality, using (4) and (21) we have,

$$\begin{aligned} d(gx, F(x, y)) &\leq \lim_{n\to\infty} d(gx, g(gx_{n+1})) + \lim_{n\to\infty} d(g(F(x_n, y_n)), F(x, y)) \\ &\leq \lim_{n\to\infty} d(F(gx_n, gy_n)), F(x, y)) \end{aligned} \tag{30}$$

Using the property of φ, we get

$$\phi(d(gx, F(x, y))) \leq \lim_{n\to\infty} \phi(d(F(gx_n, gy_n)), F(x, y)))$$

Since the mapping g is monotone increasing, using (3), (27) and (30), we have for all $n \geq 0$,

$$\begin{aligned} \phi(d(gx, F(x, y))) &\leq \lim_{n\to\infty} \frac{1}{2}\phi(d(ggx_n, gx) + d(gy_n, ggy)) \\ &\quad - \lim_{n\to\infty} \psi\left(\frac{d(ggx_n, gx) + d(ggy_n, gy)}{2}\right). \end{aligned}$$

Using the above inequality, using (24) and the property of ψ, we get $\varphi(d(gx, F(x, y))) = 0$, thus $d(gx, F(x, y)) = 0$. Hence $gx = F(x, y)$.

Similarly, we can show that $gy = F(y, x)$. Thus we proved that F and g have a coupled coincidence point.

Corollary 3.1 [11]*Let $(X, \preccurlyeq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$$x_0 \preccurlyeq F(x_0, y_0) \quad \textit{and} \quad y_0 \succcurlyeq F(y_0, x_0).$$

Suppose there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\phi(d(F(x, y), F(u, v))) \leq \frac{1}{2}\phi(d(x, u) + d(y, v)) - \psi\left(\frac{d(x, u) + d(y, v))}{2}\right)$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. Suppose either
(a) F is continuous or
(b) X has the following property.

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preccurlyeq x$, for all n,
(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preccurlyeq y_n$, for all n,

then there exist $x, y \in X$ such that

$$x = F(x, y) \quad and \quad y = F(y, x)$$

that is, F has a coupled fixed point in X.

Corollary 3.2 [11] *Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose there is a metric d on X such that* (X, d) *is a complete metric space. Let* $F : X \times X \to X$ *be a mapping having the mixed monotone property on X such that there exist two elements* $x_0, y_0 \in X$ *with*

$$x_0 \preccurlyeq F(x_0, y_0) \quad and \quad y_0 \succcurlyeq F(y_0, x_0).$$

Suppose there exists $\psi \in \Psi$ *such that*

$$d(F(x, y), F(u, v)) \le \frac{d(x, u) + d(y, v)}{2} - \psi\left(\frac{d(x, u) + d(y, v)}{2}\right)$$

for all $x, y, u, v \in X$ *with* $x \geq u$ *and* $y \leq v$*. Suppose either*

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$*, then* $x_n \preccurlyeq x$*, for all n,*

(ii) if a non-increasing sequence $\{y_n\} \to y$*, then* $y \preccurlyeq y_n$*, for all n,*

then there exist $x, y \in X$ *such that*

$$x = F(x, y) \quad and \quad y = F(y, x)$$

that is, F has a coupled fixed point in X.

Proof. Take $\varphi(t) = t$ in Corollary 3.1, we get Corollary 3.2.

Corollary 3.3 [5] eses of Corollary 3.1, suppose that for *Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose there is a metric d on X such that* (X, d) *is a complete metric space. Let* $F : X \times X \to X$ *be a mapping having the mixed monotone property on X such that there exist two elements* $x_0, y_0 \in X$ *with*

$$x_0 \preccurlyeq F(x_0, y_0) \quad and \quad y_0 \succcurlyeq F(y_0, x_0).$$

Suppose there exists a real number $k \in [0, 1)$ *such that*

$$d(F(x, y), F(u, v)) \le \frac{k}{2}[d(x, u) + d(y, v)]$$

for all $x, y, u, v \in X$ *with* $x \geq u$ *and* $y \geq v$*. Suppose either*

(a) F is continuous or

(b) X has the following property.

