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Strong convergence theorems for variational inequalities and fixed points of a countable family of nonexpansive mappings

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Abstract

A new general iterative method for finding a common element of the set of solutions of variational inequality and the set of common fixed points of a countable family of nonexpansive mappings is introduced and studied. A strong convergence theorem of the proposed iterative scheme to a common fixed point of a countable family of nonexpansive mappings and a solution of variational inequality of an inverse strongly monotone mapping are established. Moreover, we apply our main result to obtain strong convergence theorems for a countable family of nonexpansive mappings and a strictly pseudocontractive mapping, and a countable family of uniformly *k*-strictly pseudocontractive mappings and an inverse strongly monotone mapping. Our main results improve and extend the corresponding result obtained by Klin-eam and Suantai (J Inequal Appl 520301, 16 pp, 2009).

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1 Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. In this paper, we always assume that a bounded linear operator A on H is *strongly positive*, that is, there is a constant $\bar{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \bar{\gamma}||x||^2$ for all $x \in H$. Recall that a mapping T of H into itself is called *nonexpansive* if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in H$. The set of all fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$. A self-mapping $f : H \to H$ is a *contraction* on H if there is a constant $\alpha \in [0, 1)$ such that $||f(x) - f(y)|| \leq \alpha ||x - y||$ for all $x, y \in H$.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on H:

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.1}$$

where F is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. A mapping B of C into H is called *monotone* if $\langle Bx - By, x - y\rangle \geq 0$ for all $x, y \in C$. The variational inequality problem is to find $x \in C$ such that $\langle Bx, y - x\rangle \geq 0$

for all $y \in C$. The set of solutions of the variational inequality is denoted by $VI(C, B)$. A mapping B of C to H is called *inverse strongly monotone* if there exists a positive real number β such that $\langle x - y, Bx - By\rangle \geq \beta \,||Bx - By||^2$ for all $x, y \in C$.

Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b \qquad n \geq 0. \tag{1.2}$$

It is proved by Xu [1] that the sequence $\{x_n\}$ generated by (1.2) converges strongly to the unique solution of the minimization problem (1.1) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

On the other hand, Moudafi [2] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n) \qquad n \geq 0, \tag{1.3}$$

where $\{\sigma_n\}$ is a sequence in (0, 1). It is proved by Moudafi [2] and Xu [3] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.3) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0 \qquad x \in C.$$

Recently, Marino and Xu [4] combined the iterative method (1.2) with the viscosity approximation method (1.3) and considered the following general iteration process:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n) \qquad n \geq 0 \tag{1.4}$$

and proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0 \qquad x \in C$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Chen, Zhang and Fan [5] introduced the following iterative process: $x_0 \in C$,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)TP_C(x_n - \lambda_n Bx_n), \qquad n \geq 0, \tag{1.5}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < 2\beta$.

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by (1.5) converges strongly to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for an inverse strongly monotone mapping (say $\bar{x} \in C$), which solves the variational inequality

$$\langle (I - f)\bar{x}, x - \bar{x}\rangle \geq 0 \qquad \forall x \in F(T) \cap VI(C, B).$$

Klin-eam and Suantai [6] modify the iterative methods (1.4) and (1.5) by proposing the following general iterative method: $x_0 \in C$,

$$x_{n+1} = P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)TP_C(x_n - \lambda_n Bx_n)), \qquad n \geq 0, \tag{1.6}$$

where P_C is the projection of H onto C, f is a contraction, A is a strongly positive linear bounded operator, B is a β-inverse strongly monotone mapping, $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < 2\beta$. They noted that when $A = I$ and $\gamma = 1$, the iterative scheme (1.6) reduced to the iterative scheme (1.5).

Wangkeeree, Petrot and Wangkeeree [7] introduced the following iterative process:

$$\begin{cases} x_0 = x \in H, \\ y_n = \beta_n x_n + (1 - \beta_n) T_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, n \geq 0 \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$and$\{\beta_n\} \subset (0, 1)$ and T_n is a countable family of nonexpansive mappings, f is a contraction, and A is a strongly positive linear bounded operator. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{T_n\}$, the sequence $\{x_n\}$ converges strongly to $\tilde{x}$, which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - z \rangle \leq 0 \qquad z \in F(T).$$

In this paper, motivated and inspired by Klin-eam and Suantai [6], we introduced the following iteration to find some solutions of variational inequality and fixed points of countable family of nonexpansive mappings in a Hilbert spaces H: $x_0 \in C$,

$$x_{n+1} = P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A) T_n P_C(x_n - \lambda_n B x_n)), \qquad n \geq 0, \tag{1.8}$$

where P_C is the projection of H onto C, f is a contraction, A is a strongly positive linear bounded operator, T_n is a countable family of nonexpansive mappings of C into itself, B is a β-inverse strongly monotone mapping, $\{\alpha_n\} \subset (0, 1)$, and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < 2\beta$.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $|| \cdot ||$, and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x, and $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $||x - P_C x|| \leq ||x - y||$ for all $y \in C$ and $P_C x$ is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies $\langle x - y, P_C x - P_C y \rangle \geq ||P_C x - P_C y||^2$ for every $x, y \in H$. Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and $\langle x - P_C x, P_C x - y \rangle \geq 0$ for all $y \in C$. In the context of the variational inequality problem, this implies that

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u), \quad \forall \lambda > 0.$$

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse strongly monotone mapping of C into H, and let $N_C v$ be the *normal cone* to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}$, and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$.

Lemma 2.1 *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$, then*

(i) $y = P_C x$ if and only if the inequality $\langle x - y, y - z\rangle \geq 0$ for all $z \in C$,
(ii) P_C is nonexpansive,
(iii) $\langle x - y, P_C x - P_C y\rangle \geq ||P_C x - P_C y||^2$ for all $x, y \in H$,
(iv) $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $x \in H$ and $y \in C$.

