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# Nonlinear approximation of an ACQ-functional equation in nan-spaces

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**Abstract**

In this paper, using the fixed point and direct methods, we prove the generalized Hyers-Ulam stability of an additive-cubic-quartic functional equation in NAN-spaces.

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## 1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?" If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940. In the next year, Hyers [2] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [3] proved a generalization of the Hyers' theorem for additive mappings. The result of Rassias has provided a lot of influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations (see [4-8]). Furthermore, in 1994, a generalization of the Rassias' theorem was obtained by Găvruta [9] by replacing the bound $\varepsilon(||x||^p + ||y||^p)$ by a general control function $\phi(x, y)$.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. In 1983, a generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [10] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. In 1984, Cholewa [11] noticed that the theorem of Skof is still true if the relevant domain $X$ is replaced by an Abelian group and, in 2002, Czerwik [12] proved the generalized Hyers-Ulam stability of the quadratic functional equation.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [13-32]).

In 1897, Hensel [33] has introduced a normed space that does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [34-37]).

Now, we give some definitions and lemmas for the main results in this paper.

A *valuation* is a function $|\cdot|$ from a field $\mathbb{K}$ into $[0, \infty)$ such that, for all $r, s \in \mathbb{K}$, the following conditions hold:

(a) $|r| = 0$ if and only if $r = 0$;

(b) $|rs| = |r||s|$;

(c) $|r + s| \leq |r| + |s|$.

A field $\mathbb{K}$ is called a *valued field* if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation that satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq \max\{|r|, |s|\}$$

for all $r, s \in \mathbb{K}$, then the function $|\cdot|$ is called a *non-Archimedean valuation* and the field is called a *non-Archimedean field*. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

**Definition 1.1**. Let $X$ be a vector space over a field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $||\cdot|| : X \to [0, \infty)$ is called a *non-Archimedean norm* if the following conditions hold:

(a) $||x|| = 0$ if and only if $x = 0$ for all $x \in X$;

(b) $||rx|| = |r|\, ||x||$ for all $r \in K$ and $x \in X$;

(c) the strong triangle inequality holds:

$$||x+y|| \leq \max\{||x||, ||y||\}$$

for all $x, y \in X$.

Then $(X, ||\cdot||)$ is called a *non-Archimedean normed space* (briefly NAN-space).

**Definition 1.2**. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$.

(1) The sequence $\{x_n\}$ is called a *Cauchy sequence* if, for any $\varepsilon > 0$, there is a positive integer $N$ such that

$$||x_n - x_m|| \leq \varepsilon$$

for all $n, m \geq N$.

(2) The sequence $\{x_n\}$ is said to be *convergent* if, for any $\varepsilon > 0$, there are a positive integer $N$ and $x \in X$ such that

$$||x_n - x|| \leq \varepsilon$$

for all $n \geq N$. Then, the point $x \in X$ is called the *limit* of the sequence $\{x_n\}$, which is denoted by $\lim_{n\to\infty} x_n = x$.

(3) If every Cauchy sequence in $X$ converges, then the non-Archimedean normed space $X$ is called a *non-Archimedean Banach space*.

Note that $||x_n - x_m|| \le \max\{||x_{j+1} - x_j|| : m \le j \le n - 1\}$ for all $m, n \ge 1$ with $n > m$.

**Definition 1.3**. Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies the following conditions:

(a) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Theorem 1.1**. [38,39]*Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

*for all nonnegative integers n or there exists a positive integer $n_0$ such that*

(a) $d(J^n x, J^{n+1} x) < \infty$ *for all $n_0 \ge n_0$;*

(b) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of J;*

(c) *$y^*$ is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*

(d) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$*for all $y \in Y$.*

In this paper, using the fixed point and direct methods, we prove the generalized Hyers-Ulam stability of the following functional equation

$$\begin{aligned} 11f(x+2y) + 11f(x-2y) = 44\{f(x+y) + f(x-y)\} + 12f(3y) \\ - 48f(2y) + 60f(y) - 66f(x) \end{aligned} \tag{1.1}$$

in non-Archimedean normed spaces.

## 2. Non-Archimedean stability of the equation (1.1): a fixed point method-odd case

Using the fixed point alternative approach, we prove the generalized Hyers-Ulam stability of functional Equation (1.1) in non-Archimedean normed spaces for an odd case.

In [40], Lee et al. considered the following quartic functional equation:

$$f(2x+y) + f(2x-y) = 4\{f(x+y) + f(x-y)\} + 24f(x) - 6f(y) \tag{2.1}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional Equation (2.1), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping.

One can easily show that an even mapping $f : X \to Y$ satisfies (1.1) if and only if the even mapping $f : X \to Y$ is a quartic mapping, that is,

$$f(2x+y) + f(2x-y) = 4\{f(x+y) + f(x-y)\} + 24f(x) - 6f(y) \tag{2.2}$$

and an odd mapping $f : X \to Y$ satisfies (1.1) if and only if the odd mapping $f : X \to Y$ is a additive-cubic mapping, that is,

$$f(2x+y) + f(2x-y) = 4\{f(x+y) + f(x-y)\} - 6f(x) \tag{2.3}$$

It was shown in [[41], Lemma 2.2] that $g(x) = f(2x) - 2f(x)$ and $h(x) = f(2x) - 8f(x)$ are cubic and additive, respectively, and that $f(x) := \frac{1}{16} g(x) - \frac{1}{16} h(x)$.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned}\Phi_f(x,y) &= 11f(x+2y) + 11f(x-2y) - 44\{f(x+y) + f(x-y)\}\\ &\quad - 12f(3y) + 48f(2y) - 60f(y) + 66f(x)\end{aligned}$$

for all $x, y \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $\Phi_f(x, y) = 0$ in non-Archimedean normed spaces: an odd case.

Throughout this section, let $|8| \neq 1$.

**Theorem 2.1**. *Let X be a non-Archimedean normed space and Y a non-Archimedean Banach space. Assume that* $\gamma : X^2 \to [0, \infty)$ *is a function such that there exists an* $L < 1$ *with*

$$\gamma\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{|8|}\gamma(x,y) \tag{2.4}$$

*for all* $x, y \in X$. *If* $f : X \to Y$ *is an odd mapping satisfying*

$$||\Phi_f(x,y)|| \le \gamma(x,y) \tag{2.5}$$

*for all* $x, y \in X$, *then the limit*

$$C(x) := \lim_{n\to\infty} 8^n \left(f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right)$$

*exists for all* $x \in X$ *and defines a unique cubic mapping* $C : X \to Y$ *such that*

$$||f(2x) - 2f(x) - C(x)|| \le \frac{L}{|8| - |8|L} max\left\{\frac{1}{|11|}\gamma(2x,x),\ \left|\frac{14}{33}\right|\gamma(x,0)\right\}. \tag{2.6}$$

*Proof.* Putting $x = 0$ in (2.5), we have

$$||12f(3y) - 48f(2y) + 60f(y)|| \le \gamma(y,0) \tag{2.7}$$

for all $y \in X$.