(i) if a non-decreasing sequence $\{x_n\} \to x$*, then* $x_n \preccurlyeq x$*, for all n,*

(ii) if a non-increasing sequence $\{y_n\} \to y$*, then* $y \preccurlyeq y_n$*, for all n,*

then there exist $x, y \in X$ *such that*

$$x = F(x, y) \quad and \quad y = F(y, x)$$

that is, F has a coupled fixed point in X.

Proof. Taking $\psi(t) = \frac{1-k}{2}t$ in Corollary 3.2.

4 Uniqueness of coupled coincidence point

In this section, we will prove the uniqueness of the coupled coincidence point. Note that if $(X, \preccurlyeq)$ is a partially ordered set, then we endow the product $X \times X$ with the following partial order relation, for all $(x, y), (u, v) \in X \times X$,

$$(x, y) \preccurlyeq (u, v) \quad \Leftrightarrow \quad x \preccurlyeq u, y \succcurlyeq v.$$

Theorem 4.1 *In addition to hypotheses of Theorem 3.1, suppose that for every* (x, y), (z, t) *in* $X \times X$, *if there exists a* (u, v) *in* $X \times X$ *that is comparable to* (x, y) *and* (z, t), *then F has a unique coupled coincidence point.*

Proof. From Theorem 3.1, the set of coupled coincidence points of F and g is non-empty. Suppose (x, y) and (z, t) are coupled coincidence points of F and g, that is $gx = F(x, y)$, $gy = F(y, x)$, $gz = F(z, t)$ and $gt = F(t, z)$. We are going to show that $gx = gz$ and $gy = gt$. By assumption, there exists $(u, v) \subset X \times X$ that is comparable to (x, y) and (z, t). We define sequences $\{gu_n\}$, $\{gv_n\}$ as follows

$$u_0 = u \quad v_0 = v. \quad gu_{u+1} = F(u_n, v_n) \text{ and } gv_{n+1} = F(v_n, u_n) \quad \text{for all } n.$$

Since (u, v) is comparable with (x, y), we may assume that $(x, y) \succcurlyeq (u, v) = (u_0, v_0)$. Using the mathematical induction, it is easy to prove that

$$(x, y) \succcurlyeq (u_n, v_n) \quad \text{for all } n. \tag{31}$$

Using (3) and (31), we have

$$\begin{aligned} \varphi(d(gx, gu_{n+1})) &= \varphi(d(F(x, y), F(u_n, v_n))) \\ &< \frac{1}{2}\varphi(d(x, u_n) + d(y, v_n)) - \psi\left(\frac{d(x, u_n) + d(y, v_n)}{2}\right) \end{aligned} \tag{32}$$

Similarly

$$\begin{aligned} \varphi(d(gv_{n+1}, gy)) &= \varphi(d(F(v_n, u_n), F(y, x))) \\ &< \frac{1}{2}\varphi(d(v_n, y) + d(u_n, x)) - \psi\left(\frac{d(v_n, y) + d(u_n, x)}{2}\right) \end{aligned} \tag{33}$$

Using (32), (33) and the property of ϕ, we have

$$\begin{aligned} \varphi(d(gx, gu_{n+1}) + d(gy, gv_{n+1})) &\le \varphi(d(gx, gu_{n+1})) + \varphi(d(gy, gv_{n+1})) \\ &\le \varphi(d(gx, gu_n) + d(gy, gv_n)) \\ &\quad - 2\psi\left(\frac{d(gx, gu_n) + d(gy, gv_n)}{2}\right). \end{aligned} \tag{34}$$

which implies, using the property of ψ,

$$\varphi(d(gx, gu_{n+1}) + d(gy, gv_{n+1})) \le \varphi(d(gx, gu_n) + d(gy, gv_n)).$$

Thus, using the property of φ,

$$d(gx, gu_{n+1}) + d(gy, gv_{n+1}) \le d(gx, gu_n) + d(gy, gv_n).$$

That is the sequence $\{d(gx, gu_n)+ d(gy, gv_n)\}$ is decreasing. Therefore, there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty}[d(gx, gu_n) + d(gy, gv_n)] = \alpha. \tag{35}$$