Lemma 2.2 [4]*Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$and $0 < \rho \leq ||A||^{-1}$, then $||I - \rho A|| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.3 [8]*Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, n \geq 0$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 [9]*Let C be a closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. If a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z = Tz$.*

To deal with a family of mappings, the following conditions are introduced: Let C be a subset of a real Banach space E, and let $\{T_n\}_{n=1}^{\infty}$ be a family of mappings of C such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then, $\{T_n\}$ is said to satisfy the *AKTT-condition* [10] if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} sup\{||T_{n+1}z - T_n z|| \ : z \in B\} < \infty.$$

Lemma 2.5 [10]*Let C be a nonempty and closed subset of a Banach space E and let $\{T_n\}$ be a family of mappings of C into itself which satisfies the AKTT-condition. Then, for each $x \in C$, $\{T_n x\}$ converges strongly to a point in C. Moreover, let the mapping T be defined by*

$$Tx = \lim_{n\to\infty} T_n x \quad \forall x \in C.$$

Then, for each bounded subset B of C,

$$\lim_{n\to\infty} \sup\{||Tz - T_n z|| \ : z \in B\} = 0.$$

In the sequel, we will write $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}$ satisfies the AKTT-condition, and T is defined by Lemma 2.5 with $F(T) = \cap_{n=1}^{\infty} F(T_n)$.

3 Main results

In this section, we prove a strong convergence theorem for a countable family of nonexpansive mappings.

Theorem 3.1 *Let C be a closed convex subset of a real Hilbert space H, and let* $B : C \to H$ *be a* β*-inverse strongly monotone mapping, also let A be a strongly positive linear bounded operator of H into itself with coefficient* $\bar{\gamma} > 0$*such that* $||A|| = 1$ *and let* $f : C \to C$ *be a contraction with coefficient* $\alpha(0 < \alpha < 1)$*. Assume that* $0 < \gamma < \bar{\gamma}/\alpha$*. Let* $\{T_n\}$ *be a countable family of nonexpansive mappings from a subset C into itself with* $F = \cap_{n=1}^{\infty} F(T_n) \cap VI(C, B) \neq \emptyset$*. Suppose* $\{x_n\}$ *is the sequence generated by the following algorithm:* $x_0 \in C$,

$$x_{n+1} = P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)T_n P_C(x_n - \lambda_n B x_n))$$

for all $n = 0, 1, 2, ...,$ *where* $\{\alpha_n\} \subset (0, 1)$ *and* $\{\lambda_n\} \subset (0, 2\beta)$*. If* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are chosen so that* $\lambda_n \in [a, b]$ *for some a, b with* $0 < a < b < 2\beta$ *,*

$$(C1) \lim_{n \to 0} \alpha_n = 0; \qquad (C2) \sum_{n=1}^{\infty} \alpha_n = \infty;$$
$$(C3) \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty; \ (C4) \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

Suppose that $(\{T_n\}, T)$ *satisfies the AKTT-condition. Then,* $\{x_n\}$ *converges strongly to* $q \in F$*, where* $q = P_F(\gamma f + I - A)(q)$ *which solves the following variational inequality:*

$$\langle (\gamma f - A)q, p - q \rangle \leq 0 \quad \forall p \in F.$$

Proof. First, we show that the sequence $\{x_n\}$ is bounded. Consider the mapping $I - \lambda_n B$. Since B is a β-inverse strongly monotone mapping, we have that for all $x, y \in C$,

$$\begin{aligned} ||(I - \lambda_n B)x - (I - \lambda_n B)y||^2 &= ||(x - y) - \lambda_n(Bx - By)||^2 \\ &= ||x - y||^2 - 2\lambda_n \langle x - y, Bx - By \rangle + \lambda_n^2 ||Bx - By||^2 \\ &\leq ||x - y||^2 + \lambda_n(\lambda_n - 2\beta)||Bx - By||^2. \end{aligned}$$

For $0 < \lambda_n < 2\beta$, implies that $||k(I - \lambda_n B)x - (I - \lambda_n B)y||^2 \leq ||x - y||^2$.
So, the mapping $I - \lambda_n B$ is nonexpansive.
Put $y_n = P_C(x_n - \lambda_n B x_n)$ for all $n \geq 0$. Let $u \in F$. Then $u = P_C(u - \lambda_n B u)$.
From P_C is nonexpansive implies that

$$\begin{aligned} ||y_n - u|| &= ||P_C(x_n - \lambda_n B x_n) - P_C(u - \lambda_n B u)|| \\ &\leq ||(x_n - \lambda_n B x_n) - (u - \lambda_n B u)|| \\ &= ||(I - \lambda_n B)x_n - (I - \lambda_n B)u||. \end{aligned}$$

Since $I - \lambda_n B$ is nonexpansive, we have that $||y_n - u|| \leq ||x_n - u||$. Then

$$\begin{aligned} ||x_{n+1} - u|| &= ||P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)T_n y_n) - u|| \\ &\leq ||\alpha_n \gamma f(x_n) + (I - \alpha_n A)T_n y_n - u|| \\ &= ||\alpha_n(\gamma f(x_n) - Au) + (I - \alpha_n A)(T_n y_n - u)||. \end{aligned}$$

Since A is strongly positive linear bounded operator, we have

$$\begin{aligned} ||x_{n+1} - u|| &\leq \alpha_n ||\gamma f(x_n) - Au|| + (1 - \alpha_n \bar{\gamma})||T_n y_n - u|| \\ &\leq \alpha_n ||\gamma f(x_n) - \gamma f(u)|| + \alpha_n ||\gamma f(u) - Au|| + (1 - \alpha_n \bar{\gamma})||T_n y_n - u||. \end{aligned}$$

