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# On $\varepsilon$-optimality conditions for multiobjective fractional optimization problems

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**Abstract**

A multiobjective fractional optimization problem (MFP), which consists of more than two fractional objective functions with convex numerator functions and convex denominator functions, finitely many convex constraint functions, and a geometric constraint set, is considered. Using parametric approach, we transform the problem (MFP) into the non-fractional multiobjective convex optimization problem $(\mathrm{NMCP})_v$ with parametric $v \in \mathbb{R}^p$, and then give the equivalent relation between (weakly) $\varepsilon$-efficient solution of (MFP) and (weakly) $\bar{\varepsilon}$-efficient solution of $(\mathrm{NMCP})_{\bar{v}}$. Using the equivalent relations, we obtain $\varepsilon$-optimality conditions for (weakly) $\varepsilon$-efficient solution for (MFP). Furthermore, we present examples illustrating the main results of this study.
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**Keywords:** Weakly $\varepsilon$-efficient solution, $\varepsilon$-optimality condition, Multiobjective fractional optimization problem

## 1 Introduction

We need constraint qualifications (for example, the Slater condition) on convex optimization problems to obtain optimality conditions or $\varepsilon$-optimality conditions for the problem.

To get optimality conditions for an efficient solution of a multiobjective optimization problem, we often formulate a corresponding scalar problem. However, it is so difficult that such scalar program satisfies a constraint qualification which we need to derive an optimality condition. Thus, it is very important to investigate an optimality condition for an efficient solution of a multiobjective optimization problem which holds without any constraint qualification.

Jeyakumar et al. [1,2], Kim et al. [3], and Lee et al. [4], gave optimality conditions for convex (scalar) optimization problems, which hold without any constraint qualification. Very recently, Kim et al. [5] obtained $\varepsilon$-optimality theorems for a convex multiobjective optimization problem. The purpose of this article is to extend the $\varepsilon$-optimality theorems of Kim et al. [5] to a multiobjective fractional optimization problem (MFP).

Recently, many authors [5-15] have paid their attention to investigate properties of (weakly) $\varepsilon$-efficient solutions, $\varepsilon$-optimality conditions, and $\varepsilon$-duality theorems for multiobjective optimization problems, which consist of more than two objective functions and a constrained set.

In this article, an MFP, which consists of more than fractional objective functions with convex numerator functions, and convex denominator functions and finitely many convex constraint functions and a geometric constraint set, is considered. We discuss $\varepsilon$-efficient solutions and weakly $\varepsilon$-efficient solutions for (MFP) and obtain $\varepsilon$-optimality theorems for such solutions of (MFP) under weakened constraint qualifications. Furthermore, we prove $\varepsilon$-optimality theorems for the solutions of (MFP) which hold without any constraint qualifications and are expressed by sequences, and present examples illustrating the main results obtained.

## 2 Preliminaries

Now, we give some definitions and preliminary results. The definitions can be found in [16-18]. Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The subdifferential of $g$ at $a$ is given by

$$\partial g(a) := \{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x - a\rangle, \quad \forall x \in \mathrm{dom}g\},$$

where dom$g$: = $\{x \in \mathbb{R}^n \mid g(x) < \infty\}$ and $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^n$. Let $\varepsilon \geqq 0$. The $\varepsilon$-subdifferential of $g$ at $a \in$ dom$g$ is defined by

$$\partial_\varepsilon g(a) := \{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x - a\rangle - \varepsilon, \quad \forall x \in \mathrm{dom}g\}.$$

The conjugate function of $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$g^*(v) = \sup\{ \langle v, x\rangle - g(x) \mid x \in \mathbb{R}^n\}.$$

The epigraph of $g$, epi$g$, is defined by

$$\mathrm{epi}g = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid g(x) \leqq r\}.$$

For a nonempty closed convex set $C \subset \mathbb{R}^n$, $\delta_C : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called the indicator of $C$ if $\delta_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise} \end{cases}$.

**Lemma 2.1** [19]*If $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function and if $a \in$ dom$h$, then*

$$\mathrm{epi}h^* = \bigcup_{\varepsilon \geqq 0} \{(v, \langle v, a\rangle + \varepsilon - h(a)) \mid v \in \partial_\varepsilon h(a)\}.$$

**Lemma 2.2** [20]*Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous convex function and $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then*

$$\mathrm{epi}(h + u)^* = \mathrm{epi}h^* + \mathrm{epi}u^*.$$

Now, we give the following Farkas lemma which was proved in [2,5], but for the completeness, we prove it as follows:

**Lemma 2.3** *Let $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 0, 1, \ldots, l$ be convex functions. Suppose that $\{x \in \mathbb{R}^n \mid h_i(x) \leqq 0, i = 1, \ldots, l\} \neq \emptyset$. Then the following statements are equivalent:*

*(i)* $\{x \in \mathbb{R}^n \mid h_i(x) \leqq 0, i = 1, \ldots, l\} \subseteq \{x \in \mathbb{R}^n \mid h_0(x) \geqq 0\}$

*(ii)* $0 \in \mathrm{epi}h_0^* + \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^{l} \lambda_i h_i)^*$.

*Proof.* Let $Q = \{x \in \mathbb{R}^n \mid h_i(x) \leqq 0,\ i = 1, ..., l\}$. Then $Q \neq \emptyset$ and by Lemma 2.1 in [2], $\mathrm{epi}\delta_Q^* = \mathrm{cl} \bigcup_{\lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^{l} \lambda_i h_i)^*$. Hence, by Lemma 2.2, we can verify that (i) if and only if (ii).

**Lemma 2.4** [16]*Let* $h_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i =, 1, ..., m$ *be proper lower semi-continuous convex functions. Let* $\varepsilon \geqq 0$. *if* $\bigcap_{i=1}^{m} \mathrm{ri\ dom} h_i \neq \emptyset$, *where* ri dom$h_i$ *is the relative interior of* dom$h_i$, *then for all* $x \in \bigcup_{i=1}^{m} \mathrm{dom} h_i$,

$$\partial_\varepsilon(\sum_{i=1}^{m} h_i)(x) = \bigcup\{\sum_{i=1}^{m} \partial_{\varepsilon_i} h_i(x) \mid \varepsilon_i \geqq 0,\ i = 1, \cdots, m,\ \sum_{i=1}^{m} \varepsilon_i = \varepsilon\}.$$

## 3 $\varepsilon$-optimality theorems

Consider the following MFP:

$$\begin{array}{lll} \text{(MFP)} & \text{Minimize} & \dfrac{f(x)}{g(x)} := \left(\dfrac{f_1(x)}{g_1(x)}, \cdots, \dfrac{f_p(x)}{g_p(x)}\right) \\ & \text{subject to} & x \in Q := \{x \in \mathbb{R}^n | h_j(x) \leqq 0,\ j = 1, \ldots, m\}. \end{array}$$

Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., p$ be convex functions, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., p$, concave functions such that for any $x \in Q$, $f_i(x) \geqq 0$ and $g_i(x) > 0$, $i = 1, ..., p$, and $h_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, ..., m$, convex functions. Let $\varepsilon = (\varepsilon_1, ..., \varepsilon_p)$, where $\varepsilon_i \geqq 0$, $i = 1, ..., p$.

Now, we give the definition of $\varepsilon$-efficient solution of (MFP) which can be found in [11].

**Definition 3.1** *The point* $\bar{x} \in Q$*is said to be an* $\varepsilon$*-efficient solution of (MFP) if there does not exist* $x \in Q$ *such that*

$$\begin{aligned} &\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i, \text{ for all } i = 1, \ldots, p, \\ &\frac{f_j(x)}{g_j(x)} < \frac{f_j(\bar{x})}{g_j(\bar{x})} - \varepsilon_j, \text{ for some } j \in \{1, \ldots, p\}. \end{aligned}$$

When $\varepsilon = 0$, then the $\varepsilon$-efficiency becomes the efficiency for (MFP) (see the definition of efficient solution of a multiobjective optimization problem in [21]).