Replacing $x$ by $2y$ in (2.5), we get

$$||11f(4y) - 56f(3y) + 114f(2y) - 104f(y)|| \le \gamma(2y,y) \tag{2.8}$$

for all $y \in X$. By (2.7) and (2.8), we have

$$\begin{aligned}\|f(4y) - 10f(2y) + 16f(y)\| &= \left\| \frac{1}{11}\left[11f(4y) - 56f(3y) + 114f(2y) - 104f(y)\right]\right.\\ &\quad \left. + \frac{14}{33}[12f(3y) - 48f(2y) + 60f(y)]\right\| \\ &\le \max\left\{\frac{1}{|11|}\gamma(2y,y), \left|\frac{14}{33}\right|\gamma(y,0)\right\}\end{aligned} \tag{2.9}$$

for all $y \in X$. Letting $y := \frac{x}{2}$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 8g\left(\frac{x}{2}\right)\right\| \le \max\left\{\frac{1}{|11|}\gamma\left(x, \frac{x}{2}\right), \left|\frac{14}{33}\right|\gamma\left(\frac{x}{2}, 0\right)\right\}. \tag{2.10}$$

Consider the set

$$S := \{g : X \to Y\}$$

and the generalized metric $d$ in $S$ defined by

$$d(f,g) = \inf_{\mu\in(0,+\infty)} \left\{||g(x) - h(x)|| \le \mu \max\left\{\frac{1}{|11|}\gamma(2x,x), \left|\frac{14}{33}\right|\gamma(x,0)\right\}, \forall x \in X\right\},$$

where $\inf \varnothing = +\infty$. It is easy to show that $(S, d)$ is complete (see [[42], Lemma 2.1]). Now, we consider a linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right) \tag{2.11}$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \varepsilon$. Then we have

$$||g(x) - h(x)|| \le \varepsilon \max\left\{\frac{1}{|11|}\gamma(2x,x), \left|\frac{14}{33}\right|\gamma(x,0)\right\}$$

for all $x \in X$ and so

$$\begin{aligned} ||Jg(x) - Jh(x)|| &= \left\|8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right)\right\| \\ &\le |8| \max\left\{\frac{1}{|11|}\gamma\left(x,\frac{x}{2}\right), \left|\frac{14}{33}\right|\gamma\left(\frac{x}{2},0\right)\right\} \\ &\le |8| \cdot \frac{L}{|8|}\varepsilon \max\left\{\frac{1}{|11|}\gamma(2x,x), \left|\frac{14}{33}\right|\gamma(x,0)\right\} \end{aligned}$$

for all $x \in X$. Thus $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$. It follows from (2.10) that

$$d(g, Jg) \le \frac{L}{|8|}. \tag{2.12}$$

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

(1) $C$ is a fixed point of $J$, that is,

$$\frac{1}{8}C(x) = C\left(\frac{x}{2}\right) \tag{2.13}$$

for all $x \in X$. The mapping $C$ is a unique fixed point of $J$ in the set

$$\Omega = \{h \in S : d(g,h) < \infty\}.$$

This implies that $C$ is a unique mapping satisfying (2.13) such that there exists $\mu \in (0, \infty)$ satisfying

$$||g(x) - C(x)|| \le \mu \max\left\{\frac{1}{|11|}\gamma(2x,x), \left|\frac{14}{33}\right|\gamma(x,0)\right\}$$

for all $x \in X$.

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n g\left(\frac{x}{2^n}\right) = \lim_{n\to\infty} 8^n \left(f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right) = C(x)$$

for all $x \in X$.

(3) $d(g, C) \leq \frac{d(g, Jg)}{1-L}$ with $g \in \Omega$, which implies the inequality

$$d(g, C) \leq \frac{L}{|8| - |8|L}. \tag{2.14}$$

This implies that the inequality (2.6) holds.

Since $\Phi_g(x, y) = \Phi_f(2x, 2y) - 2\Phi_f(x, y)$, using (2.4) and (2.5), we have

$$\begin{aligned}
||\Phi_C(x, y)|| &= \lim_{n\to\infty} |8|^n \left\| \Phi_g\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \\
&= \lim_{n\to\infty} |8|^n \left\| \Phi_f\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) - 2\Phi_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \\
&\leq \lim_{n\to\infty} |8|^n \max\left\{ \left\| \Phi_f\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) \right\|, |2| \left\| \Phi_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \right\} \\
&\leq \lim_{n\to\infty} |8|^n \max\left\{ \gamma\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right), |2|\gamma\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\} \\
&\leq \lim_{n\to\infty} |8|^n \max\left\{ \frac{L^{n-1}}{|8|^{n-1}}\gamma(x, y), \frac{|2|L^n}{|8|^n}\gamma(x, y) \right\} \\
&= 0
\end{aligned}$$

for all $x, y \in X$ and $n \geq 1$ and so $||\Phi_C(x, y)|| = 0$ for all $x, y \in X$. Therefore, the mapping $C : X \to Y$ is cubic. This completes the proof. □

**Corollary 2.1**. *Let $\theta \geq 0$ and $r$ be a real number with $r > 1$. Let $f : X \to Y$ be an odd mapping satisfying*

$$||\Phi_f(x, y)|| \leq \theta(||x||^r + ||y||^r) \tag{2.15}$$

*for all $x, y \in X$. Then the limit $C(x) = \lim_{n\to\infty} 8^n \left(f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right)$ exists for all $x \in X$ and $C : X \to Y$ is a unique cubic mapping such that*

$$||f(2x) - 2f(x) - C(x)|| \leq \frac{|8|^r}{|8| - |8|^{r+1}} \max\left\{ \frac{(|2|^r + 1)\theta||x||^r}{|11|}, \left|\frac{14}{33}\right| \theta||x||^r \right\}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 2.1 if we take

$$\gamma(x, y) = \theta(||x||^r + ||y||^r)$$

for all $x, y \in X$. In fact, if we choose $L = |8|^r$, then we get the desired result. □

**Theorem 2.2**. *Let $X$ be a non-Archimedean normed space and $Y$ a non-Archimedean Banach space. Assume that $\gamma : X^2 \to [0, \infty)$ is a function such that there exists an $L < 1$ with*

$$\gamma(2x, 2y) \leq |8|L\gamma(x, y) \tag{2.16}$$

*for all $x, y \in X$. If $f : X \to Y$ is an odd mapping satisfying (2.5), then the limit*

$$C(x) = \lim_{n\to\infty} \frac{f(2^{n+1}x) - 2f(2^n x)}{8^n}$$

*exists for all $x \in X$ and defines a unique cubic mapping $C : X \to Y$ such that*

$$||f(2x) - 2f(x) - C(x)|| \leq \frac{1}{|8| - |8|L} \max\left\{ \frac{1}{|11|}\gamma(2x, x), \left|\frac{14}{33}\right| \gamma(x, 0) \right\}. \tag{2.17}$$

*Proof*. Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.1. Consider the mapping $J : (S, d) \to (S, d)$ such that

$$Jg(x) := \frac{1}{8}g(2x) \tag{2.18}$$

for all $x \in X$.

Proceeding as in the proof of Theorem 2.1, we find that $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that $d(Jg, Jh) \le Ld(g, h)$ for all $g, h \in S$.

It follows from (2.10) that

$$\left\| \frac{g(2x)}{8} - g(x) \right\| \le \frac{1}{|8|} \max\left\{ \frac{1}{|11|}\gamma(2x, x), \left|\frac{14}{33}\right| \gamma(x, 0) \right\}$$

for all $x \in X$. So

$$d(g, Jg) \le \frac{1}{|8|}. \tag{2.19}$$

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

(1) $C$ is a fixed point of $J$, that is,

$$8C(x) = C(2x) \tag{2.20}$$

for all $x \in X$. The mapping $C$ is a unique fixed point of $J$ in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}.$$

This implies that $C$ is a unique mapping satisfying (2.20) such that there exists $\mu \in (0, \infty)$ satisfying

$$||g(x) - C(x)|| \le \mu \max\left\{ \frac{1}{|11|}\gamma(2x, x), \left|\frac{14}{33}\right| \gamma(x, 0) \right\}$$

for all $x \in X$.

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{g(2^n x)}{8^n} = \lim_{n\to\infty} \frac{f(2^{n+1}x) - 2f(2^n x)}{8^n} = C(x)$$

for all $x \in X$.

(3) $d(g, C) \le \frac{d(g,Jg)}{1-L}$ with $g \in \Omega$, which implies the inequality

$$d(g, C) \le \frac{1}{|8| - |8|L}. \tag{2.21}$$

This implies that the inequality (2.17) holds. The rest of the proof is similar to the proof of Theorem 2.1. □

**Corollary 2.2**. *Let* $\theta \ge 0$ *and* $r$ *be a real number with* $0 < r < 1$. *Let* $f : X \to Y$ *be an odd mapping satisfying (2.15). Then the limit* $C(x) = \lim_{n\to\infty} \frac{f(2^{n+1}x) - 2f(2^n x)}{8^n}$ *exists for all* $x \in X$ *and* $C : X \to Y$ *is a unique cubic mapping such that*

$$||f(2x) - 2f(x) - C(x)|| \le \frac{|8|^r}{|8|^{r+1} - |8|^2} \max\left\{ \frac{(|2|^r + 1)\theta||x||^r}{|11|}, \left|\frac{14}{33}\right| \theta||x||^r \right\}$$

*for all* $x \in X$.