By contraction of f, we have

$$\begin{aligned}
||x_{n+1} - u|| &\le \alpha\gamma\alpha_n||x_n - u|| + \alpha_n||\gamma f(u) - Au|| + (1 - \alpha_n\bar{\gamma})||T_n y_n - u|| \\
&= \alpha\gamma\alpha_n||x_n - u|| + \alpha_n||\gamma f(u) - Au|| + (1 - \alpha_n\bar{\gamma})||T_n y_n - T_n u|| \\
&\le \alpha\gamma\alpha_n||x_n - u|| + \alpha_n||\gamma f(u) - Au|| + (1 - \alpha_n\bar{\gamma})||y_n - u|| \\
&\le \alpha\gamma\alpha_n||x_n - u|| + \alpha_n||\gamma f(u) - Au|| + (1 - \alpha_n\bar{\gamma})||x_n - u|| \\
&\le (\alpha\gamma\alpha_n + 1 - \alpha_n\bar{\gamma})||x_n - u|| + \alpha_n||\gamma f(u) - Au|| \\
&\le (1 - \alpha_n(\bar{\gamma} - \alpha\gamma))||x_n - u|| + \alpha_n(\bar{\gamma} - \alpha\gamma)\frac{||\gamma f(u) - Au||}{\bar{\gamma} - \alpha\gamma} \\
&\le \max\left\{||x_n - u||, \frac{||\gamma f(u) - Au||}{\bar{\gamma} - \alpha\gamma}\right\}.
\end{aligned}$$

It follows from induction that $||x_n - u|| \le max\left\{||x_0 - u||, \frac{||\gamma f(u) - Au||}{\bar{\gamma} - \alpha\gamma}\right\}$, $n \ge 0$.

Therefore, $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{T_n y_n\}$, $\{Bx_n\}$, and $\{f(x_n)\}$.

Next, we show that $||x_{n+1} - x_n|| \to 0$ and $||y_n - T_n y_n|| \to 0$ as $n \to \infty$.

Since P_C is nonexpansive, we also have

$$\begin{aligned}
||y_{n+1} - y_n|| &= ||P_C(x_{n+1} - \lambda_{n+1}Bx_{n+1}) - P_C(x_n - \lambda_n Bx_n)|| \\
&\le ||x_{n+1} - \lambda_{n+1}Bx_{n+1} - (x_n - \lambda_n Bx_n)|| \\
&\le ||x_{n+1} - \lambda_{n+1}Bx_{n+1} - (x_n - \lambda_{n+1}Bx_n)|| + |\lambda_n - \lambda_{n+1}|\ ||Bx_n|| \\
&= ||(I - \lambda_{n+1}B)x_{n+1} - (I - \lambda_{n+1}B)x_n|| + |\lambda_n - \lambda_{n+1}|\ ||Bx_n||.
\end{aligned}$$

Since $I - \lambda_n B$ is nonexpansive, we have

$$||y_{n+1} - y_n|| \le ||x_{n+1} - x_n|| + |\lambda_n - \lambda_{n+1}|\ ||Bx_n||.$$

So we obtain

$$\begin{aligned}
||x_{n+1} - x_n|| &= ||P_C(\alpha_n\gamma f(x_n) + (I - \alpha_n A)T_n y_n) - P_C(\alpha_{n-1}\gamma f(x_{n-1}) + (I - \alpha_{n-1}A)T_{n-1}y_{n-1})|| \\
&\le ||\alpha_n\gamma(f(x_n) - f(x_{n-1})) + \gamma(\alpha_n - \alpha_{n-1})f(x_{n-1}) + (I - \alpha_n A)(T_n y_n - T_{n-1}y_{n-1}) \\
&\quad + (\alpha_n - \alpha_{n-1})AT_{n-1}y_{n-1}|| \\
&\le \alpha_n\alpha\gamma||x_n - x_{n-1}|| + \gamma|\alpha_n - \alpha_{n-1}|\ ||f(x_{n-1})|| + (1 - \alpha_n\bar{\gamma})||T_n y_n - T_{n-1}y_{n-1}|| \\
&\quad + |\alpha_n - \alpha_{n-1}|\ ||AT_{n-1}y_{n-1}|| \\
&\le \alpha_n\alpha\gamma||x_n - x_{n-1}|| + \gamma|\alpha_n - \alpha_{n-1}|\ ||f(x_{n-1})|| + (1 - \alpha_n\bar{\gamma})(||T_n y_n - T_n y_{n-1}|| \\
&\quad + ||T_n y_{n-1} - T_{n-1}y_{n-1}||) + |\alpha_n - \alpha_{n-1}|\ ||AT_{n-1}y_{n-1}|| \\
&\le \alpha_n\alpha\gamma||x_n - x_{n-1}|| + \gamma|\alpha_n - \alpha_{n-1}|\ ||f(x_{n-1})|| + (1 - \alpha_n\bar{\gamma})(||y_n - y_{n-1}|| \\
&\quad + ||T_n y_{n-1} - T_{n-1}y_{n-1}||) + |\alpha_n - \alpha_{n-1}|\ ||AT_{n-1}y_{n-1}|| \\
&= \alpha_n\alpha\gamma||x_n - x_{n-1}|| + \gamma|\alpha_n - \alpha_{n-1}|\ ||f(x_{n-1})|| + (1 - \alpha_n\bar{\gamma})||y_n - y_{n-1}|| \\
&\quad + (1 - \alpha_n\bar{\gamma})||T_n y_{n-1} - T_{n-1}y_{n-1}|| + |\alpha_n - \alpha_{n-1}|\ ||AT_{n-1}y_{n-1}|| \\
&\le \alpha_n\alpha\gamma||x_n - x_{n-1}|| + \gamma|\alpha_n - \alpha_{n-1}|\ ||f(x_{n-1})|| + (1 - \alpha_n\bar{\gamma})||x_n - x_{n-1}|| \\
&\quad + (1 - \alpha_n\bar{\gamma})|\lambda_{n-1} - \lambda_n|\ ||Bx_{n-1}|| + (1 - \alpha_n\bar{\gamma})||T_n y_{n-1} - T_{n-1}y_{n-1}|| \\
&\quad + |\alpha_n - \alpha_{n-1}|\ ||AT_{n-1}y_{n-1}|| \\
&= (1 - (\bar{\gamma} - \alpha\gamma)\alpha_n)||x_n - x_{n-1}|| + \gamma|\alpha_n - \alpha_{n-1}|\ ||f(x_{n-1})|| \\
&\quad + (1 - \alpha_n\bar{\gamma})|\lambda_{n-1} - \lambda_n|\ ||Bx_{n-1}|| + (1 - \alpha_n\bar{\gamma})||T_n y_{n-1} - T_{n-1}y_{n-1}|| \\
&\quad + |\alpha_n - \alpha_{n-1}|\ ||AT_{n-1}y_{n-1}|| \\
&\le (1 - (\bar{\gamma} - \alpha\gamma)\alpha_n)||x_n - x_{n-1}|| + 2L|\alpha_n - \alpha_{n-1}| + M|\lambda_{n-1} - \lambda_n| \\
&\quad + \sup_{y\in\{y_n\}} ||T_n y - T_{n-1}y||,
\end{aligned}$$