Now, we give the definition of weakly $\varepsilon$-efficient solution of (MFP) which is weaker than $\varepsilon$-efficient solution of (MFP).

**Definition 3.2** *A point* $\bar{x} \in Q$*is said to be a weakly* $\varepsilon$*-efficient solution of (MFP) if there does not exist* $x \in Q$ *such that*

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i, \text{ for all } i = 1, \ldots, p.$$

When $\varepsilon = 0$, then the weak $\varepsilon$-efficiency becomes the weak efficiency for (MFP) (see the definition of efficient solution of a multiobjective optimization problem in [21]).

Using parametric approach, we transform the problem (MFP) into the nonfractional multiobjective convex optimization problem $(\mathrm{NMCP})_v$ with parametric $v \in \mathbb{R}^p$:

$$\begin{array}{lll} (\mathrm{NMCP})_v & \text{Minimize} & (f(x) - vg(x)) := (f_1(x) - v_1 g_1(x), \ldots, f_p(x) - v_p g_p(x)) \\ & \text{subject to} & x \in Q. \end{array}$$

Adapting Lemma 4.1 in [22] and modifying Proposition 3.1 in [12], we can obtain the following proposition:

**Proposition 3.1** *Let $\bar{x} \in Q$. Then the following are equivalent:*

*(i) $\bar{x}$ is an $\varepsilon$-efficient solution of (MFP).*

*(ii) $\bar{x}$ is an $\bar{\varepsilon}$-efficient solution of $(NMCP)_{\bar{v}}$, where $\bar{v} := \left(\frac{f_1(\bar{x})}{g_1(\bar{x})} - \varepsilon_1, \ldots, \frac{f_p(\bar{x})}{g_p(\bar{x})} - \varepsilon_p\right)$ and $\bar{\varepsilon} = (\varepsilon_1 g_1(\bar{x}), \ldots, \varepsilon_p g_p(\bar{x}))$.*

*(iii) $Q \cap S(\bar{x}) = \emptyset$ or*

$$\sum_{i=1}^{p}\left[f_i(x) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(x)\right] \geqq 0 = \sum_{i=1}^{p}\left[f_i(\bar{x}) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(\bar{x})\right] - \sum_{i=1}^{p}\varepsilon_i g_i(\bar{x}) \text{ for any } x \in Q \cap S(\bar{x}),$$

*where* $S(\bar{x}) = \{x \in \mathbb{R}^n \mid f_i(x) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(x) \leqq 0 = f_i(\bar{x}) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(\bar{x}) - \bar{\varepsilon}_i,\ i = 1, \ldots, p\}$.

*Proof.* (i) ⇔ (ii): It follows from Lemma 4.1 in [22].

(ii) ⇒ (iii): Let $\bar{x}$ be an $\bar{\varepsilon}$-efficient solution of $(\text{NMCP})_{\bar{v}}$, where $\bar{v} := \left(\frac{f_1(\bar{x})}{g_1(\bar{x})} - \varepsilon_1, \ldots, \frac{f_p(\bar{x})}{g_p(\bar{x})} - \varepsilon_p\right)$ and $\bar{\varepsilon} = (\varepsilon_1 g_1(\bar{x}), \ldots, \varepsilon_p g_p(\bar{x}))$. Then $Q \cap S(\bar{x}) = \emptyset$ or $Q \cap S(\bar{x}) \neq \emptyset$. Suppose that $Q \cap S(\bar{x}) \neq \emptyset$. Then for any $x \in Q \cap S(\bar{x})$ and all $i = 1, \ldots p$,

$$f_i(x) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(x) \leqq f_i(\bar{x}) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(\bar{x}) - \bar{\varepsilon}_i.$$

Hence the $\bar{\varepsilon}$-efficiency of $\bar{x}$ yields

$$f_i(x) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(x) = f_i(\bar{x}) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(\bar{x}) - \bar{\varepsilon}_i$$

for any $x \in Q \cap S(\bar{x})$ and all $i = 1, \ldots, p$. Thus we have, for all $x \in Q \cap S(\bar{x})$,

$$\sum_{i=1}^{p}\left[f_i(x) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(x)\right] = \sum_{i=1}^{p}\left[f_i(\bar{x}) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(\bar{x})\right] - \sum_{i=1}^{p}\bar{\varepsilon}_i.$$

(iii) ⇒ (ii): Suppose that $Q \cap S(\bar{x}) = \emptyset$. Then there does not exist $x \in Q$ such that $x \in S(\bar{x})$; that is, there does not exist $x \in Q$ such that

$$f_i(x) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(x) \leqq f_i(\bar{x}) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(\bar{x}) - \bar{\varepsilon}_i$$

for all $i = 1, \ldots, p$. Hence, there does not exist $x \in Q$ such that

$$f_i(x) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(x) \leqq f_i(\bar{x}) - \left(\frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i\right)g_i(\bar{x}) - \bar{\varepsilon}_i, \quad i = 1, \ldots, p,$$
$$f_j(x) - \left(\frac{f_j(\bar{x})}{g_j(\bar{x})} - \varepsilon_j\right)g_j(x) < f_j(\bar{x}) - \left(\frac{f_j(\bar{x})}{g_j(\bar{x})} - \varepsilon_j\right)g_j(\bar{x}) - \bar{\varepsilon}_j, \quad \text{for some } j \in \{1, \ldots, p\}.$$

Therefore, $\bar{x}$ is an $\bar{\varepsilon}$-efficient solution of $(\text{NMCP})_{\bar{v}}$, where $\bar{v} := \left(\frac{f_1(\bar{x})}{g_1(\bar{x})} - \varepsilon_1, \ldots, \frac{f_p(\bar{x})}{g_p(\bar{x})} - \varepsilon_p\right)$.

Assume that $Q \cap S(\bar{x}) \neq \emptyset$. Then, from this assumption

$$\sum_{i=1}^{p} \left[ f_i(x) - \left( \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i \right) g_i(x) \right] \geqq \sum_{i=1}^{p} \left[ f_i(\bar{x}) - \left( \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i \right) g_i(\bar{x}) \right] - \sum_{i=1}^{p} \bar{\varepsilon}_i, \quad (3.1)$$

for any $x \in Q \cap S(\bar{x})$. Suppose to the contrary that $\bar{x}$ is not an $\bar{\varepsilon}$-efficient solution of $(\text{NMCP})_{\bar{v}}$. Then, there exist $\hat{x} \in Q$ and an index $j$ such that

$$f_i(\hat{x}) - \bar{v}_i g_i(\hat{x}) \leqq f_i(\bar{x}) - \bar{v} g_i(\bar{x}) - \bar{\varepsilon}_i, \; i = 1, \dots, p,$$
$$f_j(\hat{x}) - \bar{v}_j g_j(\hat{x}) < f_j(\bar{x}) - \bar{v}_j g_j(\bar{x}) - \bar{\varepsilon}_j, \quad \text{for some } j \in \{1, \dots, p\}.$$

Therefore, $\hat{x} \in Q \cap S(\bar{x})$ and $\sum_{i=1}^{p} \left[ f_i(\hat{x}) - \left( \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i \right) g_i(\hat{x}) \right] < \sum_{i=1}^{p} \left[ f_i(\bar{x}) - \left( \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i \right) g_i(\bar{x}) \right] - \sum_{i=1}^{p} \bar{\varepsilon}_i$, which contradicts the above inequality. Hence, $\bar{x}$ is an $\bar{\varepsilon}$-efficient solution of $(\text{NMCP})_{\bar{v}}$.