*Proof*. The proof follows from Theorem 2.2 if we take

$$\gamma(x,y) = \theta(||x||^r + ||y||^r)$$

for all $x, y \in X$. In fact, if we choose $L = |8|^{1-r}$, then we get the desired result. □

**Theorem 2.3**. *Let X be a non-Archimedean normed space and Y a non-Archimedean Banach space. Assume that* $\gamma : X^2 \to [0, \infty)$ *is a function such that there exists an* $L < 1$ *with*

$$\gamma\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{|2|}\gamma(x,y) \tag{2.22}$$

*for all* $x, y \in X$. *If* $f : X \to Y$ *is an odd mapping satisfying (2.5), then the limit*

$$A(x) := \lim_{n\to\infty} 2^n \left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right)$$

*exists for all* $x \in X$ *and defines a unique additive mapping* $A : X \to Y$ *such that*

$$||f(2x) - 8f(x) - A(x)|| \leq \frac{L}{|2| - |2|L} \max\left\{\frac{1}{|11|}\gamma(2x,x), \left|\frac{14}{33}\right|\gamma(x,0)\right\}. \tag{2.23}$$

*Proof*. Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.1. Letting $y := \frac{x}{2}$ and $h(x) := f(2x) - 8f(x)$ for all $x \in X$ in (2.9), we get

$$\left\|h(x) - 2h\left(\frac{x}{2}\right)\right\| \leq \max\left\{\frac{1}{|11|}\gamma\left(x, \frac{x}{2}\right), \left|\frac{14}{33}\right|\gamma\left(\frac{x}{2}, 0\right)\right\}. \tag{2.24}$$

Now, we consider a linear mapping $J : S \to S$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right) \tag{2.25}$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \varepsilon$. Then we have

$$\|g(x) - h(x)\| \leq \varepsilon \max\left\{\frac{1}{|11|}\gamma(2x,x), \left|\frac{14}{33}\right|\gamma(x,0)\right\}$$

for all $x \in X$ and so

$$\begin{aligned} \| Jg(x) - Jh(x) \| = \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| &\leq |2| \max\left\{\frac{1}{|11|}\gamma\left(x, \frac{x}{2}\right), \left|\frac{14}{33}\right|\gamma\left(\frac{x}{2}, 0\right)\right\} \\ &\leq |2| \cdot \frac{L}{|2|}\varepsilon \max\left\{\frac{1}{|11|}\gamma(2x,x), \left|\frac{14}{33}\right|\gamma(x,0)\right\} \end{aligned}$$

for all $x \in X$. Thus, $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$. It follows from (2.24) that

$$d(g, Jg) \leq \frac{L}{|2|}. \tag{2.26}$$

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, that is,

$$\frac{1}{2}A(x) = A\left(\frac{x}{2}\right) \tag{2.27}$$

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set

$$\Omega = \{h \in S : d(g,h) < \infty\}.$$

This implies that $A$ is a unique mapping satisfying (2.27) such that there exists $\mu \in (0, \infty)$ satisfying

$$\| h(x) - A(x) \| \leq \mu \max\left\{\frac{1}{|11|}\gamma(2x,x), \left|\frac{14}{33}\right|\gamma(x,0)\right\}$$

for all $x \in X$.

(2) $d(J^n h, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n h\left(\frac{x}{2^n}\right) = \lim_{n\to\infty} 2^n\left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right) = A(x)$$

for all $x \in X$.

(3) $d(h,A) \leq \frac{d(h,Jh)}{1-L}$ with $h \in \Omega$, which implies the inequality

$$d(h,A) \leq \frac{L}{|2| - |2|L}. \tag{2.28}$$

This implies that the inequality (2.23) holds. The rest of the proof is similar to the proof of Theorem 2.1. □

**Corollary 2.3**. *Let $\theta \geq 0$ and $r$ be a real number with $r > 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.15). Then, the limit $A(x) = \lim_{n\to\infty} 2^n\left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that*

$$\| f(2x) - 8f(x) - A(x) \| \leq \frac{|2|^r}{|2| - |2|^{r+1}} \max\left\{\frac{(|2|^r + 1)\theta \| x\|^r}{|11|}, \left|\frac{14}{33}\right|\theta \| x\|^r\right\}$$

*for all $x \in X$.*

*Proof*. The proof follows from Theorem 2.3 if we take

$$\gamma(x,y) = \theta(\| x\|^r + \| y\|^r)$$

for all $x, y \in X$. In fact, if we choose $L = |2|^r$, then we get the desired result. □

**Theorem 2.4**. *Let $X$ be a non-Archimedean normed space and $Y$ a non-Archimedean Banach space. Assume that $\gamma : X^2 \to [0, \infty)$ is a function such that there exists an $L < 1$ with*

$$\gamma(2x, 2y) \leq |2|L\gamma(x,y) \tag{2.29}$$

*for all $x, y \in X$. If $f : X \to Y$ is an odd mapping satisfying (2.5), then the limit*

$$A(x) = \lim_{n\to\infty} \frac{f(2^{n+1}x) - 8f(2^n x)}{2^n}$$

*exists for all $x \in X$ and defines a unique additive mapping $A : X \to Y$ such that*

$$\| f(2x) - 8f(x) - A(x) \| \le \frac{1}{|2| - |2|L} \max\left\{ \frac{1}{|11|}\gamma(2x, x),\ \left|\frac{14}{33}\right| \gamma(x, 0) \right\}. \qquad (2.30)$$

*Proof.* Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.1. Consider the mapping $J : (S, d) \to (S, d)$ such that

$$Jg(x) := \frac{1}{2}g(2x) \qquad (2.31)$$

for all $x \in X$. By (2.24), we obtain

$$\left\| \frac{h(2x)}{2} - g(x) \right\| \le \frac{1}{|2|} \max\left\{ \frac{1}{|11|}\gamma(2x, x),\ \left|\frac{14}{33}\right| \gamma(x, 0) \right\}$$

for all $x \in X$. So

$$d(g, Jg) \le \frac{1}{|2|}. \qquad (2.32)$$

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, that is,

$$2A(x) = A(2x) \qquad (2.33)$$

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}.$$

This implies that $A$ is a unique mapping satisfying (2.33) such that there exists $\mu \in (0, \infty)$ satisfying

$$\| h(x) - A(x) \| \le \mu \max\left\{ \frac{1}{|11|}\gamma(2x, x),\ \left|\frac{14}{33}\right| \gamma(x, 0) \right\}$$

for all $x \in X$.

(2) $d(J^n h, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{h(2^n x)}{2^n} = \lim_{n\to\infty} \frac{f(2^{n+1}x) - 8f(2^n x)}{2^n} = A(x)$$

for all $x \in X$.

(3) $d(h, A) \le \frac{d(h, Jh)}{1-L}$ with $h \in \Omega$, which implies the inequality

$$d(h, A) \le \frac{1}{|2| - |2|L}. \qquad (2.34)$$

This implies that the inequality (2.30) holds. The rest of the proof is similar to the proof of Theorem 2.1. □

**Corollary 2.4**. *Let $\theta \ge 0$ and $r$ be a real number with $0 < r < 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.15). Then, the limit $A(x) = \lim_{n\to\infty} \frac{f(2^{n+1}x) - 8f(2^n x)}{2^n}$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive mapping such that*

$$\| f(2x) - 8f(x) - A(x) \| \le \frac{1}{|2| - |2|^{r+2}} \max\left\{ \frac{(|2|^r + 1)\theta \| x\|^r}{|11|},\ \left|\frac{14}{33}\right| \theta \| x\|^r \right\}$$

*for all* $x \in X$.