where $L = \max\{\sup_{n\in\mathbb{N}} ||AT_{n-1}y_{n-1}||, \sup_{n\in\mathbb{N}} \gamma\,||f(x_{n-1})||\}$ and $M = \sup\{||Bx_{n-1}|| : n\in\mathbb{N}\}$.

Since $\{T_n\}$ satisfies the AKTT-condition, we get that

$$\sum_{n=1}^{\infty} \sup_{y\in\{y_n\}} ||T_n y - T_{n-1}y|| < \infty.$$

From condition (C3), (C4) and by Lemma 2.3, we have $||x_{n+1} - x_n|| \to 0$.
For $u \in F$ and $u = P_C(u - \lambda_n Bu)$, we have

$$\begin{aligned} ||x_{n+1} - u||^2 &= ||P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)T_n y_n) - P_C(u)||^2 \\ &\le ||\alpha_n(\gamma f(x_n) - Au) + (I - \alpha_n A)(T_n y_n - u)||^2 \\ &\le (\alpha_n||\gamma f(x_n) - Au|| + ||I - \alpha_n A||\,||T_n y_n - u||)^2 \\ &\le (\alpha_n||\gamma f(x_n) - Au|| + (1 - \alpha_n\bar{\gamma})||y_n - u||)^2 \\ &\le \alpha_n||\gamma f(x_n) - Au||^2 + (1 - \alpha_n\bar{\gamma})||y_n - u||^2 \\ &\quad + 2\alpha_n(1 - \alpha_n\overline{\gamma})||\gamma f(x_n) - Au||\,||y_n - u|| \\ &\le \alpha_n||\gamma f(x_n) - Au||^2 + (1 - \alpha_n\overline{\gamma})||(I - \lambda_n B)x_n - (I - \lambda_n B)u||^2 \\ &\quad + 2\alpha_n(1 - \alpha_n\overline{\gamma})||\gamma f(x_n) - Au||\,||y_n - u|| \\ &\le \alpha_n||\gamma f(x_n) - Au||^2 + (1 - \alpha_n\bar{\gamma})(||x_n - u||^2 - 2\lambda_n\langle x_n - u, Bx_n - Bu\rangle \\ &\quad + \lambda_n^2||Bx_n - Bu||^2) + 2\alpha_n(1 - \alpha_n\bar{\gamma})||\gamma f(x_n) - Au||\,||y_n - u|| \\ &\le \alpha_n||\gamma f(x_n) - Au||^2 + (1 - \alpha_n\bar{\gamma})(||x_n - u||^2 - 2\lambda_n\beta||Bx_n - Bu||^2 \\ &\quad + \lambda_n^2||Bx_n - Bu||^2) + 2\alpha_n(1 - \alpha_n\bar{\gamma})||\gamma f(x_n) - Au||\,||y_n - u|| \\ &= \alpha_n||\gamma f(x_n) - Au||^2 + (1 - \alpha_n\bar{\gamma})\left(||x_n - u||^2 + \lambda_n(\lambda_n - 2\beta)||Bx_n - Bu||^2\right) \\ &\quad + 2\alpha_n(1 - \alpha_n\bar{\gamma})||\gamma f(x_n) - Au||\,||y_n - u|| \\ &\le \alpha_n||\gamma f(x_n) - Au||^2 + ||x_n - u||^2 + (1 - \alpha_n\bar{\gamma})b(b - 2\beta)||Bx_n - Bu||^2 \\ &\quad + 2\alpha_n(1 - \alpha_n\bar{\gamma})||\gamma f(x_n) - Au||\,||y_n - u||. \end{aligned}$$