We can easily obtain the following proposition:

**Proposition 3.2** *Let $\bar{x} \in Q$ and suppose that $f_i(\bar{x}) \geqq \varepsilon_i g_i(\bar{x}), \;\; i = 1, \dots, p$. Then the following are equivalent:*

*(i) $\bar{x}$ is a weakly $\varepsilon$-efficient solution of (MFP).*

*(ii) $\bar{x}$ is a weakly $\bar{\varepsilon}$-efficient solution of $(NMCP)_{\bar{v}}$, where $\bar{\varepsilon} = (\varepsilon_1 g_1(\bar{x}), \dots, \varepsilon_p g_p(\bar{x}))$ and $\bar{\varepsilon} = (\varepsilon_1 g_1(\bar{x}), \dots, \varepsilon_p g_p(\bar{x}))$.*

*(iii) there exists $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_p) \in \mathbb{R}^p_+ \setminus \{0\}$ such that*

$$\sum_{i=1}^{p} \bar{\lambda}_i \left[ f_i(x) - \left( \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i \right) g_i(x) \right]$$
$$\geqq 0 = \sum_{i=1}^{p} \bar{\lambda}_i \left[ f_i(\bar{x}) - \left( \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i \right) g_i(\bar{x}) \right] - \sum_{i=1}^{p} \bar{\lambda}_i \varepsilon_i g_i(\bar{x}) \text{ for any } x \in Q.$$

*Proof.* (i) ⇔ (ii): The proof is also following the similar lines of Proposition 3.1.

(ii) ⇒ (iii): Let $\phi(x) = (\phi_1(x), ..., \phi_p(x)), \; \forall x \in Q$, where $\varphi_i(x) = f_i(\bar{x}) - \left( \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i \right) g_i(x), \;\; i = 1, \cdots, p$. Then, $\phi_i(x)$, $i = 1,..., p$, are convex. Since $\bar{x} \in Q$ is a weakly $\varepsilon$-efficient solution of $(\text{NMCP})_{\bar{v}}$, where $(\varphi(Q) + \mathbb{R}^p_+) \cap (-\text{int}\mathbb{R}^p_+) = \emptyset$, $(\varphi(Q) + \mathbb{R}^p_+) \cap (-\text{int}\mathbb{R}^p_+) = \emptyset$, and hence, it follows from separation theorem that there exist $\bar{\lambda}_i \geqq 0$, $i = 1, ..., p$, $(\bar{\lambda}_1, \dots, \bar{\lambda}_p) \neq 0$ such that

$$\sum_{i=1}^{p} \bar{\lambda}_i \varphi_i(x) \geqq 0 \quad \forall x \in Q.$$

Thus (iii) holds.

(iii) ⇒ (ii): If (ii) does not hold, that is, $\bar{x}$ is not a weakly $\bar{\varepsilon}$-efficient solution of $(\text{NMCP})_{\bar{v}}$, then (iii) does not hold. □

We present a necessary and sufficient $\varepsilon$-optimality theorem for $\varepsilon$-efficient solution of (MFP) under a constraint qualification, which will be called the closedness assumption.

**Theorem 3.1** *Let $\bar{x} \in Q$ and assume that $Q \cap S(\bar{x}) \neq \emptyset$ and $f_i(\bar{x}) \geqq \varepsilon_i g_i(\bar{x}), \;\; i = 1, \dots, p$ $i = 1, ..., p$. Suppose that*

$$\bigcup_{\lambda_j \geqq 0} \sum_{j=1}^{m} \text{epi}(\lambda_j h_j)^* + \bigcup_{\mu_i \geqq 0} \sum_{i=1}^{p} \left[ \text{epi}(\mu_i f_i)^* + \text{epi}(-\bar{v}_i \mu_i g_i)^* \right]$$

*is closed, where $\bar{v}_i = \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i$, $i = 1, ..., p$. Then the following are equivalent.*

*(i)* $\bar{x}$ *is an* $\varepsilon$*-efficient solution of (MFP).*

*(ii)*
$$\begin{pmatrix}0\\0\end{pmatrix}^{\mathrm{T}} \in \sum_{i=1}^{p}\left[\mathrm{epi}f_i^* + \mathrm{epi}(-\bar{v}_i g_i)^*\right] + \bigcup_{\lambda_j \geqq 0}\sum_{j=1}^{m}\mathrm{epi}(\lambda_j h_j)^* + \bigcup_{\mu_i \geqq 0}\sum_{i=1}^{p}\left[\mathrm{epi}(\mu_i f_i)^* + \mathrm{epi}(-\bar{v}_i\mu_i g_i)^*\right].$$

*(iii) there exist* $\alpha_i \geqq 0$, $u_i \in \partial_{\alpha_i} f_i(\bar{x})$, $\beta_i \geqq 0$, $\gamma_i \in \partial_{\beta_i}(-\bar{v}_i\mu_i g_i)(\bar{x})$, $i = 1, ..., p$, $\lambda_j \geqq 0$, $\gamma_j \geqq 0$, $w_j \in \partial_{\gamma_j}(\lambda_j h_j)(\bar{x})$, $j = 1, ..., m$, $\mu_i \geqq 0$, $q_i \geqq 0$, $s_i \in \partial_{q_i}(\mu_i f_i)(\bar{x})$, $z_i \geqq 0$, $t_i \in \partial_{z_i}(-\bar{v}_i\mu_i g_i)(\bar{x}) i = 1, ..., p$ *such that*

$$0 = \sum_{i=1}^{p}(u_i + \gamma_i) + \sum_{j=1}^{m} w_j + \sum_{i=1}^{p}(s_i + t_i)$$

*and*

$$\sum_{i=1}^{p}(\alpha_i + \beta_i + q_i + z_i) + \sum_{j=1}^{m}\gamma_j = \sum_{i=1}^{p}\varepsilon_i(1+\mu_i)g_i(\bar{x}) + \sum_{j=1}^{m}\lambda_j h_j(\bar{x}).$$

*Proof.* Let $h_0(x) = \sum_{i=1}^{p}\left[f_i(x) - \bar{v}_i g_i(x)\right]$.

(i) ⇔ (by Proposition 3.1) $h_0(x) \geqq 0, \forall x \in Q \cap S(\bar{x})$.

⇔ $\{x | f_i(x) - \bar{v}_i g_i(x) \leqq 0, i = 1, ..., p, h_j(x) \leqq 0, j = 1, ..., m\} \subset \{x \mid h_0(x) \geqq 0\}$.

⇔ (by lemma 2.3)

$$\begin{pmatrix}0\\0\end{pmatrix}^{\mathrm{T}} \in \sum_{i=1}^{p}\left[\mathrm{epi}f_i^* + \mathrm{epi}(-\bar{v}_i g_i)^*\right] + \mathrm{cl}\left\{\bigcup_{\lambda_j \geqq 0}\sum_{j=1}^{m}\mathrm{epi}(\lambda_j h_j)^* + \bigcup_{\mu_i \geqq 0}\sum_{i=1}^{p}\left[\mathrm{epi}(\mu_i f_i)^* + \mathrm{epi}(-\bar{v}_i\mu_i g_i)^*\right]\right\}.$$

Thus by the closedness assumption, (i) is equivalent to (ii).

(ii) ⇔ (iii): (ii) ⇔ (by Lemma 2.1), there exist $\alpha_i \geqq 0$, $u_i \in \partial_{\alpha_i}(\mu_i f_i)(\bar{x})$, $i = 1, ..., p$, $\beta_i \geqq 0$, $\gamma_i \in \partial_{\beta_i}(-\bar{v}_i\mu_i g_i)(\bar{x})$, $i = 1, ..., p$, $\lambda_j \geqq 0$, $\gamma_j \geqq 0$, $w_j \in \partial_{\gamma_j}(\lambda_j h_j)(\bar{x})$, $j = 1, ..., m$, $\mu_i \geqq 0$, $q_i \geqq 0$, $s_i \in \partial_{q_i}(\mu_i f_i)(\bar{x})$, $i = 1, ..., p$, $z_i \geqq 0$, $t_i \in \partial_{z_i}(-\bar{v}_i\mu_i g_i)(\bar{x})$, $i = 1, ..., p$ such that

$$\begin{aligned}\begin{pmatrix}0\\0\end{pmatrix}^{\mathrm{T}} = &\sum_{i=1}^{p}\left[\begin{pmatrix}u_i\\ \langle u_i,\bar{x}\rangle + \alpha_i - f_i(\bar{x})\end{pmatrix}^{\mathrm{T}} + \begin{pmatrix}\gamma_i\\ \langle \gamma_i,\bar{x}\rangle + \beta_i - (-\bar{v}_i g_i)(\bar{x})\end{pmatrix}^{\mathrm{T}}\right]\\ &+\sum_{j=1}^{m}\begin{pmatrix}w_j\\ \langle w_j,\bar{x}\rangle + \gamma_j - (\lambda_j h_j)(\bar{x})\end{pmatrix}^{\mathrm{T}}\\ &+\sum_{i=1}^{p}\left[\begin{pmatrix}s_i\\ \langle s_i,\bar{x}\rangle + q_i - (\mu_i f_i)(\bar{x})\end{pmatrix}^{\mathrm{T}} + \begin{pmatrix}t_i\\ \langle t_i,\bar{x}\rangle + z_i - (-\bar{v}_i\mu_i g_i)(\bar{x})\end{pmatrix}^{\mathrm{T}}\right].\end{aligned}$$

