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On rate of convergence of various iterative schemes

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Abstract

In this note, by taking an counter example, we prove that the iteration process due to Agarwal et al. (J. Nonlinear Convex. Anal. **8** (1), 61-79, 2007) is faster than the Mann and Ishikawa iteration processes for Zamfirescu operators.

Keywords: iteration processes, Zamfirescu operator

1 Introduction

For a nonempty convex subset C of a normed space E and $T : C \to C$, (a) the Mann iteration process [1] is defined by the following sequence$\{x_n\}$:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - b_n)\, x_n + b_n T x_n, \; n \geq 0, \end{cases} \qquad (M_n,)$$

where $\{b_n\}$ is a sequence in [0, 1].

(b) the sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in C, \\ y_n = (1 - b'_n)x_n + b'_n T x_n, \\ x_{n+1} = (1 - b_n)x_n + b_n T y_n, \; n \geq 0, \end{cases} \qquad (I_n,)$$

where $\{b_n\}$, $\{b'_n\}$ are sequences in [0, 1] is known as the Ishikawa [2] iteration process.

(c) the sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in C, \\ y_n = (1 - b'_n)x_n + b'_n T x_n, \\ x_{n+1} = (1 - b_n)T x_n + b_n T y_n, \; n \geq 0, \end{cases} \qquad (ARS_n,)$$

where $\{b_n\}$, $\{b'_n\}$ are sequences in [0, 1] is known as the Agarwal et al. [3] iteration process.

Definition 1. [4]*Suppose that* $\{a_n\}$ *and* $\{b_n\}$ *are two real convergent sequences with limits* a *and* b*, respectively. Then,* $\{a_n\}$ *is said to converge faster than* $\{b_n\}$ *if*

$$\lim_{n\to\infty} \left| \frac{a_n - a}{b_n - b} \right| = 0.$$

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Theorem 2. [5]*Let* (X, d) *be a complete metric space, and* $T : X \to X$ *a mapping for which there exist real numbers, a, b, and c satisfying* $0 < a < 1$, $0 < b, c < \frac{1}{2}$*such that for each pair* $x, y \in X$*, at least one of the following is true:*

(z1) $d(Tx, Ty) \leq ad(x, y)$,
(z2) $d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)]$,
(z3) $d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)]$.

Then, T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=1}^{\infty}$*defined by*

$$x_{n+1} = Tx_n, \; n = 0, 1, 2, \; \ldots,$$

converges to p, for any $x_0 \in X$.

Remark 3. *An operator T that satisfies the contraction conditions* $(z1)$ - $(z3)$ *of Theorem 2 will be called a Zamfirescu operator* [[4,6,7]] *and is denoted by Z.*

In [6,7], Berinde introduced a new class of operators on a normed space E satisfying

$$||Tx - Ty|| \; \leq \delta||x - y|| + L||Tx - x|| \quad (B)$$

for any $x, y \in E$, $0 \leq \delta < 1$ and $L \geq 0$. He proved that this class is wider than the class of Zamfiresu operators.

The following results are proved in [6,7].

Theorem 4. [7]*Let C be a nonempty closed convex subset of a normed space E. Let T* $: C \to C$ *be an operator satisfying* (B)*. Let* $\{x_n\}$ *be defined through the iterative process* (M_n)*. If* $F(T) \neq \emptyset$ *and* $\sum b_n = \infty$*, then* $\{x_n\}$ *converges strongly to the unique fixed point of T.*

Theorem 5. [6]*Let C be a nonempty closed convex subset of an arbitrary Banach space E and* $T : C \to C$ *be an operator satisfying* (B)*. Let* $\{x_n\}$ *be defined through the iterative process* I_n *and* $x_0 \in C$*, where* $\{b_n\}$ *and* $\{b'_n\}$*are sequences of positive numbers in* $[0, 1]$ *with* $\{b_n\}$ *satisfying* $\sum b_n = \infty$*. Then* $\{x_n\}$ *converges strongly to the fixed point of T.*

The following theorem was presented in [8].

Theorem 6. *Let C be a closed convex subset of an arbitrary Banach space E. Let the Mann and Ishikawa iteration processes denoted by* M_n *and* I_n*, respectively, with* $\{b_n\}$ *and* $\{b'_n\}$*be real sequences satisfying (i)* $0 \leq b_n, b'_n \leq 1$*, and (ii)* $\sum b_n = \infty$*. Then,* M_n *and* I_n *converge strongly to the unique fixed point of a Zamfirescu operator* $T : C \to C$*, and moreover, the Mann iteration process converges faster than the Ishikawa iteration process to the fixed point of T.*

Remark 7. *In* [9]*, Qing and Rhoades, by taking a counter example, showed that the Ishikawa iteration process is faster than the Mann iteration process for Zamfirescu operators. Thus, Theorem in* [8]*and the presentation in* [9]*contradict to each other (see also* [10]*).*

In this note, we establish a general theorem to approximate fixed points of quasi-contractive

operators in a Banach space through the iteration process ARS_n, due to Agarwal et al. [3]. Our result generalizes and improves upon, among others, the corresponding results of Babu and Prasad [8] and Berinde [4,6,7].