So, we obtain

$$\begin{aligned} &-(1 - \alpha_n\bar{\gamma})b(b - 2\beta)||Bx_n - Bu||^2 \\ &\quad \le \alpha_n||\gamma f(x_n) - Au||^2 + (||x_n - u|| + ||x_{n+1} - u||)(||x_n - u|| - ||x_{n+1} - u||) + \varepsilon_n \\ &\quad \le \alpha_n||\gamma f(x_n) - Au||^2 + \varepsilon_n + ||x_n - x_{n+1}||(||x_n - u|| + ||x_{n+1} - u||), \end{aligned}$$

where $\varepsilon_n = 2\alpha_n(1 - \alpha_n\bar{\gamma})||\gamma f(x_n) - Au||\,||y_n - u||$.
Since $\alpha_n \to 0$ and $||x_{n+1} - x_n|| \to 0$, we obtain $||Bx_n - Bu|| \to 0$ as $n \to \infty$.
Further, by Lemma 2.1, we have

$$\begin{aligned} ||y_n - u||^2 &= ||P_C(x_n - \lambda_n Bx_n) - P_C(u - \lambda_n Bu)||^2 \\ &\le \langle (x_n - \lambda_n Bx_n) - (u - \lambda_n Bu), y_n - u\rangle \\ &= \frac{1}{2}(||(x_n - \lambda_n Bx_n) - (u - \lambda_n Bu)||^2 + ||y_n - u||^2 \\ &\quad - ||(x_n - \lambda_n Bx_n) - (u - \lambda_n Bu) - (y_n - u)||^2) \\ &\le \frac{1}{2}(||x_n - u||^2 + ||y_n - u||^2 - ||(x_n - y_n) - \lambda_n(Bx_n - Bu)||^2) \\ &\le ||x_n - u||^2 - ||x_n - y_n||^2 + 2\lambda_n\langle x_n - y_n, Bx_n - Bu\rangle - \lambda_n^2||Bx_n - Bu||^2. \end{aligned}$$

So, we have

$$\begin{aligned}||x_{n+1}-u||^2 &= ||P_C(\alpha_n\gamma f(x_n)+(I-\alpha_nA)T_ny_n)-P_C(u)||^2\\ &\le ||\alpha_n(\gamma f(x_n)-Au)+(I-\alpha_nA)(T_ny_n-u)||^2\\ &\le (\alpha_n||\gamma f(x_n)-Au||+||I-\alpha_nA||\,||T_ny_n-u||)^2\\ &\le (\alpha_n||\gamma f(x_n)-Au||+(1-\alpha_n\bar{\gamma})||y_n-u||)^2\\ &\le \alpha_n||\gamma f(x_n)-Au||^2+(1-\alpha_n\bar{\gamma})||y_n-u||^2\\ &\quad +2\alpha_n(1-\alpha_n\bar{\gamma})||\gamma f(x_n)-Au||\,||y_n-u||\\ &\le \alpha_n||\gamma f(x_n)-Au||^2+(1-\alpha_n\overline{\gamma})||x_n-u||^2-(1-\alpha_n\overline{\gamma})||x_n-y_n||^2\\ &\quad +2(1-\alpha_n\bar{\gamma})\lambda_n\langle x_n-y_n,Bx_n-Bu\rangle-(1-\alpha_n\bar{\gamma})\lambda_n^2||Bx_n-Bu||^2\\ &\quad +2\alpha_n(1-\alpha_n\bar{\gamma})||\gamma f(x_n)-Au||\,||y_n-u||,\end{aligned}$$

which implies

$$\begin{aligned}(1-\alpha_n\bar{\gamma})||x_n-y_n||^2 \le{}& \alpha_n||\gamma f(x_n)-Au||^2+(||x_n-u||+||x_{n+1}-u||)||x_n-x_{n+1}||\\ &+2(1-\alpha_n\bar{\gamma})\lambda_n\langle x_n-y_n,Bx_n-Bu\rangle-(1-\alpha_n\bar{\gamma})\lambda_n^2||Bx_n-Bu||^2\\ &+2\alpha_n(1-\alpha_n\bar{\gamma})||\gamma f(x_n)-Au||\,||y_n-u||.\end{aligned}$$

Since $\alpha_n \to 0$, $||x_{n+1} - x_n|| \to 0$, and $||Bx_n - Bu|| \to 0$, we obtain $||x_n - y_n|| \to 0$ as $n \to \infty$.

Next, we have

$$\begin{aligned}||x_{n+1}-T_ny_n|| &= ||P_C(\alpha_n\gamma f(x_n)+(I-\alpha_nA)T_ny_n)-P_C(T_ny_n)||\\ &\le ||\alpha_n\gamma f(x_n)+(I-\alpha_nA)T_ny_n-T_ny_n||\\ &= \alpha_n||\gamma f(x_n)-AT_ny_n||.\end{aligned}$$

Since $\alpha_n \to 0$ and $\{f(x_n)\}$, $\{AT_ny_n\}$ are bounded, we have $||x_{n+1} - T_ny_n|| \to 0$ as $n \to \infty$. Since

$$||x_n-T_ny_n|| \le ||x_n-x_{n+1}||+||x_{n+1}-T_ny_n||,$$

it implies that $||x_n - T_ny_n|| \to 0$ as $n \to \infty$. Since

$$\begin{aligned}||x_n-T_nx_n|| &\le ||x_n-T_ny_n||+||T_ny_n-T_nx_n||\\ &\le ||x_n-T_ny_n||+||y_n-x_n||,\end{aligned}$$

we obtain $||x_n - T_nx_n|| \to 0$ as $n \to \infty$. Moreover, from

$$||y_n-T_ny_n|| \le ||y_n-x_n||+||x_n-T_ny_n||,$$

it follows that $||y_n - T_ny_n|| \to 0$ as $n \to \infty$.

By $||y_n - x_n|| \to 0$, $||T_ny_n - x_n|| \to 0$ and Lemma 2.5, we have

$$\begin{aligned}||Tx_n-x_n|| &\le ||Tx_n-Ty_n||+||Ty_n-T_ny_n||+||T_ny_n-x_n||\\ &\le ||x_n-y_n||+\sup\{||T_nz-Tz|| : z\in\{y_n\}\}+||T_ny_n-x_n||.\end{aligned}$$

Hence, $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$. Observe that $P_F(\gamma f + I - A)$ is a contraction.