⇔ there exist $\alpha_i \geqq 0$, $u_i \in \partial_{\alpha_i}(\mu_i f_i)(\bar{x})$, $\beta_i \geqq 0$, $\gamma_i \in \partial_{\beta_i}(-\bar{v}_i\mu_i g_i)(\bar{x})$, $i = 1, ..., p$, $\lambda_j \geqq 0$, $\gamma_j \geqq 0$, $w_j \in \partial_{\gamma_j}(\lambda_j h_j)(\bar{x})$, $j = 1, ..., m$, $\mu_i \geqq 0$, $q_i \geqq 0$, $s_i \in \partial_{q_i}(\mu_i f_i)(\bar{x})$, $z_i \geqq 0$, $t_i \in \partial_{z_i}(-\bar{v}_i\mu_i g_i)(\bar{x}) i = 1, ..., p$ such that

$$0=\sum_{i=1}^{p}(u_i+\gamma_i)+\sum_{j=1}^{m}w_j+\sum_{i=1}^{p}(s_i+t_i)$$

and$\sum_{i=1}^{p}(\alpha_i+\beta_i+q_i+z_i)+\sum_{j=1}^{m}\gamma_j=\sum_{i=1}^{p}\left[f_i(\bar{x})-\bar{v}_i g_i(\bar{x})+(\mu_i f_i)(\bar{x})-(\bar{v}_i\mu_i g_i)(\bar{x})+\sum_{j=1}^{m}\lambda_j h_j(\bar{x})\right]$.

$\Leftrightarrow$ (iii) holds. □

Now we give a necessary and sufficient $\varepsilon$-optimality theorem for $\varepsilon$-efficient solution of (MFP) which holds without any constraint qualification.

**Theorem 3.2** *Let $\bar{x}\in Q$. Suppose that $Q\cap S(\bar{x})\neq\emptyset$and $f_i(\bar{x})\geqq\varepsilon_i g_i(\bar{x}),\ \ i=1,\dots,p$, i = 1, ..., p. Then $\bar{x}$is an $\varepsilon$-efficient solution of (MFP) if and only if there exist $\alpha_i\geqq 0$, $u_i\in\partial_{\alpha_i}(\mu_i f_i)(\bar{x})$, i = 1, ..., p, $\beta_i\geqq 0$, $\gamma_i\in\partial_{\beta_i}(-\bar{v}_i\mu_i g_i)(\bar{x})$, i = 1, ..., p, $\lambda_j^n\geqq 0$, $\gamma_j^n\geqq 0$, $w_j^n\in\partial_{\gamma_j^n}(\lambda_j^n h_j)(\bar{x})$, j = 1, ..., m, $\mu_k^n\geqq 0$, $q_k^n\geqq 0$, $s_k^n\in\partial_{q_k^n}(\mu_k^n f_k)(\bar{x})$, $z_k^n\geqq 0$, $t_k^n\in\partial_{z_k^n}(-\bar{v}_k\mu_k^n g_k)(\bar{x})$, k = 1, ..., p such that*

$$0=\sum_{i=1}^{p}(u_i+\gamma_i)+\lim_{n\to\infty}\left[\sum_{j=1}^{m}w_j^n+\sum_{k=1}^{p}(s_k^n+t_k^n)\right]$$

*and*

$$\sum_{i=1}^{p}\varepsilon_i g_i(\bar{x}) = \sum_{i=1}^{p}(\alpha_i+\beta_i)+\lim_{n\to\infty}\left\{\sum_{j=1}^{m}\left[\gamma_j^n-(\lambda_j^n h_j)(\bar{x})\right]+\sum_{k=1}^{p}\left[q_k^n+z_k^n-\mu_k^n\varepsilon_k g_k(\bar{x})\right]\right\}.$$

*Proof.* $\bar{x}$ is an $\varepsilon$-efficient solution of (MFP)

$\Leftrightarrow$ (from the proof of Theorem 3.1)

$$\begin{pmatrix}0\\0\end{pmatrix}^{\mathrm{T}}\in\sum_{i=1}^{p}\left[\mathrm{epi}f_i^*+\mathrm{epi}(-\bar{v}_i g_i)^*\right]+\mathrm{cl}\left\{\bigcup_{\lambda_j\geqq 0}\sum_{j=1}^{m}\mathrm{epi}(\lambda_j h_j)^*+\bigcup_{\mu_i\geqq 0}\sum_{i=1}^{p}\left[\mathrm{epi}(\mu_i f_i)^*+\mathrm{epi}(-\bar{v}_i\mu_i g_i)^*\right]\right\}.$$

$\Leftrightarrow$ (by Lemma 2.1) there exist $\alpha_i\geqq 0$, $u_i\in\partial_{\alpha_i}(\mu_i f_i)(\bar{x})$, i = 1, ..., p, $\beta_i\geqq 0$, $\gamma_i\in\partial_{\beta_i}(-\bar{v}_i\mu_i g_i)(\bar{x})$, i = 1, ..., p, $\lambda_j^n\geqq 0$, $\gamma_j^n\geqq 0$, $w_j^n\in\partial_{\gamma_j^n}(\lambda_j^n h_j)(\bar{x})$, j = 1, ..., m, $\mu_k^n\geqq 0$, $s_k^n\in\partial_{q_k^n}(\mu_k^n f_k)(\bar{x})$, $s_k^n\in\partial_{q_k^n}(\mu_k^n f_k)(\bar{x})$, $z_k^n\geqq 0$, $t_k^n\in\partial_{z_k^n}(-\bar{v}_k\mu_k^n g_k)(\bar{x})$, k = 1, ..., p, such that

$$\begin{aligned}\begin{pmatrix}0\\0\end{pmatrix}^{\mathrm{T}}=&\sum_{i=1}^{p}\left[\begin{pmatrix}u_i\\ \langle u_i,\bar{x}\rangle+\alpha_i-f_i(\bar{x})\end{pmatrix}^{\mathrm{T}}+\begin{pmatrix}\gamma_i\\ \langle\gamma_i,\bar{x}\rangle+\beta_i-(-\bar{v}_i g_i)(\bar{x})\end{pmatrix}^{\mathrm{T}}\right]\\&+\lim_{n\to\infty}\left\{\sum_{j=1}^{m}\begin{pmatrix}w_j^n\\ \langle w_j^n,\bar{x}\rangle+\gamma_j^n-(\lambda_j^n h_j)(\bar{x})\end{pmatrix}^{\mathrm{T}}\right.\\&\left.+\sum_{k=1}^{p}\left[\begin{pmatrix}s_k^n\\ \langle s_k^n,\bar{x}\rangle+q_k^n-(\mu_k^n f_k)(\bar{x})\end{pmatrix}^{\mathrm{T}}+\begin{pmatrix}t_k^n\\ \langle t_k^n,\bar{x}\rangle+z_k^n-(-\bar{v}_k\mu_k^n g_i)(\bar{x})\end{pmatrix}^{\mathrm{T}}\right]\right\}.\end{aligned}$$