*Proof.* The proof follows from Theorem 2.4 if we take

$$\gamma(x, y) = \theta(\| x\|^r + \| y\|^r)$$

for all $x, y \in X$. In fact, if we choose $L = |2|^{r+1}$, then we get the desired result. □

## 3. Non-Archimedean stability of the equation (1.1): a fixed point method-even case

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional Equation (1.1) in non-Archimedean normed spaces for an even case. Throughout this section, let $|16| \neq 1$.

**Theorem 3.1**. *Let X be a non-Archimedean normed space and Y a non-Archimedean Banach space. Assume that* $\gamma : X^2 \to [0, \infty)$ *is a function such that there exists an* $L < 1$ *with*

$$\gamma(2x, 2y) \leq |16| L\gamma(x, y) \tag{3.1}$$

*for all* $x, y \in X$. *If* $f : X \to Y$ *is an even mapping with* $f(0) = 0$ *satisfying (2.5), then the limit*

$$Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{16^n}$$

*exists for all* $x \in X$ *and defines a unique quartic mapping* $Q : X \to Y$ *such that*

$$\| f(x) - Q(x) \| \leq \frac{1}{|16| - |16|L} \max\left\{\frac{1}{|22|}\gamma(0, x),\ \left|\frac{6}{11}\right| \gamma(x, x)\right\}. \tag{3.2}$$

*Proof.* Putting $x = 0$ in (2.5), we have

$$\left\|12f(3y) - 70f(2y) + 148f(y)\right\| \leq \gamma(0, y) \tag{3.3}$$

for all $y \in X$.

Substituting $x = y$ in (2.5), we get

$$\left\|f(3y) - 4f(2y) - 17f(y)\right\| \leq \gamma(y, y) \tag{3.4}$$

for all $y \in X$. By (3.3) and (3.4), we have

$$\begin{aligned} \left\|f(2y) - 16f(y)\right\| &= \left\|\frac{-1}{22}\left[12f(3y) - 70f(2y) + 148f(y)\right] + \frac{6}{11}[f(3y) - 4f(2y) - 17f(y)]\right\| \\ &\leq \max\left\{\frac{1}{|22|}\gamma(0, y),\ \left|\frac{6}{11}\right| \gamma(y, y)\right\} \end{aligned} \tag{3.5}$$

for all $y \in X$. Consider the set

$$S := \{g : X \to Y,\ g(0) = 0\}$$

and the generalized metric $d$ in $S$ defined by

$$d(f, g) = \inf_{\mu \in (0, +\infty)} \left\{\| g(x) - h(x) \| \leq \mu \max\left\{\frac{1}{|22|}\gamma(0, x),\ \left|\frac{6}{11}\right| \gamma(x, x)\right\}, \forall x \in X\right\}$$

where $\inf \emptyset = +\infty$. It is easy to show that $(S, d)$ is complete (see [[42], Lemma 2.1]).

Now, we consider a linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{16}g(2x) \tag{3.6}$$

for all $x \in X$. It follows from (3.5) that

$$d(f, Jf) \le \frac{1}{|16|}. \tag{3.7}$$

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:
(1) $Q$ is a fixed point of $J$, that is,

$$16Q(x) = Q(2x) \tag{3.8}$$

for all $x \in X$. The mapping $Q$ is a unique fixed point of $J$ in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}.$$

This implies that $Q$ is a unique mapping satisfying (3.8) such that there exists $\mu \in (0, \infty)$ satisfying

$$\| f(x) - Q(x) \| \le \mu \max\left\{\frac{1}{|22|}\gamma(0, x),\ \left|\frac{6}{11}\right|\gamma(x, x)\right\}$$

for all $x \in X$.
(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty}\frac{f(2^n x)}{16^n} = Q(x)$$

for all $x \in X$.
(3) $d(f, Q) \le \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, C) \le \frac{1}{|16| - |16|L}. \tag{3.9}$$

This implies that the inequality (3.2) holds. The rest of the proof is similar to the proof of Theorem 2.1. □

**Corollary 3.1**. *Let $\theta \ge 0$ and $r$ be a real number with $r > 1$. Let $f : X \to Y$ be an even mapping with $f(0) = 0$ satisfying (2.15). Then, the limit $Q(x) = \lim_{n\to\infty}\frac{f(2^n x)}{16^n}$ exists for all $x \in X$ and $Q : X \to Y$ is a unique quartic mapping such that*

$$\| f(x) - Q(x) \| \le \frac{1}{|16| - |16|^{r+1}}\max\left\{\frac{\theta \| x\|^r}{|22|},\ 2\left|\frac{6}{11}\right|\theta \| x\|^r\right\}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.1 if we take

$$\gamma(x, y) = \theta(\| x\|^r + \| y\|^r)$$

for all $x, y \in X$. In fact, if we choose $L = |16|^r$, then we get the desired result. □

Similarly, we can obtain the following. We will omit the proof.

**Theorem 3.2**. *Let $X$ be a non-Archimedean normed space and $Y$ a non-Archimedean Banach space. Assume that $\gamma : X^2 \to [0, \infty)$ is a function such that there exists an $L < 1$ with*

$$\gamma\left(\frac{x}{2},\frac{y}{2}\right) \le \frac{L}{|16|}\gamma(x,y) \tag{3.10}$$

*for all $x, y \in X$. If $f : X \to Y$ is an even mapping with $f(0) = 0$ satisfying (2.5), then the limit*

$$Q(x) := \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right)$$

*exists for all $x \in X$ and defines a unique quartic mapping $Q : X \to Y$ such that*

$$\| f(x) - Q(x) \| \le \frac{L}{|16| - |16|L} \max\left\{\frac{1}{|22|}\gamma(0,x),\ \left|\frac{6}{11}\right|\gamma(x,x)\right\}. \tag{3.11}$$

**Corollary 3.2**. *Let $\theta \ge 0$ and $r$ be a real number with $0 < r < 1$. Let $f : X \to Y$ be an even mapping with $f(0) = 0$ satisfying (2.15). Then, the limit $Q(x) = \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right)$ exists for all $x \in X$ and $Q : X \to Y$ is a unique quartic mapping such that*

$$\| f(x) - Q(x) \| \le \frac{|16|}{|16|^{r+1} - |16|^2} \max\left\{\frac{\theta \| x\|^r}{|22|},\ 2\left|\frac{6}{11}\right|\theta \| x\|^r\right\}$$

*for all $x \in X$.*

*Proof*. The proof follows from Theorem 3.2 if we take

$$\gamma(x,y) = \theta(\| x\|^r + \| y\|^r)$$

for all $x, y \in X$. In fact, if we choose $L = |16|^{1-r}$, then we get the desired result. □

## 4. Non-Archimedean stability of Equation (1.1): a direct method-odd case

Throughout this section, using direct method, we prove the generalized Hyers-Ulam stability of the functional Equation (1.1) in non-Archimedean spaces for an odd case.