By Lemma 2.2, we have that $||I - A|| \leq 1 - \bar{\gamma}$, and since $0 < \gamma < \bar{\gamma}/\alpha$, we get

$$\begin{aligned}||P_F(\gamma f + I - A)x - P_F(\gamma f + I - A)y|| &\leq ||(\gamma f + I - A)x - (\gamma f + I - A)y|| \\ &\leq \gamma ||f(x) - f(y)|| + ||I - A|| \, ||x - y|| \\ &\leq \gamma\alpha ||x - y|| + (1 - \bar{\gamma})||x - y|| \\ &= (1 - (\bar{\gamma} - \gamma\alpha))||x - y||.\end{aligned}$$

Then, Banach's contraction mapping principle guarantees that $P_F(\gamma f + I - A)$ has a unique fixed point, say $q \in H$. That is, $q = P_F(\gamma f + I - A)q$. By Lemma 2.1, we obtain

$$\langle(\gamma f - A)q, x - q\rangle \leq 0 \text{ for all } x \in F. \tag{3.1}$$

Choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)q, T_n y_n - q\rangle = \lim_{k\to\infty}\langle(\gamma f - A)q, T_{n_k} y_{n_k} - q\rangle.$$

As $\{y_{n_k}\}$ is bounded, there exists a subsequence $\{y_{n_{k_i}}\}$ of $\{y_{n_k}\}$ which converges weakly to p. Without loss of generality, we may assume that $y_{n_k} \rightharpoonup p$.

Since $||y_n - T_n y_n|| \to 0$, we obtain $T_{n_k} y_{n_k} \rightharpoonup p$. Since $||x_n - Tx_n|| \to 0$, $||x_n - y_n|| \to 0$ and by Lemma 2.4-2.5, we have $p \in \cap_{n=1}^{\infty} F(T_n)$. Let

$$Sv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

where $N_C v$ is normal cone to C at $v \in C$, that is $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$. Then S is a maximal monotone. Let $(v, w) \in G(S)$. Since $w - Bv \in N_C v$ and $y_n \in C$, we have $\langle v - y_n, w - Bv\rangle \geq 0$. On the other hand, by Lemma 2.1 and from $y_n = P_C(x_n - \lambda_n Bx_n)$, we have

$$\begin{aligned}&\langle v - y_n, y_n - (x_n - \lambda_n Bx_n)\rangle \geq 0 \\ &\langle v - y_n, (y_n - x_n)/\lambda_n + Bx_n\rangle \geq 0.\end{aligned}$$

Hence,

$$\begin{aligned}\langle v - y_{n_k}, w\rangle &\geq \langle v - y_{n_k}, Bv\rangle \\ &\geq \langle v - y_{n_k}, Bv\rangle - \left\langle v - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_n} + Bx_{n_k}\right\rangle \\ &= \left\langle v - y_{n_k}, Bv - Bx_{n_k} - \frac{y_{n_k} - x_{n_k}}{\lambda_n}\right\rangle \\ &= \langle v - y_{n_k}, Bv - By_{n_k}\rangle + \langle v - y_{n_k}, By_{n_k} - Bx_{n_k}\rangle - \left\langle v - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_n}\right\rangle \\ &\geq \langle v - y_{n_k}, By_{n_k} - Bx_{n_k}\rangle - \left\langle v - y_{n_k}, \frac{y_{n_k} - x_{n_k}}{\lambda_n}\right\rangle.\end{aligned}$$

This implies $\langle v - p, w\rangle \geq 0$. Since S is maximal monotone, we have $p \in S^{-1}0$ and hence $p \in VI(C, B)$. We obtain that $p \in F$. By (3.1), we have $\langle(\gamma f - A)q, p - q\rangle \leq 0$. It follows that

$$\limsup_{n\to\infty}\langle(\gamma f - A)q, T_n y_n - q\rangle = \lim_{k\to\infty}\langle(\gamma f - A)q, T_{n_k} y_{n_k} - q\rangle = \langle(\gamma f - A)q, p - q\rangle \leq 0.$$

Finally, we prove $x_n \to q$. By $||y_n - u|| \le ||x_n - u||$ and Schwarz inequality, we have