⇔ there exist $\alpha_i \geqq 0$, $u_i \in \partial_{\alpha_i}(\mu_i f_i)(\bar{x})$, $i = 1, ..., p$, $\beta_i \geqq 0$, $\gamma_i \in \partial_{\beta_i}(-\bar{v}_i \mu_i g_i)(\bar{x})$, $i = 1, ...,$ $p$, $\lambda_j^n \geqq 0$, $\gamma_j^n \geqq 0$, $w_j^n \in \partial_{\gamma_j^n}(\lambda_j^n h_j)(\bar{x})$, $j = 1, ..., m$, $\mu_k^n \geqq 0$, $q_k^n \geqq 0$, $s_k^n \in \partial_{q_k^n}(\mu_k^n f_k)(\bar{x})$, $t_k^n \in \partial_{z_k^n}(-\bar{v}_k \mu_k^n g_k)(\bar{x})$, $t_k^n \in \partial_{z_k^n}(-\bar{v}_k \mu_k^n g_k)(\bar{x})$, $k = 1, ..., p$, such that

$$0 = \sum_{i=1}^{p}(u_i + \gamma_i) + \lim_{n\to\infty}\left[\sum_{j=1}^{m} w_j^n + \sum_{k=1}^{p}(s_k^n + t_k^n)\right]$$

and

$$\sum_{i=1}^{p}\varepsilon_i g_i(\bar{x}) = \sum_{i=1}^{p}(\alpha_i + \beta_i) + \lim_{n\to\infty}\left\{\sum_{j=1}^{m}\left[\gamma_j^n - (\lambda_j^n h_j)(\bar{x})\right]\right.$$
$$\left. + \sum_{k=1}^{p}\left[q_k^n + z_k^n - \mu_k^n \varepsilon_k g_k(\bar{x})\right]\right\}.$$

We present a necessary and sufficient $\varepsilon$-optimality theorem for weakly $\varepsilon$-efficient solution of (MFP) under a constraint qualification.

**Theorem 3.3** *Let* $\bar{x} \in Q$*and assume that* $f_i(\bar{x}) \geqq \varepsilon_i g_i(\bar{x}),\ \ i = 1, \dots, p$, $i = 1, ..., p$, *and* $\bigcup_{\lambda_j \geqq 0}\sum_{j=1}^{m}\text{epi}(\lambda_j h_j)^*$*is closed. Then the following are equivalent.*

*(i)* $\bar{x}$*is a weakly* $\varepsilon$*-efficient solution of (MFP).*

*(ii) there exist* $\mu_i \geqq 0$, $i = 1, ..., p$, $\sum_{i=1}^{p}\mu_i = 1$*such that*

$$\begin{pmatrix}0\\0\end{pmatrix}^{\mathrm{T}} \in \sum_{i=1}^{p}\left[\text{epi}(\mu_i f_i)^* + \text{epi}(-\bar{v}_i \mu_i g_i)^*\right] + \bigcup_{\lambda_j \geqq 0}\sum_{j=1}^{m}\text{epi}(\lambda_j h_j)^*,$$

*where* $\bar{v}_i = \dfrac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i$, $i = 1, ..., p$.

*(iii) there exist* $\mu_i \geqq 0$, $\sum_{i=1}^{p}\mu_i = 1$, $\alpha_i \geqq 0$, $u_i \in \partial_{\alpha_i}(\mu_i f_i)(\bar{x})$, $\beta_i \geqq 0$, $\gamma_i \in \partial_{\beta_i}(-\bar{v}_i \mu_i g_i)(\bar{x})$, $i = 1, ..., p$, $\lambda_j \geqq 0$, $\gamma_j \geqq 0$, $w_j \in \partial_{\gamma_j}(\lambda_j h_j)(\bar{x})$, $j = 1, ..., m$, *such that*

$$0 = \sum_{i=1}^{p}(u_i + \gamma_i) + \sum_{j=1}^{m} w_j$$

*and*

$$\sum_{i=1}^{p}\mu_i \varepsilon_i g_i(\bar{x}) = \sum_{i=1}^{p}(\alpha_i + \beta_i) + \sum_{j=1}^{m}\left[\gamma_j - (\lambda_j h_j)(\bar{x})\right].$$

*Proof.* (i) ⇔ (ii): $\bar{x}$ is a weakly $\varepsilon$-efficient solution of (MFP)

⇔ (by Proposition 3.2) there exist $\mu_i \geqq 0$, $i = 1, ..., p$, $\sum_{i=1}^{p}\mu_i = 1$ such that

$$\sum_{i=1}^{p}\mu_i[f_i(x) - \bar{v}_i g_i(x)] \geqq 0 \quad \forall x \in Q$$

⇔ there exist $\mu_i \geqq 0$, $i = 1, ..., p$, $\sum_{i=1}^{p}\mu_i = 1$ such that

$$\{x | h_j(x) \leqq 0, j = 1, \dots, m\} \subset \{x | \sum_{i=1}^{p}\mu_i\left[f_i(x) - \bar{v}_i g_i(x)\right] \geqq 0\}$$

⇔ (by Lemma 2.3) there exist $\mu_i \geqq 0$, $i = 1, ..., p$, $\sum_{i=1}^{p} \mu_i = 1$ such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{\mathrm{T}} \in \sum_{i=1}^{p} \left[ \mathrm{epi}(\mu_i f_i)^* + \mathrm{epi}(-\bar{v}_i \mu_i g_i)^* \right] + \mathrm{cl} \left\{ \bigcup_{\lambda_j \geqq 0} \sum_{j=1}^{m} \mathrm{epi}(\lambda_j h_j)^* \right\}.$$

Thus, by the closedness assumption, (i) is equivalent to (ii).

(ii) ⇔ (iii): (ii) ⇔ (by Lemma 2.1) there exist $\mu_i \geqq 0$, $\sum_{i=1}^{p} \mu_i = 1$, $\alpha_i \geqq 0$, $u_i \in \partial_{\alpha_i}(\mu_i f_i)(\bar{x})$, $\beta_i \geqq 0$, $\gamma_i \in \partial_{\beta_i}(-\bar{v}_i \mu_i g_i)(\bar{x})$, $i = 1, ..., p$, $\lambda_j \geqq 0$, $\gamma_j \geqq 0$, $w_j \in \partial_{\gamma_j}(\lambda_j h_j)(\bar{x})$, $j = 1, ..., m$, *such that*

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{\mathrm{T}} = \sum_{i=1}^{p} \left[ \begin{pmatrix} u_i \\ \langle u_i, \bar{x} \rangle + \alpha_i - (\mu_i f_i)(\bar{x}) \end{pmatrix}^{\mathrm{T}} + \begin{pmatrix} \gamma_i \\ \langle \gamma_i, \bar{x} \rangle + \beta_i - (-\bar{v}_i \mu_i g_i)(\bar{x}) \end{pmatrix}^{\mathrm{T}} \right] + \sum_{j=1}^{m} \begin{pmatrix} w_j \\ \langle w_j, \bar{x} \rangle + \gamma_j - (\lambda_j h_j)(\bar{x}) \end{pmatrix}^{\mathrm{T}}.$$

⇔ (iii) holds. □

Now, we propose a necessary and sufficient $\varepsilon$-optimality theorem for weakly $\varepsilon$-efficient solution of (MFP) which holds without any constraint qualification.