We also prove that the iteration process ARS_n is faster than the Mann iteration process M_n and the Ishikawa iteration process I_n for Zamfirescu operators.

2 Main results

We now prove our main results.

Theorem 8. *Let C be a nonempty closed convex subset of an arbitrary Banach space E and $T : C \to C$ be an operator satisfying (B). Let $\{x_n\}$ be defined through the iterative process ARS_n and $x_0 \in C$, where $\{b_n\}$, $\{b'_n\}$are sequences in $[0, 1]$ satisfying $\sum b_n = \infty$. Then, $\{x_n\}$ converges strongly to the fixed point of T.*

Proof. Assume that $F(T) \neq \emptyset$ and $w \in F(T)$, then using (ARS_n), we have

$$\begin{aligned} ||x_{n+1} - w|| &= ||(1 - b_n)Tx_n + b_nTy_n - w|| \\ &= ||(1 - b_n)(Tx_n - w) + b_n(Ty_n - w)|| \\ &\leq (1 - b_n)||Tx_n - w|| + b_n||Ty_n - w||. \end{aligned} \tag{2.1}$$

Now using (B) with $x = w$, $y = x_n$, and then with $x = w$, $y = y_n$, we obtain the following two inequalities,

$$||Tx_n - w|| \leq \delta||x_n - w||, \tag{2.2}$$

and

$$||Ty_n - w|| \leq \delta||y_n - w||. \tag{2.3}$$

By substituting (2.2) and (2.3) in (2.1), we obtain

$$||x_{n+1} - w|| \leq (1 - b_n)\delta||x_n - w|| + b_n\delta||y_n - w||. \tag{2.4}$$

In a similar fashion, again by using (ARS_n), we can get

$$||y_n - w|| \leq (1 - (1 - \delta)b'_n)||x_n - w||. \tag{2.5}$$

From (2.4) and (2.5), we have

$$||x_{n+1} - w|| \leq [1 - (1 - \delta)b_n(1 + \delta b'_n)]||x_n - w||. \tag{2.6}$$

It may be noted that for $\delta \in [0, 1)$ and $\{\eta_n\} \in [0, 1]$, the following inequality holds:

$$1 \leq 1 + \delta\eta_n \leq 1 + \delta. \tag{2.7}$$

From (2.6) and (2.7), we get

$$||x_{n+1} - w|| \leq (1 - (1 - \delta)b_n)||x_n - w||. \tag{2.8}$$

By (2.8) we inductively obtain

$$||x_{n+1} - w|| \leq \prod_{k=0}^{n} [1 - \delta(1 - \delta)b_k]||x_0 - w||, \; n = 0, 1, 2, \ldots \tag{2.9}$$

Using the fact that $0 \leq \delta < 1$, $0 \leq b_n \leq 1$, and $\sum b_n = \infty$, it results that

$$\lim_{n\to\infty} \prod_{k=0}^{n}[1 - \delta(1 - \delta)b_k] = 0,$$

which by (2.9) implies

$$\lim_{n\to\infty} ||x_{n+1} - w|| = 0.$$

Consequently $x_n \to w \in F$ and this completes the proof. □

Now by an counter example, we prove that the iteration process ARS_n due to Agarwal et al. [3] is faster than the Mann and Ishikawa iteration processes for Zamfirescu operators.

Example 9. [9]*Suppose* $T : [0, 1] \to [0, 1] := \frac{1}{2}x$, $b_n = 0 = b'_n$, $n = 1, 2, \ldots, 15$. $b_n = \frac{4}{\sqrt{n}} = b'_n$, $n \geq 16$.

It is clear that T is a Zamfirescu operator with a unique fixed point 0. Also, it is easy to see that Example 9 satisfies all the conditions of Theorem 8.

Proof. Since $b_n = 0 = b'_n$, $n = 1, 2, \ldots, 15$, so $M_n = x_0 = I_n = ARS_n$, $n = 1, 2, \ldots, 16$. Suppose $x_0 \neq 0$. For M_n, I_n and ARS_n iteration processes, we have

$$\begin{aligned} M_n &= (1 - b_n)x_n + b_n T x_n \\ &= \left(1 - \frac{4}{\sqrt{n}}\right) x_n + \frac{4}{\sqrt{n}} \frac{1}{2} x_n \\ &= \left(1 - \frac{2}{\sqrt{n}}\right) x_n \\ &= \cdots \\ &= \prod_{i=16}^{n} \left(1 - \frac{2}{\sqrt{i}}\right) x_0, \end{aligned}$$