$$\begin{aligned}||x_{n+1} - q||^2 &= ||P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)T_n y_n) - P_C(q)||^2 \\ &\le ||\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(T_n y_n - q)||^2 \\ &\le ||(I - \alpha_n A)(T_n y_n - q)||^2 + \alpha_n^2 ||\gamma f(x_n) - Aq||^2 \\ &\quad + 2\alpha_n \langle (I - \alpha_n A)(T_n y_n - q), \gamma f(x_n) - Aq\rangle \\ &\le (1 - \alpha_n \bar{\gamma})^2 ||y_n - q||^2 + \alpha_n^2 ||\gamma f(x_n) - Aq||^2 \\ &\quad + 2\alpha_n \langle T_n y_n - q, \gamma f(x_n) - Aq\rangle - 2\alpha_n^2 \langle A(T_n y_n - q), \gamma f(x_n) - Aq\rangle \\ &\le (1 - \alpha_n \bar{\gamma})^2 ||x_n - q||^2 + \alpha_n^2 ||\gamma f(x_n) - Aq||^2 \\ &\quad + 2\alpha_n \langle T_n y_n - q, \gamma f(x_n) - \gamma f(q)\rangle + 2\alpha_n \langle T_n y_n - q, \gamma f(q) - Aq\rangle \\ &\quad - 2\alpha_n^2 \langle A(T_n y_n - q), \gamma f(x_n) - Aq\rangle \\ &\le (1 - \alpha_n \bar{\gamma})^2 ||x_n - q||^2 + \alpha_n^2 ||\gamma f(x_n) - Aq||^2 \\ &\quad + 2\alpha_n ||T_n y_n - q||\, ||\gamma f(x_n) - \gamma f(q)|| + 2\alpha_n \langle T_n y_n - q, \gamma f(q) - Aq\rangle \\ &\quad - 2\alpha_n^2 \langle A(T_n y_n - q), \gamma f(x_n) - Aq\rangle \\ &\le (1 - \alpha_n \bar{\gamma})^2 ||x_n - q||^2 + \alpha_n^2 ||\gamma f(x_n) - Aq||^2 \\ &\quad + 2\gamma\alpha\alpha_n ||y_n - q||\, ||x_n - q|| + 2\alpha_n \langle T_n y_n - q, \gamma f(q) - Aq\rangle \\ &\quad - 2\alpha_n^2 \langle A(T_n y_n - q), \gamma f(x_n) - Aq\rangle \\ &\le (1 - \alpha_n \bar{\gamma})^2 ||x_n - q||^2 + \alpha_n^2 ||\gamma f(x_n) - Aq||^2 \\ &\quad + 2\gamma\alpha\alpha_n ||x_n - q||^2 + 2\alpha_n \langle T_n y_n - q, \gamma f(q) - Aq\rangle \\ &\quad - 2\alpha_n^2 \langle A(T_n y_n - q), \gamma f(x_n) - Aq\rangle \\ &\le ((1 - \alpha_n \bar{\gamma})^2 + 2\gamma\alpha\alpha_n) ||x_n - q||^2 + \alpha_n (2\langle T_n y_n - q, \gamma f(q) - Aq\rangle \\ &\quad + \alpha_n ||\gamma f(x_n) - Aq||^2 + 2\alpha_n ||A(T_n y_n - q)||\, ||\gamma f(x_n) - Aq||) \\ &= (1 - 2(\bar{\gamma} - \gamma\alpha)\alpha_n) ||x_n - q||^2 + \alpha_n (2\langle T_n y_n - q, \gamma f(q) - Aq\rangle \\ &\quad + \alpha_n ||\gamma f(x_n) - Aq||^2 + 2\alpha_n ||A(T_n y_n - q)||\, ||\gamma f(x_n) - Aq|| \\ &\quad + \alpha_n \overline{\gamma}^2 ||x_n - q||^2).\end{aligned}$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{T_n y_n\}$ are bounded, we can take a constant $\eta > 0$ such that

$$\eta \ge ||\gamma f(x_n) - Aq||^2 + 2||A(T_n y_n - q)||\, ||\gamma f(x_n) - Aq|| + \bar{\gamma}^2 ||x_n - q||^2$$

for all $n \ge 0$. It follows that

$$||x_{n+1} - q||^2 \le (1 - 2(\bar{\gamma} - \gamma\alpha)\alpha_n) ||x_n - q||^2 + \alpha_n \beta_n, \tag{3.2}$$

where $\beta_n = 2\langle T_n y_n - q, \gamma f(q) - Aq\rangle + \eta\alpha_n$. By $\limsup_{n\to\infty} \langle (\gamma f - A)q, T_n y_n - q\rangle \le 0$, we get $\limsup_{n\to\infty} \beta_n \le 0$. By Lemma 2.3 and (3.2), we can conclude that $x_n \to q$. This completes the proof. ■

Corollary 3.2 *Let C be a closed convex subset of a real Hilbert space H, and let* $B : C \to H$ *be a* β*-inverse strongly monotone mapping, also let* $f : C \to C$ *be a contraction with coefficient* $\alpha (0 < \alpha < 1)$. *Let* $\{T_n\}$ *be a countable family of nonexpansive mappings from a subset C into itself with* $F = \cap_{n=1}^{\infty} F(T_n) \cap VI(C, B) \neq \emptyset$. *Suppose* $\{x_n\}$ *is the sequence generated by the following algorithm:* $x_0 \in C$,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n P_C(x_n - \lambda_n B x_n)$$

for all $n = 0, 1, 2, \ldots$, *where* $\{\alpha_n\} \subset (0, 1)$ *and* $\{\lambda_n\} \subset (0, 2\beta)$. *If* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are chosen so that* $\lambda_n \in [a, b]$ *for some a, b with* $0 < a < b < 2\beta$,

$$(C1)\ \lim_{n\to 0}\ \alpha_n = 0; \qquad (C2)\sum_{n=1}^{\infty}\ \alpha_n = \infty;$$
$$(C3)\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty;\ (C4)\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty.$$

Suppose that ($\{T_n\}$, T) *satisfies the AKTT-condition. Then* $\{x_n\}$ *converges strongly to* $q \in F$, *where* $q = P_F(\gamma f + I - A)(q)$ *which solves the following variational inequality:*

$$\langle(\gamma f - I)q, p - q\rangle \le 0 \quad \forall p \in F.$$

Proof. Taking $A = I$ and $\gamma = 1$ in Theorem 3.1, we get the results. ■

4 Applications

In this section, we apply the iterative scheme (1.8) and Theorem 3.1 for finding a common fixed point of countable family of nonexpansive mappings and strictly pseudocontractive mapping and inverse strongly monotone mapping.