**Theorem 3.4** *Let* $\bar{x} \in Q$ *and assume that* $f_i(\bar{x}) \geqq \varepsilon_i g_i(\bar{x}), \quad i = 1, \ldots, p$. *Then* $\bar{x}$ *is a weakly* $\varepsilon$*-efficient solution of (MFP) if and only if there exist* $\mu_i \geqq 0$, $i = 1, ..., p$, $\sum_{i=1}^{p} \mu_i = 1$, $\alpha_i \geqq 0$, $u_i \in \partial_{\alpha_i}(\mu_i f_i)(\bar{x})$, $i = 1, ..., p$, $\beta_i \geqq 0$, $\gamma_i \in \partial_{\beta_i}(-\bar{v}_i \mu_i g_i)(\bar{x})$, $i = 1, ..., p$, $\gamma_j^n \geqq 0$, $\gamma_j^n \geqq 0$, $w_j^n \in \partial_{\gamma_j^n}(\lambda_j^n h_j)(\bar{x})$, $j = 1, ..., m$, *such that*

$$0 = \sum_{i=1}^{p} (u_i + \gamma_i) + \lim_{n \to \infty} \sum_{j=1}^{m} w_j^n$$

*and*

$$\sum_{i=1}^{p} \mu_i \varepsilon_i g_i(\bar{x}) = \sum_{i=1}^{p} (\alpha_i + \beta_i) + \lim_{n \to \infty} \sum_{j=1}^{m} \left[ \gamma_j^n - (\lambda_j^n h_j)(\bar{x}) \right].$$

*Proof.* $\bar{x}$ is a weakly $\varepsilon$-efficient solution of (MFP)

⇔ ((from the proof of Theorem 3.3) there exist $\mu_i \geqq 0$, $i = 1, ..., p$, $\sum_{i=1}^{p} \mu_i = 1$ such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{\mathrm{T}} \in \sum_{i=1}^{p} \left[ \mathrm{epi}(\mu_i f_i)^* + \mathrm{epi}(-\bar{v}_i \mu_i g_i)^* \right] + \mathrm{cl} \left\{ \bigcup_{\lambda_j \geqq 0} \sum_{j=1}^{m} \mathrm{epi}(\lambda_j h_j)^* \right\}.$$

⇔ (by Lemma 2.1) there exist $\mu_i \geqq 0$, $i = 1, ..., p$, $\sum_{i=1}^{p} \mu_i = 1$, $\alpha_i \geqq 0$, $u_i \in \partial_{\alpha_i}(\mu_i f_i)(\bar{x})$, $i = 1, ..., p$, $\beta_i \geqq 0$, $\gamma_i \in \partial_{\beta_i}(-\bar{v}_i \mu_i g_i)(\bar{x})$, $i = 1, ..., p$, $\lambda_j^n \geqq 0$, $\gamma_j^n \geqq 0$, $w_j^n \in \partial_{\gamma_j^n}(\lambda_j^n h_j)(\bar{x})$, $j = 1, ..., m$, such that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}^{\mathrm{T}} = \sum_{i=1}^{p} \left[ \begin{pmatrix} u_i \\ \langle u_i, \bar{x} \rangle + \alpha_i - (\mu_i f_i)(\bar{x}) \end{pmatrix}^{\mathrm{T}} + \begin{pmatrix} \gamma_i \\ \langle \gamma_i, \bar{x} \rangle + \beta_i - (-\bar{v}_i \mu_i g_i)(\bar{x}) \end{pmatrix}^{\mathrm{T}} \right] + \lim_{n \to \infty} \left\{ \sum_{j=1}^{m} \begin{pmatrix} w_j^n \\ \langle w_j^n, \bar{x} \rangle + \gamma_j^n - (\lambda_j^n h_j)(\bar{x}) \end{pmatrix}^{\mathrm{T}} \right\}.$$

$\Leftrightarrow$ there exist $\mu_i \geqq 0$, $i = 1, ..., p$, $\sum_{i=1}^{p} \mu_i = 1$, $\alpha_i \geqq 0$, $u_i \in \partial_{\alpha_i}(\mu_i f_i)(\bar{x})$, $i = 1, ..., p$, $\beta_i \geqq 0$, $\gamma_i \in \partial_{\beta_i}(-\bar{v}_i \mu_i g_i)(\bar{x})$, $i = 1, ..., p$, $\lambda_j^n \geqq 0$, $\gamma_j^n \geqq 0$, $w_j^n \in \partial_{\gamma_j^n}(\lambda_j^n h_j^n)(\bar{x})$, $j = 1, ..., m$, such that

$$0 = \sum_{i=1}^{p} (u_i + \gamma_i) + \lim_{n\to\infty} \sum_{j=1}^{m} w_j^n$$

and

$$\sum_{i=1}^{p} \mu_i \varepsilon_i g_i(\bar{x}) = \sum_{i=1}^{p} (\alpha_i + \beta_i) + \lim_{n\to\infty} \sum_{j=1}^{m} \left[\gamma_j^n - (\gamma_j^n h_j)(\bar{x})\right].$$

□

Now, we give examples illustrating Theorems 3.1, 3.2, 3.3, and 3.4.

**Example 3.1** *Consider the following MFP:*

$$\begin{aligned} &(\text{MFP})_1 \ \textit{Minimize} \quad \left(x_1, \frac{x_2}{x_1}\right) \\ &\qquad\qquad \textit{subject to} \quad (x_1, x_2) \in Q := \{(x_1, x_2) \in \mathbb{R}^2 \mid -x_1 + 1 \leqq 0, \quad -x_2 + 1 \leqq 0\}. \end{aligned}$$

*Let* $\varepsilon = (\varepsilon_1, \varepsilon_2) = (\frac{1}{2}, \frac{1}{2})$, *and* $f_1(x_1, x_2) = x_1$, $g_1(x_1, x_2) = 1$, $f_2(x_1, x_2) = x_2$, $g_2(x_1, x_2) = x_1$, $h_1(x_1, x_2) = -x_1 + 1$ *and* $h_2(x_1, x_2) = -x_2 + 1$.

*(1)Let* $(\bar{x}_1, \bar{x}_2) = (\frac{3}{2}, \frac{9}{4})$. *Then* $(\bar{x}_1, \bar{x}_2)$*is an* $\varepsilon$*-efficient solution of* $(\text{MFP})_1$.

*Let* $\bar{v}_1 = \dfrac{f_1(\bar{x}_1, \bar{x}_2)}{g_1(\bar{x}_1, \bar{x}_2)} - \varepsilon_1$*and* $\bar{v}_2 = \dfrac{f_2(\bar{x}_1, \bar{x}_2)}{g_2(\bar{x}_1, \bar{x}_2)} - \varepsilon_2$. *Then* $\bar{v}_1 = \bar{v}_2 = 1$, *and*

$$\begin{aligned} &Q \cap S(\bar{x}_1, \bar{x}_2) \\ &= Q \cap \{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 | f_1(\bar{x}_1, \bar{x}_2) - \bar{v}_1 g_1(\bar{x}_1, \bar{x}_2) \leqq 0, f_2(\bar{x}_1, \bar{x}_2) - \bar{v}_2 g_2(\bar{x}_1, \bar{x}_2) \leqq 0\} \\ &= \{(1, 1)\}. \end{aligned}$$

*Thus,* $Q \cap S(\bar{x}_1, \bar{x}_2) \neq \emptyset$. *It is clear that* $f_1(\bar{x}_1, \bar{x}_2) \geqq \varepsilon_1 g_1(\bar{x}_1, \bar{x}_2)$*and* $f_2(\bar{x}_1, \bar{x}_2) \geqq \varepsilon_2 g_2(\bar{x}_1, \bar{x}_2)$. *Let* $A = \bigcup_{\lambda_1 \geq 0, \lambda_2 \geq 0} \sum_{j=1}^{2} \text{epi}(\lambda_j h_j)^* + \bigcup_{\mu_1 \geq 0, \mu_2 \geq 0} \sum_{j=1}^{2} [\text{epi}(\mu_j f_j)^* + \text{epi}(-\bar{v}_i \mu_i g_i)^*]$. *Then*

$$\begin{aligned} A &= \bigcup_{\substack{\lambda_1 \geq 0,\, \lambda_2 \geq 0 \\ \mu_1 \geq 0,\, \mu_2 \geq 0}} \text{epi}\left(\sum_{j=1}^{2} \lambda_j h_j + \sum_{i=1}^{2} \mu_i (f_i - \bar{v}_i g_i)\right)^* \\ &= \text{cone co}\{(-1, 0, -1), (0, -1, -1), (1, 0, 1), (-1, 1, 0), (0, 0, 1)\}, \end{aligned}$$