**Theorem 4.1**. *Let $G$ be an additive semigroup and $X$ a complete non-Archimedean space. Assume that $\phi : G^2 \to [0, +\infty)$ is a function such that*

$$\lim_{n\to\infty} |8|^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right) = 0 \tag{4.1}$$

*for all $x, y \in G$. Let for all $x \in G$*

$$\Phi(x) = \lim_{n\to\infty} \max\left\{|8|^{k+1} \max\left\{\frac{1}{|11|}\varphi\left(\frac{x}{2^k},\frac{x}{2^{k+1}}\right), \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2^{k+1}},0\right)\right\}; 0 \le k < n\right\} \tag{4.2}$$

*exist. Suppose that $f : G \to X$ is an odd mapping satisfying the inequality*

$$\left\|\Phi_f(x,y)\right\|_X \le \varphi(x,y) \tag{4.3}$$

*for all $x, y \in G$. Then the limit*

$$C(x) := \lim_{n\to\infty} 8^n \left(f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right)$$

*exists for all $x \in G$ and $C : G \to X$ is a cubic mapping satisfying*

$$||f(2x) - 2f(x) - C(x)||_X \le \frac{1}{|8|}\Phi(x) \tag{4.4}$$

*for all* $x \in G$. *Moreover, if*

$$\lim_{j\to\infty}\lim_{n\to\infty}\max\left\{|8|^{k+1}\max\left\{\frac{1}{|11|}\varphi\left(\frac{x}{2^k},\frac{x}{2^{k+1}}\right),\left|\frac{14}{33}\right|\varphi\left(\frac{x}{2^{k+1}},0\right)\right\};j\le k<n+j\right\}=0,$$

*then C is the unique mapping satisfying (4.4).*

*Proof*. Proceeding as in the proof of Theorem 2.1, we obtain

$$\|f(4y)-10f(2y)+16f(y)\|_X\le\max\left\{\frac{1}{|11|}\varphi(2y,y),\ \left|\frac{14}{33}\right|\varphi(y,0)\right\}\tag{4.5}$$

for all $y\in X$. Letting $y:=\frac{x}{2}$ and $g(x):=f(2x)-2f(x)$ for all $x\in X$, we get

$$\left\|g(x)-8g\left(\frac{x}{2}\right)\right\|_X\le\max\left\{\frac{1}{|11|}\varphi\left(x,\frac{x}{2}\right),\ \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2},0\right)\right\}.\tag{4.6}$$

Replacing $x$ by $\frac{x}{2^n}$ in (4.6), we get

$$\left\|8^ng\left(\frac{x}{2^n}\right)-8^{n+1}g\left(\frac{x}{2^{n+1}}\right)\right\|_X\le|8|^n\max\left\{\frac{1}{|11|}\varphi\left(\frac{x}{2^n},\frac{x}{2^{n+1}}\right),\ \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2^{n+1}},0\right)\right\}.\tag{4.7}$$

It follows from (4.1) and (4.7) that the sequence $\left\{8^ng\left(\frac{x}{2^n}\right)\right\}_{n=1}^\infty$ is a Cauchy sequence. Since $X$ is complete, so $\left\{8^ng\left(\frac{x}{2^n}\right)\right\}_{n=1}^\infty$ is convergent. Set

$$C(x):=\lim_{n\to\infty}8^ng\left(\frac{x}{2^n}\right)=\lim_{n\to\infty}8^n\left(f\left(\frac{x}{2^{n-1}}\right)-2f\left(\frac{x}{2^n}\right)\right).$$

Using induction, we see that

$$\begin{aligned}&\left\|8^ng\left(\frac{x}{2^n}\right)-g(x)\right\|_X\\&\le\frac{1}{|8|}\max\left\{|8|^{k+1}\max\left\{\frac{1}{|11|}\varphi\left(\frac{x}{2^k},\frac{x}{2^{k+1}}\right),\ \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2^{k+1}},0\right)\right\};0\le k<n\right\}.\end{aligned}\tag{4.8}$$

By taking $n$ to approach infinity in (4.8), one obtains (4.4). If $L$ is another mapping satisfying (4.4), then, for $x\in G$, we get

$$\begin{aligned}&\|C(x)-L(x)\|_X\\&=\lim_{j\to\infty}\left\|8^jL\left(\frac{x}{2^j}\right)-8^jC\left(\frac{x}{2^j}\right)\right\|_X\\&=\lim_{j\to\infty}\left\|8^jL\left(\frac{x}{2^j}\right)\pm8^jg\left(\frac{x}{2^j}\right)-8^jC\left(\frac{x}{2^j}\right)\right\|_X\\&\le\lim_{j\to\infty}\max\left\{\left\|8^j\left[L\left(\frac{x}{2^j}\right)-g\left(\frac{x}{2^j}\right)\right]\right\|_X,\ \left\|8^j\left[g\left(\frac{x}{2^j}\right)-C\left(\frac{x}{2^j}\right)\right]\right\|_X\right\}\\&\le\frac{1}{|8|}\lim_{j\to\infty}\lim_{n\to\infty}\max\left\{|8|^{k+1}\max\left\{\frac{1}{|11|}\varphi\left(\frac{x}{2^k},\frac{x}{2^{k+1}}\right),\ \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2^{k+1}},0\right)\right\};j\le k<n+j\right\}\\&=0.\end{aligned}$$

Therefore, $L=C$. This completes the proof. □

**Corollary 4.1**. *Let* $\xi:[0,\infty)\to[0,\infty)$ *be a function satisfying*

$$\xi\left(\frac{t}{|2|}\right)\le\xi\left(\frac{1}{|2|}\right)\xi(t),\quad\xi\left(\frac{1}{|2|}\right)<\frac{1}{|8|}$$

*for all* $t \geq 0$. *Let* $\delta > 0$ *and* $f : G \to X$ *be an odd mapping satisfying the inequality*

$$\left\| \Phi_f(x, y) \right\|_X \leq \delta(\xi(|x|) + \xi(|y|)) \tag{4.9}$$

*for all* $x, y \in G$. *Then the limit* $C(x) = \lim_{n\to\infty} 8^n \left(f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right)$*exists for all* $x \in G$ *and* $C : G \to X$ *is a unique cubic mapping such that*

$$||f(2x) - 2f(x) - C(x)||_X \leq \max\left\{\frac{1}{|11|}\delta\xi(|x|)\left(1 + \frac{1}{|8|}\right), \left|\frac{7}{132}\right|\xi(|x|)\right\}$$

*for all* $x \in G$.

*Proof.* Defining $\phi : G^2 \to [0, \infty)$ by $\phi(x, y) := \delta(\xi(|x|) + \xi(|y|))$. Since $|8|\xi\left(\frac{1}{|2|}\right) < 1$, we have

$$\lim_{n\to\infty} |8|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \leq \lim_{n\to\infty}\left[|8|\xi\left(\frac{1}{|2|}\right)\right]^n \varphi(x, y) = 0$$

for all $x, y \in G$. Also for all $x \in G$

$$\begin{aligned}
\Phi(x) &= \lim_{n\to\infty} \max\left\{|8|^{k+1} \max\left\{\frac{1}{|11|}\varphi\left(\frac{x}{2^k}, \frac{x}{2^{k+1}}\right), \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2^{k+1}}, 0\right)\right\}; 0 \leq k < n\right\} \\
&= |8| \max\left\{\frac{1}{|11|}\varphi\left(x, \frac{x}{2}\right), \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2}, 0\right)\right\} \\
&= |8| \max\left\{\frac{1}{|11|}\delta\xi(|x|)\left(1 + \frac{1}{|8|}\right), \left|\frac{7}{132}\right|\xi(|x|)\right\}
\end{aligned}$$

exists for all $x \in G$. On the other hand,

$$\begin{aligned}
&\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{|8|^{k+1} \max\left\{\frac{1}{|11|}\varphi\left(\frac{x}{2^k}, \frac{x}{2^{k+1}}\right), \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2^{k+1}}, 0\right)\right\}; j \leq k < n + j\right\} \\
&\lim_{j\to\infty} |8|^{j+1} \max\left\{\frac{1}{|11|}\varphi\left(\frac{x}{2^j}, \frac{x}{2^{j+1}}\right), \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2^{j+1}}, 0\right)\right\} \\
&= 0.
\end{aligned}$$

Applying Theorem 4.1, we get the desired result. □

**Theorem 4.2**. *Let* $G$ *be an additive semigroup and* $X$ *a complete non-Archimedean space. Assume that* $\phi : G^2 \to [0, +\infty)$ *is a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|8|^n} = 0 \tag{4.10}$$

*for all* $x, y \in G$. *Let for each* $x \in G$

$$\Phi(x) = \lim_{n\to\infty} \max\left\{\frac{1}{|8|^{k+1}} \max\left\{\frac{1}{|11|}\varphi(2^{k+1}x, 2^k x), \left|\frac{14}{33}\right|\varphi(2^k x, 0)\right\}; 0 \leq k < n\right\} \tag{4.11}$$