$$\begin{aligned} I_n &= (1 - b_n)x_n + b_n T((1 - b'_n)x_n + b'_n T x_n) \\ &= \left(1 - \frac{4}{\sqrt{n}}\right) x_n + \frac{4}{\sqrt{n}} \frac{1}{2} \left(1 - \frac{2}{\sqrt{n}}\right) x_n \\ &= \left(1 - \frac{2}{\sqrt{n}} - \frac{4}{n}\right) x_n \\ &= \cdots \\ &= \prod_{i=16}^{n} \left(1 - \frac{2}{\sqrt{i}} - \frac{4}{i}\right) x_0, \end{aligned}$$

and

$$\begin{aligned} ARS_n &= (1 - b_n)T x_n + b_n T((1 - b'_n)x_n + b'_n T x_n) \\ &= \left(1 - \frac{4}{\sqrt{n}}\right) \frac{x_n}{2} + \frac{4}{\sqrt{n}} \frac{1}{2} \left(1 - \frac{2}{\sqrt{n}}\right) x_n \\ &= \left(\frac{1}{2} - \frac{4}{n}\right) x_n \\ &= \cdots \\ &= \prod_{i=16}^{n} \left(\frac{1}{2} - \frac{4}{i}\right) x_0. \end{aligned}$$

Now consider

$$\begin{aligned} \left|\frac{ARS_n - 0}{M_n - 0}\right| &= \left|\frac{\prod_{i=16}^{n} (\frac{1}{2} - \frac{4}{i})x_0}{\prod_{i=16}^{n} (1 - \frac{2}{\sqrt{i}})x_0}\right| \\ &= \left|\frac{\prod_{i=16}^{n} (\frac{1}{2} - \frac{4}{i})}{\prod_{i=16}^{n} (1 - \frac{2}{\sqrt{i}})}\right| \\ &= \left|\prod_{i=16}^{n} \left(1 - \frac{\frac{1}{2} - \frac{2}{\sqrt{i}} + \frac{4}{i}}{(1 - \frac{2}{\sqrt{i}})}\right)\right| \\ &= \left|\prod_{i=16}^{n} \left(1 - \frac{1}{2\sqrt{i}} \frac{i - 4\sqrt{i} + 8}{\sqrt{i} - 2}\right)\right|. \end{aligned}$$

It is easy to see that

$$\begin{aligned}
0 &\le \lim_{n\to\infty} \prod_{i=16}^{n} \left(1 - \frac{1}{2\sqrt{i}} \frac{i - 4\sqrt{i} + 8}{\sqrt{i} - 2}\right) \\
&\le \lim_{n\to\infty} \prod_{i=16}^{n} \left(1 - \frac{1}{i}\right) \\
&= \lim_{n\to\infty} \frac{15}{n} \\
&= 0.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \left| \frac{ARS_n - 0}{M_n - 0} \right| = 0.$$

Thus, the iteration process due to Agarwal et al. [3] converges faster than the Mann iteration process to the fixed point 0 of T.

Similarly

$$\begin{aligned}
\left| \frac{ARS_n - 0}{I_n - 0} \right| &= \left| \frac{\prod_{i=16}^{n} (\frac{1}{2} - \frac{4}{i})x_0}{\prod_{i=16}^{n} (1 - \frac{2}{\sqrt{i}} - \frac{4}{i})x_0} \right| \\
&= \left| \frac{\prod_{i=16}^{n} (\frac{1}{2} - \frac{4}{i})}{\prod_{i=16}^{n} (1 - \frac{2}{\sqrt{i}} - \frac{4}{i})} \right| \\
&= \left| \prod_{i=16}^{n} \left(1 - \frac{\frac{1}{2} - \frac{2}{\sqrt{i}}}{(1 - \frac{2}{\sqrt{i}} - \frac{4}{i})}\right) \right| \\
&= \left| \prod_{i=16}^{n} \left(1 - \frac{\sqrt{i}}{2} \frac{i - 4}{i - 2\sqrt{i} - 4}\right) \right|,
\end{aligned}$$

with

$$\begin{aligned}
0 &\le \lim_{n\to\infty} \prod_{i=16}^{n} \left(1 - \frac{\sqrt{i}}{2} \frac{\sqrt{i} - 4}{i - 2\sqrt{i} - 4}\right) \\
&\le \lim_{n\to\infty} \prod_{i=16}^{n} \left(1 - \frac{1}{i}\right) \\
&= \lim_{n\to\infty} \frac{15}{n} \\
&= 0,
\end{aligned}$$

implies

$$\lim_{n\to\infty} \left| \frac{ARS_n - 0}{I_n - 0} \right| = 0.$$

Thus, the iteration process due to Agarwal et al. [3] converges faster than the Ishikawa iteration process to the fixed point 0 of T. □

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Authors' contributions
The four authors have equitably contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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