*where coD is the convexhull of a set D and cone coD is the cone generated by coD. Thus A is closed. Let* $B = \sum_{i=1}^{2} [\text{epi} f_i^* + \text{epi}(-\bar{v}_i g_i)^*] + A$. *Then*

$B = \{(1, 0)\} \times [0, \infty) + \{(0, 0)\} \times [1, \infty) + \{(0, 1)\} \times [0, \infty) + \{(-1, 0)\} \times [0, \infty) + A$. *Since* $(0, -1, -1) \in A$, $(0, 0, 0) \in B$. *Thus (ii) of Theorem 3.1 holds.* Let $\alpha_1 = \beta_1 = \gamma_1 = q_1 = z_1 = \alpha_2 = \beta_2 = \gamma_2 = q_2 = z_2 = 0$, *and let* $\mu_1 = \mu_2 = 1$, *and* $\lambda_1 = 0$ *and* $\lambda_1 = 2$. *Moreover,* $\partial f_2(\bar{x}_1, \bar{x}_2) = \{(0, 1)\}$, $\partial f_2(\bar{x}_1, \bar{x}_2) = \{(0, 1)\}$, $\partial(-\bar{v}_1 g_1)(\bar{x}_1, \bar{x}_2) = \{(0, 0)\}$, $\partial(-\bar{v}_2 g_2)(\bar{x}_1, \bar{x}_2) = \{(-1, 0)\}$, $\partial(\lambda_2 h_2)(\bar{x}_1, \bar{x}_2) = \{(0, -2)\}$,, $\partial(\lambda_2 h_2)(\bar{x}_1, \bar{x}_2) = \{(0, -2)\} \partial(\mu_1 f_1)(\bar{x}_1, \bar{x}_2) = \{(1, 0)\}$, $\partial(-\bar{v}_1 \mu_1 g_1)(\bar{x}_1, \bar{x}_2) = \{(0, 0)\}$, $\partial(-\bar{v}_1 \mu_1 g_1)(\bar{x}_1, \bar{x}_2) = \{(0, 0)\}$, $\partial(-\bar{v}_2 \mu_2 g_2)(\bar{x}_1, \bar{x}_2) = \{(-1, 0)\}$.

*Thus,* $\sum_{i=1}^{2} \partial(f_i - \bar{v}_i g_i)(\bar{x}_1, \bar{x}_2) + \sum_{i=1}^{2} \partial(\lambda_i h_i)(\bar{x}_1, \bar{x}_2) + \sum_{i=1}^{2} \partial(\mu_i f_i - \bar{v}_i \mu_i g_i)(\bar{x}_1, \bar{x}_2) = \{(0, 0)\}$*and* $\sum_{i=1}^{2} (\alpha_i + \beta_i + q_i + z_i) + \sum_{j=1}^{2} \gamma_j = 0 = \sum_{i=1}^{2} \varepsilon_i (1 + \mu_i) g_i(\bar{x}_1, \bar{x}_2) + \sum_{i=1}^{2} \lambda_j h_j(\bar{x}_1, \bar{x}_2)$.

*Thus, (iii) of Theorem 3.1 holds.*

*(2) Let* $(\tilde{x}_1, \tilde{x}_2) = (\frac{3}{2}, \frac{15}{4})$*. Then* $(\tilde{x}_1, \tilde{x}_2)$*is not an ε-efficient solution of* $(\text{MFP})_1$*, but* $(\tilde{x}_1, \tilde{x}_2)$*is a weakly ε-efficient solution of* $(\text{MFP})_1$.

*Let* $C = \bigcup_{\substack{\lambda_1 \geq 0, \\ \lambda_2 \geqq 0}} \sum_{i=1}^{2} \text{epi}(\lambda_i h_i)^*$*. Then*

$$C = \text{cone co}\{(-1, 0, -1), (0, -1, -1), (0, 0, 1)\}.$$

*Hence, C is closed. Moreover,* $f_1(\tilde{x}_1, \tilde{x}_2) - \varepsilon_1 g_1(\tilde{x}_1, \tilde{x}_2) = 1 \geqq 0$, *and* $f_2(\tilde{x}_1, \tilde{x}_2) - \varepsilon_2 g_2(\tilde{x}_1, \tilde{x}_2) = 3 \geqq 0$*. Let* $\bar{v}_1 = \dfrac{f_1(\bar{x}_1, \bar{x}_2)}{g_1(\bar{x}_1, \bar{x}_2)} - \varepsilon_1$*and* $\bar{v}_2 = \dfrac{f_2(\bar{x}_1, \bar{x}_2)}{g_2(\bar{x}_1, \bar{x}_2)} - \varepsilon_2$*. Then,* $\tilde{v}_2 = 2, \tilde{v}_2 = 2$*. Let* $\mu_1 = 1$ *and* $\mu_2 = 1$*. Then,*

$$\sum_{i=1}^{2} [\text{epi}(\mu_i f_i)^* + \text{epi}(-\tilde{v}_i \mu_i g_i)^*]$$
$$= \{(1, 0)\} \times \mathbb{R}_+ + \{(0, 0)\} \times [1, \infty) + \{(0, 0)\} \times \mathbb{R}_+.$$

*Since* (-1, 0,-1) ∈ *C*, $(0, 0, 0) \in \sum_{i=1}^{2} [\text{epi}(\mu_i f_i)^* + \text{epi}(-\tilde{v}_i \mu_i g_i)^*] + C$*. So, (ii) of Theorem 3.3 holds. Let* $\alpha_1 = \beta_1 = \gamma_1 = \alpha_2 = \beta_2 = \gamma_2 = 0$, $\lambda_1 = 1$ *and* $\lambda_2 = 0$*. Then,*

$$\sum_{i=1}^{2} \partial(\mu_i f_i)(\tilde{x}_1, \tilde{x}_2) + \sum_{i=1}^{2} \partial(-\tilde{v}_i \mu_i g_i)(\tilde{x}_1, \tilde{x}_2) + \sum_{j=1}^{2} \partial(\lambda_j h_j)(\tilde{x}_1, \tilde{x}_2) = \{(0, 0)\}$$

*and*

$$\sum_{i=1}^{2} \mu_i \varepsilon_i g_i(\tilde{x}_1, \tilde{x}_2) = \frac{1}{2} = \sum_{i=1}^{2} (\alpha_i + \beta_i) + \sum_{j=1}^{2} [\gamma_j - (\lambda_j h_j)(\tilde{x}_1, \tilde{x}_2)].$$

*Thus, (iii) of Theorem 3.3 holds.*

**Example 3.2** *Consider the following MFP:*

$$(\text{MFP})_2 \ \textit{Minimize} \left(-x_1 + 1, \frac{x_2}{-x_1 + 1}\right)$$
$$\textit{subject to} \ [\max\{0, x_1\}]^2 \leqq 0, \quad -x_2 + 1 \leqq 0.$$

*Let* $\varepsilon = (\varepsilon_1, \varepsilon_2) = (\frac{1}{2}, \frac{1}{2})$*, and* $f_1(x_1, x_2) = -x_1 + 1$, $g_1(x_1, x_2) = 1$, $f_2(x_1, x_2) = x_2$, $g_2(x_1, x_2) = -x_1 + 1$, $h_1(x_1, x_2) = [\max\{0, x_1\}]^2$ *and* $h_2(x_1, x_2) = -x_2 + 1$.