*exist. Suppose that* $f : G \to X$ *is an odd mapping satisfying the inequality (4.3). Then the limit*

$$C(x) := \lim_{n\to\infty} \frac{f(2^{n+1}x) - 2f(2^n x)}{8^n}$$

*exists for all* $x \in G$ *and* $C : G \to X$ *is a cubic mapping satisfying*

$$||f(2x) - 2f(x) - C(x)||_X \le \Phi(x) \tag{4.12}$$

*for all* $x \in G$. *Moreover, if*

$$\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{\frac{1}{|8|^{k+1}} \max\left\{\frac{1}{|11|}\varphi(2^{k+1}x, 2^k x),\ \left|\frac{14}{33}\right| \varphi(2^k x, 0)\right\}; j \le k < n+j\right\} = 0,$$

*then* $C$ *is the unique mapping satisfying (4.12).*

*Proof.* It follows from (4.5) that

$$\left\|\frac{g(2x)}{8} - g(x)\right\|_X \le \frac{1}{|8|} \max\left\{\frac{1}{|11|}\varphi(2x, x),\ \left|\frac{14}{33}\right| \varphi(x, 0)\right\} \tag{4.13}$$

for all $x \in G$. Replacing $x$ by $2^n x$ in (4.13), we get

$$\left\|\frac{g(2^{n+1}x)}{8^{n+1}} - \frac{g(2^n x)}{8^n}\right\|_X \le \frac{1}{|8|^{n+1}} \max\left\{\frac{1}{|11|}\varphi(2^{n+1}x, 2^n x),\ \left|\frac{14}{33}\right| \varphi(2^n x, 0)\right\}. \tag{4.14}$$

It follows from (4.10) and (4.14) that the sequence $\left\{\frac{g(2^n x)}{8^n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $X$ is complete, $\left\{\frac{g(2^n x)}{8^n}\right\}_{n=1}^{\infty}$ is convergent. It follows from (4.14) that

$$\begin{aligned}\left\|\frac{g(2^p x)}{8^p} - \frac{g(2^q x)}{8^q}\right\|_X &= \left\|\sum_{k=p}^{q-1} \frac{g(2^{k+1}x)}{8^{k+1}} - \frac{g(2^k x)}{8^k}\right\|_X \\ &\le \max\left\{\left\|\frac{g(2^{k+1}x)}{8^{k+1}} - \frac{g(2^k x)}{8^k}\right\|_X; p \le k < q-1\right\} \\ &\le \max\left\{\frac{1}{|8|^{k+1}} \max\left\{\frac{1}{|11|}\varphi(2^{k+1}x, 2^k x),\ \left|\frac{14}{33}\right| \varphi(2^k x, 0)\right\}; p \le k < q-1\right\}\end{aligned} \tag{4.15}$$

for all $x \in G$ and all non-negative integers $q$, $p$ with $q > p \ge 0$. Letting $p = 0$ and passing the limit $q \to \infty$ in the last inequality, we obtain (4.12).

The rest of the proof is similar to the proof of Theorem 4.1. □

**Corollary 4.2**. *Let* $\xi : [0, \infty) \to [0, \infty)$ *be a function satisfying*

$$\xi(|2|t) \le \xi(|2|)\xi(t), \quad \xi(|2|) < |8|$$

*for all* $t \ge 0$. *Let* $\delta > 0$ *and* $f : G \to X$ *be a mapping satisfying the inequality (4.9). Then the limit* $C(x) = \lim_{n\to\infty} \frac{f(2^{n+1}x) - 2f(2^n x)}{8^n}$ *exists for all* $x \in G$ *and* $C : G \to X$ *is a unique cubic mapping such that*

$$||f(2x) - 2f(x) - C(x)||_X \le \frac{1}{|8|} max\left\{\frac{1 + |8|}{|11|}\delta\xi(|x|),\ \left|\frac{14}{33}\right| \delta\xi(|x|)\right\} \tag{4.16}$$

*for all* $x \in G$.

*Proof.* Define $\phi : G^2 \to [0, \infty)$ by $\phi(x, y) := \delta(\xi(|x|) + \xi(|y|))$. Proceeding as in the proof of Corollary 4.1, we have

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|8|^n} = 0$$

for all $x, y \in G$. Also

$$\begin{aligned}\Phi(x) &= lim_{n\to\infty} \max\left\{\frac{1}{|8|^{k+1}} \max\left\{\frac{1}{|11|}\varphi(2^{k+1}x, 2^k x),\ \left|\frac{14}{33}\right|\varphi(2^k x, 0)\right\}; 0 \le k < n\right\} \\ &= \frac{1}{|8|}\max\left\{\frac{1}{|11|}\varphi(2x, x),\ \left|\frac{14}{33}\right|\varphi(x, 0)\right\} \\ &\le \frac{1}{|8|}\max\left\{\frac{1+|8|}{|11|}\delta\xi(|x|),\ \left|\frac{14}{33}\right|\delta\xi(|x|)\right\}\end{aligned}$$

exists for all $x \in G$. Applying Theorem 4.2, we get the desired result. □

**Theorem 4.3**. *Let G be an additive semigroup and X a complete non-Archimedean space. Assume that $\phi : G^2 \to [0, +\infty)$ is a function such that*

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{4.17}$$

*for all $x, y \in G$. Let for all $x \in G$*

$$\Phi(x) = \lim_{n\to\infty} \max\left\{|2|^k \max\left\{\frac{1}{|11|}\varphi\left(\frac{x}{2^k}, \frac{x}{2^{k+1}}\right),\ \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2^{k+1}}, 0\right)\right\}; 0 \le k < n\right\} \tag{4.18}$$

*exist. Suppose that $f : G \to X$ is an odd mapping satisfying the inequality (4.3). Then the limit*

$$A(x) := \lim_{n\to\infty} 2^n \left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right)$$

*exists for all $x \in G$ and $A : G \to X$ is an additive mapping satisfying*

$$||f(2x) - 8f(x) - A(x)||_X \le \Phi(x) \tag{4.19}$$

*for all $x \in G$. Moreover, if*

$$\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{|2|^k \max\left\{\frac{1}{|11|}\varphi\left(\frac{x}{2^k}, \frac{x}{2^{k+1}}\right),\ \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2^{k+1}}, 0\right)\right\}; j \le k < n+j\right\} = 0,$$

*then A is the unique mapping satisfying (4.19).*

*Proof*. Letting $y := \frac{x}{2}$ and $h(x) := f(2x) - 8f(x)$ for all $x \in G$ in (4.5), we get

$$\left\|h(x) - 2h\left(\frac{x}{2}\right)\right\|_X \le \max\left\{\frac{1}{|11|}\varphi\left(x, \frac{x}{2}\right),\ \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2}, 0\right)\right\}. \tag{4.20}$$

Replacing $x$ by $\frac{x}{2^n}$ in (4.20), we obtain

$$\begin{aligned}&\left\|2^n h\left(\frac{x}{2^n}\right) - 2^{n+1}h\left(\frac{x}{2^{n+1}}\right)\right\|_X \\ &\quad\le |2|^n \max\left\{\frac{1}{|11|}\varphi\left(\frac{x}{2^n}, \frac{x}{2^{n+1}}\right),\ \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2^{n+1}}, 0\right)\right\}.\end{aligned} \tag{4.21}$$

Using induction, one can easily show that

$$\begin{aligned}&\left\|2^n h\left(\frac{x}{2^n}\right) - h(x)\right\|_X \\ &\quad\le \max\left\{|2|^k \max\left\{\frac{1}{|11|}\varphi\left(\frac{x}{2^k}, \frac{x}{2^{k+1}}\right),\ \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2^{k+1}}, 0\right)\right\}; 0 \le k < n\right\}.\end{aligned} \tag{4.22}$$

The rest of the proof is similar to the proof of Theorem 4.1. □

**Corollary 4.3**. *Let* $\xi : [0, \infty) \to [0, \infty)$ *be a function satisfying*

$$\xi\left(\frac{t}{|2|}\right) \le \xi\left(\frac{1}{|2|}\right)\xi(t), \quad \xi\left(\frac{1}{|2|}\right) < \frac{1}{|2|}$$

*for all* $t \ge 0$. *Let* $\delta > 0$ *and* $f : G \to X$ *be an odd mapping satisfying the inequality (4.9). Then the limit* $A(x) = \lim_{n\to\infty} 2^n \left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right)$ exists for all $x \in G$ *and* $A : G \to X$ *is a unique additive mapping such that*

$$||f(2x) - 8f(x) - A(x)||_X \le \max\left\{\left(1 + \frac{1}{|2|}\right)\frac{\delta\xi(|x|)}{|11|}, \left|\frac{7}{33}\right|\xi(|x|)\right\}$$

*for all* $x \in G$.