A mapping $T : C \to C$ is called *strictly pseudocontractive* if there exists k with $0 \le k < 1$ such that

$$||Tx - Ty||^2 \le\ ||x - y||^2 + k||(I - T)x - (I - T)y||^2 \qquad \forall x, y \in C.$$

If $k = 0$, then T is nonexpansive. Put $B = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with k. Then, B is $((1 - k)/2)$-inverse strongly monotone and $B^{-1}(0) = F(T)$. Hence, for all $x, y \in C$,

$$||(I - B)x - (I - B)y||^2 \le\ ||x - y||^2 + k||Bx - By||^2.$$

Conversely, since H is a real Hilbert space, we have

$$||(I - B)x - (I - B)y||^2 \le\ ||x - y||^2 + ||Bx - By||^2 - 2\langle x - y, Bx - By\rangle.$$

Thus, we have

$$\langle x - y, Bx - By\rangle \ge \frac{1-k}{2}||Bx - By||^2.$$

Theorem 4.1 *Let C be a closed convex subset of a real Hilbert space H, and let A be a strongly positive linear bounded operator of H into itself with coefficient* $\bar{\gamma} > 0$*such that* $||A|| = 1$ *and let* $f : C \to C$ *be a contraction with coefficient* $\alpha(0 < \alpha < 1)$. *Assume that* $0 < \gamma < \bar{\gamma}/\alpha$. *Let* $\{T_n\}$ *be a family of nonexpansive mappings of C into itself and let S be a strictly pseudocontractive mapping of C into itself with* β *such that* $F = \cap_{n=1}^{\infty}F(T_n) \cap F(S) \neq \emptyset$. *Suppose* $\{x_n\}$ *is a sequence generated by the following algorithm:* $x_0 \in C$,

$$x_{n+1} = P_C(\alpha_n\gamma f(x_n) + (I - \alpha_n A)T_n((1 - \lambda_n)x_n - \lambda_n Sx_n))$$

for all $n = 0, 1, 2, \ldots$, *where* $\{\alpha_n\} \subset [0, 1)$ *and* $\{\lambda_n\} \subset [0, 1 - \beta)$. *If* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are chosen so that* $\lambda_n \in [a, b]$ *for some a, b with* $0 < a < b < 1 - \beta$,

$$(C1)\ \lim_{n\to 0}\alpha_n = 0; \qquad (C2)\sum_{n=1}^{\infty}\ \alpha_n = \infty;$$
$$(C3)\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty;\ (C4)\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty.$$

Suppose that ($\{T_n\}$, T) *satisfies the AKTT-condition. Then*, $\{x_n\}$ *converges strongly to* $q \in F$, *such that*

$$\langle (\gamma f - A)q, p - q\rangle \leq 0 \quad \forall p \in F.$$

Proof. Put $B = I - S$, then B is $((1 - k)/2)$-inverse strongly monotone and $F(S) = VI(C, B)$ and $P_C(x_n - \lambda_n Bx_n) = (1 - \lambda_n)x_n + \lambda_n Sx_n$. Therefore, by Theorem 3.1, the conclusion follows. ∎

Lemma 4.2 [9]*Let* $T : C \to H$ *be a k-strictly pseudocontractive, then*

(i) the fixed point set $F(T)$ *of* T *is closed convex so that the projection* $P_{F(T)}$ *is well defined;*

(ii) define a mapping $S : C \to H$ *by*

$$Sx = \mu x + (1 - \mu)Tx, x \in C. \tag{4.1}$$

If $\mu \in [k, 1)$, *then* S *is a nonexpansive mapping such that* $F(T) = F(S)$.

A family of mappings $\{T_n : C \to H\}_{n=1}^{\infty}$ is called a family of uniformly k-strictly pseudocontractions, if there exists a constant $k \in [0, 1)$ such that $||T_n x - T_n y||^2 \leq ||x - y||^2 + k||(I - T_n)x - (I - T_n)y||^2 \qquad \forall x, y \in C, \forall n \geq 1.$

Let $\{T_n : C \to C\}$ be a countable family of uniformly k-strictly pseudocontractions. Let $\{S_n : C \to C\}_{n=1}^{\infty}$ be the sequence of mappings defined by (4.1), i.e.,

$$S_n x = \mu x + (1 - \mu)T_n x, \quad x \in C, \forall n \geq 1 \text{ with } \mu \in [k, 1).$$

Corollary 4.3 *Let* C *be a closed convex subset of a real Hilbert space* H, *and let* $B : C \to H$ *be a* β-*inverse strongly monotone mapping, also let* A *be a strongly positive linear bounded operator of* H *into itself with coefficient* $\bar{\gamma} > 0$*such that* $||A|| = 1$ *and let* $f : C \to C$ *be a contraction with coefficient* $\alpha(0 < \alpha < 1)$. *Assume that* $0 < \gamma < \bar{\gamma}/\alpha$. *Let* $\{T_n\}$ *be a countable family of uniformly k-strictly pseudocontractions from a subset C into itself with* $F = \cap_{n=1}^{\infty} F(T_n) \cap VI(C, B) \neq \emptyset$. *Suppose* $\{x_n\}$ *is the sequence generated by the following algorithm:* $x_0 \in C$,

$$x_{n+1} = P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)S_n P_C(x_n - \lambda_n Bx_n))$$

for all $n = 0, 1, 2, \ldots$, *where* $\{\alpha_n\} \subset (0, 1)$ *and* $\{\lambda_n\} \subset (0, 2\beta)$. *If* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are chosen so that* $\lambda_n \in [a, b]$ *for some* a, b *with* $0 < a < b < 2\beta$,

$$(C1) \lim_{n \to 0} \alpha_n = 0; \qquad (C2) \sum_{n=1}^{\infty} \alpha_n = \infty;$$
$$(C3) \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty; \; (C4) \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

Then, $\{x_n\}$ *converges strongly to* $q \in F$, *where* $q = P_F(\gamma f + I - A)(q)$ *which solves the following variational inequality:*

$$\langle (\gamma f - A)q, p - q\rangle \leq 0 \quad \forall p \in F.$$

Proof. Let $\{T_n\}$ be a countable family of uniformly k-strictly pseudo-contractions from a subset C into itself. Set $S_n = \mu I + (1 - \mu)T_n$ where $\mu \in [k, 1)$. By Lemma 4.2, we have S_n is nonexpansive and $F(S_n) = F(T_n)$. Therefore, by Theorem 3.1, the conclusion follows. ∎

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Authors' contributions

AB study and researched nonlinear analysis and also wrote this article. SS participated in the process of the study and helped to draft the manuscript. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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