We will show that $\alpha = 0$. Suppose, to the contrary, that $\alpha > 0$. Taking the limit as $n \to \infty$ in (34), we have, using the property of ψ,

$$\varphi(\alpha) \leq \varphi(\alpha) - 2\lim_{n\to\infty}\psi\left(\frac{d(gx, gu_n) + d(gy, gv_n)}{2}\right) < \varphi(\alpha)$$

a contradiction. Thus. $\alpha = 0$, that is,

$$\lim_{n\to\infty}[d(gx, gu_n) + d(gy, gv_n)] = 0.$$

It implies

$$\lim_{n\to\infty} d(gx, gu_n) = \lim_{n\to\infty} d(gy, gv_n) = 0. \tag{36}$$

Similarly, we show that

$$\lim_{n\to\infty} d(gz, gu_n) = \lim_{n\to\infty} d(gt, gv_n) = 0. \tag{37}$$

Using (36) and (37) we have $gx = gz$ and $gy = gt$.

Corollary 4.1 [11]*In addition to hypotheses of Corollary 3.1, suppose that for every* (x, y), (z, t) *in* $X \times X$, *if there exists a* (u, v) *in* $X \times X$ *that is comparable to* (x, y) *and* (z, t), *then F has a unique coupled fixed point.*

5 Example

Example 5.1 *Let* $X = [0, 1]$. *Then* $(X, \leq)$ *is a partially ordered set with the natural ordering of real numbers. Let*

$$d(x, y) = |x - y| \quad \textit{for} \quad x, y \in [0, 1].$$

Then (X, d) *is a complete metric space.*

Let $g : X \to X$ *be defined as*

$$gx = x^2, \quad \textit{for all} \quad x \in X,$$

and let $F : X \times X \to X$ *be defined as*

$$F(x, y) = \begin{cases} \dfrac{x^2 - y^2}{3}, & \textit{if } x \geq y, \\ 0, & \textit{if } x < y. \end{cases}$$

F obeys the mixed g-monotone property.

Let $\varphi : [0, \infty) \to [0, \infty)$ *be defined as*

$$\phi(t) = \frac{3}{4}t, \quad \textit{for} \quad t \in [0, \infty).$$

and let $\psi : [0, \infty) \to [0, \infty)$ *be defined as*

$$\psi(t) = \frac{1}{4}t, \quad \textit{for} \quad t \in [0, \infty).$$

Let $\{x_n\}$ *and* $\{y_n\}$ *be two sequences in X such that* $\lim_{n\to\infty} F(x_n, y_n) = a$, $\lim_{n\to\infty} gx_n = a$, $\lim_{n\to\infty} F(y_n, x_n) = b$ *and* $\lim_{n\to\infty} gy_n = b$ *Then obviously,* $a = 0$ *and* $b = 0$. *Now, for all* $n \geq 0$,

$$gx_n = x_n^2, gy_n = y_n^2,$$

$$F(x_n, y_n) = \begin{cases} \dfrac{x_n^2 - y_n^2}{3}, & \text{if } x_n \geq y_n, \\ 0, & \text{if } x_n < y_n. \end{cases}$$

and

$$F(y_n, x_n) = \begin{cases} \dfrac{y_n^2 - x_n^2}{3}, & \text{if } y_n \geq x_n, \\ 0, & \text{if } y_n < x_n. \end{cases}$$

Then it follows that,

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0,$$

Hence, the mappings F and g are compatible in X. Also, $x_0 = 0$ *and* $y_0 = c(>0)$ *are two points in X such that*

$$gx_0 = g(0) = 0 = F(0, c) = F(x_0, y_0)$$

and

$$gy_0 = g(c) = c^2 \geq \frac{c^2}{3} = F(c, 0) = F(y_0, x_0).$$

We next verify the contraction (3). We take $x, y, u, v, \in X$, *such that* $gx \geq gu$ *and* $gy \leq gv$, *that is,* $x^2 \geq u^2$ *and* $y^2 \leq v^2$.