*Proof.* Define $\phi : G^2 \to [0, \infty)$ by $\phi(x, y) := \delta((\xi(|x|) + \xi(|y|))$. Also

$$\begin{aligned}\Phi(x) &= \lim_{n\to\infty} \max\left\{|2|^k \max\left\{\frac{1}{|11|}\varphi\left(\frac{x}{2^k}, \frac{x}{2^{k+1}}\right), \left|\frac{14}{33}\right|\varphi\left(\frac{x}{2^{k+1}}, 0\right)\right\}; 0 \le k < n\right\} \\ &= \max\left\{\left(1 + \frac{1}{|2|}\right)\frac{\delta\xi(|x|)}{|11|}, \left|\frac{7}{33}\right|\xi(|x|)\right\}\end{aligned}$$

exists for all $x \in G$. Applying Theorem 4.3, we get the desired result. □

Similarly, we can obtain the following. We will omit the proof.

**Theorem 4.4**. *Let G be an additive semigroup and X a complete non-Archimedean space. Assume that* $\phi : G^2 \to [0, +\infty)$ *is a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^n} = 0 \tag{4.23}$$

*for all* $x, y \in G$. *Let for each* $x \in G$

$$\Phi(x) = \lim_{n\to\infty} \max\left\{\frac{1}{|2|^k} \max\left\{\frac{1}{|11|}\varphi(2^{k+1}x, 2^k x), \left|\frac{14}{33}\right|\varphi(2^k x, 0)\right\}; 0 \le k < n\right\} \tag{4.24}$$

*exist. Suppose that* $f : G \to X$ *be an odd mapping satisfying the inequality (4.3). Then the limit*

$$A(x) := \lim_{n\to\infty} \frac{f(2^{n+1}x) - 8f(2^n x)}{2^n}$$

*exists for all* $x \in G$ *and* $A : G \to X$ *is an additive mapping satisfying*

$$||f(2x) - 8f(x) - A(x)||_X \le \frac{1}{|2|}\Phi(x) \tag{4.25}$$

*for all* $x \in G$. *Moreover, if*

$$\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{\frac{1}{|2|^k} \max\left\{\frac{1}{|11|}\varphi(2^{k+1}x, 2^k x), \left|\frac{14}{33}\right|\varphi(2^k x, 0)\right\}; j \le k < n + j\right\} = 0,$$

*then A is the unique mapping satisfying (4.25).*

## 5. Non-Archimedean stability of Equation (1.1): a direct method-even case

**Theorem 5.1**. *Let G be an additive semigroup and X a complete non-Archimedean space. Assume that* $\phi : G^2 \to [0, +\infty)$ *is a function such that*

$$\lim_{n\to\infty}\frac{\varphi(2^n x, 2^n y)}{|16|^n} = 0 \tag{5.1}$$

*for all x, y ∈ G. Let for all x ∈ G*

$$\Phi(x) = \lim_{n\to\infty}\max\left\{\frac{1}{|16|^k}\max\left\{\frac{1}{|22|}\varphi(0, 2^k x),\ \left|\frac{6}{11}\right|\varphi(2^k x, 2^k x)\right\}; 0 \le k < n\right\} \tag{5.2}$$

*exist. Suppose that $f : G \to X$ is an even mapping with $f(0) = 0$ satisfying the inequality (4.3). Then the limit*

$$Q(x) := \lim_{n\to\infty}\frac{f(2^n x)}{16^n}$$

*exists for all x ∈ G and $Q : G \to X$ is a quartic mapping satisfying*

$$||f(x) - Q(x)||_X \le \frac{1}{|16|}\Phi(x) \tag{5.3}$$

*for all x ∈ G. Moreover, if*

$$\lim_{j\to\infty}\lim_{n\to\infty}\max\left\{\frac{1}{|16|^k}\max\left\{\frac{1}{|22|}\varphi(0, 2^k x),\ \left|\frac{6}{11}\right|\varphi(2^k x, 2^k x)\right\}; j \le k < n + j\right\} = 0,$$

*then Q is the unique mapping satisfying (5.3).*

*Proof.* Proceeding as in the proof of Theorem 3.1, we obtain

$$\left\|\frac{f(2x)}{16} - f(x)\right\|_X \le \frac{1}{|16|}\max\left\{\frac{1}{|22|}\varphi(0, x), \left|\frac{6}{11}\right|\varphi(x, x)\right\}.$$

One can easily show that

$$\left\|\frac{f(2^n x)}{16^n} - f(x)\right\|_X \le \frac{1}{|16|}\max\left\{\frac{1}{|16|^k}\max\left\{\frac{1}{|22|}\varphi(0, 2^k x),\ \left|\frac{6}{11}\right|\varphi(2^k x, 2^k x)\right\}; 0 \le k < n\right\}.$$

The rest of the proof is similar to the proof of Theorem 4.1. □

**Corollary 5.1**. *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\xi(|2|t) \le \xi(|2|)\,\xi(t), \quad \xi(|2|) < |16|$$

*for all t ≥ 0. Let δ > 0 and $f : G \to X$ be an even mapping with $f(0) = 0$ satisfying the inequality (4.9). Then the limit $Q(x) = \lim_{n\to\infty}\frac{f(2^n x)}{16^n}$ exists for all x ∈ G and $Q : G \to X$ is a unique quartic mapping such that*

$$||f(x) - Q(x)||_X \le \frac{1}{|16|}\max\left\{\frac{1}{|22|}\delta\xi(|x|),\ 2\left|\frac{6}{11}\right|\xi(|x|)\right\}$$

*for all x ∈ G.*

*Proof.* Define $\phi : G^2 \to [0, \infty)$ by $\phi(x, y) := \delta(\xi(|x|) + \xi(|y|))$. Also

$$\Phi(x) = \max\left\{\frac{1}{|22|}\delta\xi(|x|),\ 2\left|\frac{6}{11}\right|\xi(|x|)\right\}$$

exists for all x ∈ G. Applying Theorem 5.1, we get the desired result. □

Similarly, we can obtain the following. We will omit the proof.

**Theorem 5.2**. *Let G be an additive semigroup and X a complete non-Archimedean space. Assume that $\phi : G^2 \to [0, +\infty)$ is a function such that*

$$\lim_{n\to\infty} 16|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$$

*for all $x, y \in G$. Let for all $x \in G$*

$$\Phi(x) = \lim_{n\to\infty} \max\left\{|16|^k \max\left\{\frac{1}{|22|}\varphi\left(0, \frac{x}{2^{k+1}}\right), \left|\frac{6}{11}\right|\varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right)\right\}; 0 \le k < n\right\}$$

*exist. Suppose that $f : G \to X$ is an even mapping satisfying the inequality (4.3). Then the limit*

$$Q(x) := \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right)$$

*exists for all $x \in G$ and $Q : G \to X$ is a quartic mapping satisfying*

$$||f(x) - Q(x)||_X \le \Phi(x) \tag{5.4}$$

*for all $x \in G$. Moreover, if*

$$\lim_{j\to\infty}\lim_{n\to\infty} \max\left\{|16|^k \max\left\{\frac{1}{|22|}\varphi\left(0, \frac{x}{2^{k+1}}\right), \left|\frac{6}{11}\right|\varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right)\right\}; j \le k < n+j\right\} = 0,$$

*then Q is the unique mapping satisfying (5.4).*

## 6. Conclusion

We linked here three different disciplines, namely, the non-Archimedean normed spaces, functional equations and fixed point theory. We established the generalized Hyers-Ulam stability of the functional Equation (1.1) in non-Archimedean normed spaces.