*(1) Let* $(\bar{x}_1, \bar{x}_2) = (-\frac{1}{2}, \frac{9}{4})$*. Then,* $(\bar{x}_1, \bar{x}_2)$*is an ε-efficient solution of* $(\text{MFP})_2$*. Let* $A = \bigcup_{\substack{\lambda_1 \geqq 0, \\ \lambda_2 \geqq 0}} \sum_{j=1}^{2} \text{epi}(\lambda_j h_j)^* + \bigcup_{\substack{\mu_1 \geqq 0, \\ \mu_2 \geqq 0}} \sum_{i=1}^{2} [\text{epi}(\mu_i f_i)^* + \text{epi}(-\bar{v}_i \mu_i g_i)^*]$*. Then,* cl*A* = cone co{(0, -1, -1), (1, 0, 0), (-1, 0, 0), (1, 1, 1), (0, 0, 1)}*. Here,* (1, 0, 0) ∈ cl*A, but* (1, 0, 0) ∈ *A, where* cl*A is the closure of the set A. Thus, A is not closed. Let* $Q = \{(x_1, x_2) \in \mathbb{R}^n \mid h_1(x_1, x_2) \leqq 0, h_2(x_1, x_2) \leqq 0\}$*. Then,* $Q \cap S(\bar{x}_1, \bar{x}_2) = \{(1, 1)\}$*. Let* $v_i = \frac{f_i(\bar{x}_1, \bar{x}_2)}{g_i(\bar{x}_1, \bar{x}_2)} - \varepsilon_i$, $i = 1, 2$*. Then,* $\bar{v}_1 = \bar{v}_2 = 1$*. Let* $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$, $\lambda_1^n = 0$, $\lambda_2^n = 1$, $\gamma_1^n = \gamma_2^n = 0$, $w_1^n = (0, 0)$, $w_2^n = (0, -1)$*. Let* $u_1 = (-1, 0)$ $u_2 = (0, 1)$, $y_1 = (0, 0)$ *and* $y_2 = (1, 0)$*. Let* $q_1^n = q_2^n = z_1^n = z_1^n = 0$, *and* $\mu_1^n = \mu_2^n = 0$*. Let* $s_1^n = s_2^n = (0, 0)$*and* $t_1^n = t_2^n = \{(0, 0)\}$*. Then,* $u_i \in \partial f_i(\bar{x}_1, \bar{x}_2)$, $i = 1, 2$, $y_i \in \partial(-\bar{v}_i g_i)(\bar{x}_1, \bar{x}_2)$, $i = 1, 2$, $w_j^n \in \partial(\lambda_j^n h_j)(\bar{x}_1, \bar{x}_2)$, $j = 1, 2$, $s_k^n \in \partial(\mu_k^n f_k)(\bar{x}_1, \bar{x}_2)$, $k = 1, 2$, *and* $t_k^n \in \partial(-\bar{v}_k \mu_k^n g_k)(\bar{x}_1, \bar{x}_2)$, $k = 1, 2$*. Moreover,*

$$0 = \sum_{i=1}^{2} (u_i + y_i) + \lim_{n\to\infty} \left[ \sum_{j=1}^{2} w_j^n + \sum_{i=1}^{2} (s_k^n + t_k^n) \right]$$

*and*

$$\sum_{i=1}^{2} \varepsilon_i g_i(\bar{x}_1, \bar{x}_2) = \sum_{i=1}^{2} (\alpha_i + \beta_i) + \lim_{n\to\infty} \sum_{j=1}^{2} [\gamma_j^n - (\lambda_j^n h_j)(\bar{x}_1, \bar{x}_2)] + \sum_{k=1}^{2} [q_k^n + z_k^n - \mu_k^n \varepsilon_k g_k(\bar{x}_1, \bar{x}_2)].$$

*Thus, Theorem 3.2 holds.*

*(2) Let* $(\tilde{x}_1, \tilde{x}_2) = (-\frac{1}{2}, \frac{15}{4})$*. Then,* $(\tilde{x}_1, \tilde{x}_2)$ *is a weakly ε-efficient solution of* $(\mathrm{MFP})_2$*, but not an ε-efficient solution of* $(\mathrm{MFP})_2$*. Let* $B = \bigcup_{\lambda_1 \geqq 0, \lambda_2 \geqq 0} \mathrm{epi}(\sum_{i=1}^{2} \lambda_i h_i)^*$*. Then,* $\mathrm{cl}B = \mathrm{cone\ co}$ {(0, -1, -1), (1, 0, 0), (0, 0, 1)}*. However,* (1, 0, 0) ∉ *B. Thus, B is not closed. Moreover,* $f_2(\tilde{x}_1, \tilde{x}_2) - \varepsilon_2 g_2(\tilde{x}_1, \tilde{x}_2) = 3 \geqq 0$, $f_2(\tilde{x}_1, \tilde{x}_2) - \varepsilon_2 g_2(\tilde{x}_1, \tilde{x}_2) = 3 \geqq 0$. *Let* $\tilde{v}_2 = \frac{f_2(\tilde{x}_1, \tilde{x}_2)}{g_2(\tilde{x}_1, \tilde{x}_2)} - \varepsilon_2$ *and* $\tilde{v}_2 = \frac{f_2(\tilde{x}_1, \tilde{x}_2)}{g_2(\tilde{x}_1, \tilde{x}_2)} - \varepsilon_2$*. Then,* $\tilde{v}_1 = 1$ *and* $\tilde{v}_2 = 2$*. Let* $\mu_1 = 1$, $\mu_2 = 0$, $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$ and $r_2^n = 0$, $\lambda_2^n = 0$*. Let* $\gamma_1^n = \frac{1}{2} + \frac{1}{4n}$, $\lambda_1^n = n$, $\gamma_2^n = 0$, $\lambda_2^n = 0$, $n \in \mathbb{N}$*. Then,* $\partial(\mu_1 f_1)(\tilde{x}_1, \tilde{x}_2) = \{(-1, 0)\}$, $\partial(\mu_2 f_2)(\tilde{x}_1, \tilde{x}_2) = \{(0, 0)\}$, $\partial(-\tilde{v}_1 \mu_1 g_1)(\tilde{x}_1, \tilde{x}_2) = \{(0, 0)\}$, $\partial_{\gamma_1^n}(\lambda_1^n h_1)(\tilde{x}_1, \tilde{x}_2) = \left[0, -n + \sqrt{n^2 + 4n(\frac{1}{2} + \frac{1}{4n})}\right] \times \{0\} = [0, 1] \times \{0\}$, $\partial_{\gamma_2^n}(\lambda_2^n h_2)(\tilde{x}_1, \tilde{x}_2) = \{(0, 0)\}$, $\partial_{\gamma_2^n}(\lambda_2^n h_2)(\tilde{x}_1, \tilde{x}_2) = \{(0, 0)\}$*. Let* $u_1 = (-1, 0)$ *and* $u_2 = y_1 = y_2 = (0, 0)$*. Then,* $u_1 \in \partial(\mu_1 f_1)(\tilde{x}_1, \tilde{x}_2)$, $u_2 \in \partial(\mu_2 f_2)(\tilde{x}_1, \tilde{x}_2)$, $y_1 \in \partial(-\tilde{v}_1 \mu_1 g_1)(\tilde{x}_1, \tilde{x}_2)$, $y_2 \in \partial(-\tilde{v}_2 \mu_2 g_2)(\tilde{x}_1, \tilde{x}_2)$*. Let* $w_1^n = (1, 0)$ *and* $w_2^n = (0, 0)$*. Then,* $w_1^n \in \partial_{\gamma_1^n}(\lambda_1^n h_1)(\tilde{x}_1, \tilde{x}_2)$ *and* $w_2^n \in \partial_{\gamma_2^n}(\lambda_2^n h_2)(\tilde{x}_1, \tilde{x}_2)$*. Thus,* $\sum_{i=1}^{2} (u_i + y_i) + \lim_{n\to\infty} \sum_{j=1}^{2} w_j^n = (-1, 0) + (1, 0) = (0, 0)$, $\lim_{n\to\infty} \sum_{i=1}^{2} \left[\gamma_j^n - (\lambda_j^n h_j)(\tilde{x}_1, \tilde{x}_2)\right] = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{4n}\right) = \frac{1}{2}$ *and* $\lim_{n\to\infty} \sum_{i=1}^{2} \left[\gamma_j^n - (\lambda_j^n h_j)(\tilde{x}_1, \tilde{x}_2)\right] = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{4n}\right) = \frac{1}{2}$*. Hence, Theorem 3.4 holds.*

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**Authors' contributions**
The authors, together discussed and solved the problems in the manuscript. All Authors read and approved the final manuscript.

**Competing interests**
The authors declare that they have no competing interests.

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