We consider the following cases:

Case 1. $x \geq y$, $u \geq v$. *Then,*

$$\begin{aligned}
\phi(d(F(x,y), F(u,v))) &= \frac{3}{4}[d(F(x,y), F(u,v)] \\
&= \frac{3}{4}\left[d\left(\frac{x^2-y^2}{3}, \frac{u^2-v^2}{3}\right)\right] \\
&= \frac{3}{4}\left|\frac{(x^2-y^2)-(u^2-v^2)}{3}\right| \\
&= \frac{3}{4}\frac{|x^2-u^2|+|y^2-v^2|}{3} \\
&= \frac{1}{2}\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right) \\
&= \frac{3}{4}\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right) \\
&\quad - \frac{1}{4}\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right) \\
&= \frac{3}{8}(d(gx,gu)+d(gy,gv)) \\
&\quad - \frac{1}{4}\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right) \\
&= \frac{1}{2}\phi(d(gx,gu)+d(gy,gv)) \\
&\quad - \psi\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right)
\end{aligned}$$

Case 2. $x \geq y$, $u < v$. *Then*

$$\begin{aligned}
\phi(d(F(x,y),F(u,v)) &= \frac{3}{4}[d(F(x,y),F(u,v)] \\
&= \frac{3}{4}\left[d(\frac{x^2-y^2}{3},0)\right] \\
&= \frac{3}{4}\frac{|x^2-y^2|}{3} \\
&= \frac{3}{4}\frac{|v^2+x^2-y^2-u^2|}{3} \\
&= \frac{3}{4}\frac{|(v^2-y^2)-(u^2-x^2)|}{3} \\
&\leq \frac{3}{4}\frac{|v^2-y^2|+|u^2-x^2|}{3} \\
&= \frac{3}{4}\left(\frac{|u^2-x^2|+|y^2-v^2|}{3}\right) \\
&= \frac{1}{2}\left(\frac{|u^2-x^2|+|y^2-v^2|}{2}\right) \\
&= \frac{1}{2}\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right) \\
&= \frac{3}{4}\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right) \\
&\quad - \frac{1}{4}\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right) \\
&= \frac{3}{8}(d(gx,gu)+d(gy,gv)) \\
&\quad - \frac{1}{4}(\frac{d(gx,gu)+d(gy,gv)}{2} \\
&= \frac{1}{2}\phi(d(gx,gu)+d(gy,gv)) \\
&\quad - \psi\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right)
\end{aligned}$$

Case 3. $x < y$ *and* $u \geq v$. *Then*

$$\begin{aligned}
\phi(d(F(x,y),F(u,v))) &= \frac{3}{4}\left[d(0,\frac{u^2-v^2}{3})\right] \\
&= \frac{3}{4}\frac{|u^2-v^2|}{3} \\
&= \frac{3}{4}\frac{|u^2+x^2-v^2-x^2|}{3} \\
&= \frac{3}{4}\frac{|(x^2-v^2)+(u^2-x^2)|}{3} (since\ y > x) \\
&\leq \frac{3}{4}\frac{|y^2-v^2|+|u^2-x^2|}{3} \\
&= \frac{1}{2}\left(\frac{|u^2-x^2|+|y^2-v^2|}{2}\right) \\
&= \frac{1}{2}\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right) \\
&= \frac{3}{4}\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right) \\
&\quad - \frac{1}{4}\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right) \\
&= \frac{3}{8}(d(gx,gu)+d(gy,gv)) \\
&\quad - \frac{1}{4}\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right) \\
&= \frac{1}{2}\phi(d(gx,gu)+d(gy,gv)) \\
&\quad - \psi\left(\frac{d(gx,gu)+d(gy,gv)}{2}\right)
\end{aligned}$$

Case 4. *$x < y$ and $u < v$ with $x^2 \le u^2$ and $y^2 \ge v^2$. Then, $F(x, y) = 0$ and $F(u, v) = 0$, that is,*

$$\phi(d(F(x,y),F(u,v))) = \phi(d(0,0)) = \phi(0) = 0.$$

Therefore all conditions of Theorem 3.1 are satisfied. Thus the conclusion follows.

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