## 7. Competing interests

The authors declare that they have no competing interests.

## 8. Authors' contributions

All of the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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**References**
1. Ulam, SM: Problems in Modern Mathematics, Science Editions. Wiley, New York (1964)
2. Hyers, DH: On the stability of the linear functional equation. Proc Natl Acad Sci USA. **27**, 222–224 (1941). doi:10.1073/pnas.27.4.222

3. Rassias, TM: On the stability of the linear mapping in Banach spaces. Proc Am Math Soc. **72**, 297–300 (1978). doi:10.1090/S0002-9939-1978-0507327-1
4. Arriola, LM, Beyer, WA: Stability of the Cauchy functional equation over *p*-adic fields. Real Anal Exch. **31**, 125–132 (2005)
5. Azadi Kenary, H: Non-Archimedean stability of Cauchy-Jensen type functional equation. Int J Nonlinear Anal Appl. **1**(2), 1–10 (2010)
6. Azadi Kenary, H: Stability of a Pexiderial functional equation in random normed spaces. Rend Circ Mat Palermo
7. Azadi Kenary, H, Shafaat, Kh, Shafei, M, Takbiri, G: Hyers-Ulam-Rassias stability of the Appollonius quadratic mapping in RN-spaces. J Nonlinear Sci Appl. **4**(1), 110–119 (2011)
8. Cho, Y, Kim, H: Stability of functional inequalities with Cauchy-Jensen additive mappings. Abstr Appl Anal **2007**, 13 (2007). Article ID 89180
9. Găvruta, P: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J Math Anal Appl. **184**, 431–436 (1994). doi:10.1006/jmaa.1994.1211
10. Skof, F: Local properties and approximation of operators. Rend Sem Mat Fis Milano. **53**, 113–129 (1983). doi:10.1007/BF02924890
11. Cholewa, PW: Remarks on the stability of functional equations. Aequatines Math. **27**, 76–86 (1984). doi:10.1007/BF02192660
12. Czerwik, S: Functional Equations and Inequalities in Several Variables. World Scientific, River Edge, NJ (2002)
13. Gordji, ME, Savadkouhi, MB: Stability of mixed type cubic and quartic functional equations in random normed spaces. J Inequal Appl **2009**, 9 (2009). Article ID 527462
14. Gordji, ME, Savadkouhi, MB, Park, C: Quadratic-quartic functional equations in RN-spaces. J Inequal Appl **2009**, 14 (2009). Article ID 868423
15. Gordji, ME, Khodaei, H: Stability of Functional Equations. Lap Lambert Academic Publishing, USA (2010)
16. Gordji, ME, Zolfaghari, S, Rassias, JM, Savadkouhi, MB: Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces. Abstr Appl Anal **2009**, 14 (2009). Article ID 417473
17. Fechner, W: Stability of a functional inequality associated with the Jordan-Von Neumann functional equation. Aequationes Math. **71**, 149–161 (2006). doi:10.1007/s00010-005-2775-9
18. Hyers, DH, Isac, G, Rassias, TM: Stability of Functional Equations in Several Variables. Birkhäuser, Basel (1998)
19. Jung, S: Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis. Hadronic Press, Palm Harbor (2001)
20. Khodaei, H, Rassias, TM: Approximately generalized additive functions in several variables. Int J Nonlinear Anal Appl. **1**(1), 22–41 (2010)
21. Kominek, Z: On a local stability of the Jensen functional equation. Demon Math. **22**, 499–507 (1989)
22. Mirmostafaee, AK: Approximately additive mappings in non-Archimedean normed spaces. Bull Korean Math Soc. **46**, 387–400 (2009). doi:10.4134/BKMS.2009.46.2.387
23. Moradlou, F, Vaezi, H, Park, C: Fixed point and stability of an additive functional equation of *n*-Apollonius type in *C**-algebras. Abstr Appl Anal **2008**, 13 (2008). Article ID 672618
24. Moslehian, MS, Rassias, TM: Stability of functional equations in non-Archimedean spaces. Appl Anal Discret Math. **1**, 325–334 (2007). doi:10.2298/AADM0702325M
25. Park, C: Fuzzy stability of a functional equation associated with inner product spaces. Fuzzy Sets Syst. **160**, 1632–1642 (2009). doi:10.1016/j.fss.2008.11.027
26. Park, C: Generalized Hyers-Ulam-Rassias stability of *n*-sesquilinear-quadratic mappings on Banach modules over *C**-algebras. J Comput Appl Math. **180**, 279–291 (2005). doi:10.1016/j.cam.2004.11.001
27. Park, C: Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras. Fixed Point Theory Appl **2007**, 15 (2007). Article ID 50175
28. Park, C: Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach. Fixed Point Theory Appl **2008**, 9 (2008). Art. ID 493751
29. Rassias, ThM: On the stability of functional equations and a problem of Ulam. Acta Appl Math. **62**, 23–130 (2000). doi:10.1023/A:1006499223572
30. Saadati, R, Park, C: Non-Archimedean $\mathcal{L}$-fuzzy normed spaces and stability of functional equations. Comput Math Appl. **60**, 2488–2496 (2010). doi:10.1016/j.camwa.2010.08.055
31. Saadati, R, Vaezpour, M, Cho, Y: A note to paper on the stability of cubic mappings and quartic mappings in random normed spaces. J Inequal Appl **2009**, 6 (2009). Article ID 214530
32. Saadati, R, Zohdi, MM, Vaezpour, SM: Nonlinear L-random stability of an ACQ functional equation. J Inequal Appl **2011**, 23 (2011). Article ID 194394. doi:10.1186/1029-242X-2011-23
33. Hensel, K: Ubereine news begrundung der theorie der algebraischen Zahlen. Jahresber Deutsch Math Verein. **6**, 83–88 (1897)
34. Deses, D: On the representation of non-Archimedean objects. Topol Appl. **153**, 774–785 (2005). doi:10.1016/j.topol.2005.01.010
35. Katsaras, AK, Beoyiannis, A: Tensor products of non-Archimedean weighted spaces of continuous functions. Georgian Math J. **6**, 33–44 (1999). doi:10.1023/A:1022926309318
36. Khrennikov, A: Non-Archimedean analysis: quantum paradoxes, dynamical systems and biological models. In Math Appl, vol. 427,Kluwer, Dordrecht (1997)
37. Nyikos, PJ: On some non-Archimedean spaces of Alexandrof and Urysohn. Topol Appl. **91**, 1–23 (1999). doi:10.1016/S0166-8641(97)00239-3
38. Diaz, J, Margolis, B: A fixed point theorem of the alternative for contractions on a generalized complete metric space. Bull Am Math Soc. **74**, 305–309 (1968). doi:10.1090/S0002-9904-1968-11933-0
39. Cădariu, L, Radu, V: Fixed points and the stability of Jensen's functional equation. J Inequal Pure Appl Math. **4**, 1–9 (2003)
40. Lee, S, Im, S, Hwang, I: Quartic functional equations. J Math Anal Appl. **307**, 387–394 (2005). doi:10.1016/j.jmaa.2004.12.062
41. Eshaghi-Gordji, M, Kaboli-Gharetapeh, S, Park, C, Zolfaghri, S: Stability of an additive-cubicquartic functional equation. Adv Diff Equ **2009**, 20 (2009). Article ID 395693

42. Mihet, D, Radu, V: On the stability of the additive Cauchy functional equation in random normed spaces. J Math Anal Appl. **343**, 567–572 (2008)

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