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Some Coincidence Point Theorems for Nonlinear Contraction in Ordered Metric Spaces

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Abstract

We establish new coincidence point theorems for nonlinear contraction in ordered metric spaces. Also, we introduce an example to support our results. Some applications of our obtained results are given.

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1. Introduction and Preliminaries

Generalization of the Banach principle [1] has been heavily investigated by many authors (see [2-14]). In particular, there has been a number of fixed point theorems involving altering distance functions. Such functions were introduced by Khan et al. [15].

Definition 1.1. [15]*The function* $\varphi : [0, +\infty) \to [0, +\infty)$ *is called an altering distance function if the following properties are satisfied:*

(1) φ *is continuous and nondecreasing.*
(2) $\varphi(t) = 0$ *if and only if* $t = 0$.

Khan et al. [15] proved the following theorem.

Theorem 1.1. *Let* (X, d) *be a complete metric space,* ψ *an altering distance function and* $T : X \to X$ *satisfying*

$$\psi(d(Tx, Ty)) \leq c\psi(d(x, y))$$

for $x, y \in X$ *and* $0 < c < 1$. *Then,* T *has a unique fixed point.*

Existence of fixed point in partially ordered sets has been considered by many authors. Ran and Reurings [14] studied a fixed point theorem in partially ordered sets and applied their result to matrix equations. While Nieto and Rodŕiguez-López [9] studied some contractive mapping theorems in partially ordered set and applied their main theorems to obtain a unique solution for a first order ordinary differential equation. For more works in partially ordered metric spaces, we refer the reader to [16-31].

Harjani and Sadarangani [7,8] obtained some fixed point theorems in a complete ordered metric space using altering distance functions. They proved the following theorems.

Theorem 1.2. [8]*Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose that there exists a metric d in X such that* (X, d) *is a complete metric space. Let* $f : X \to X$ *be a continuous and nondecreasing mapping such that*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y))$$

for comparable $x, y \in X$, *where* ψ *and* φ *are altering distance functions. If there exists* $x_0 \preccurlyeq f(x_0)$, *then f has a fixed point.*

Theorem 1.3. [8]*Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose that there exists a metric d in X such that* (X, d) *is a complete metric space. Assume that X satisfies if* (x_n) *is a nondecreasing sequence in X such that* $x_n \to x$, *then* $x_n \preccurlyeq x$ *for all* $n \in \mathbb{N}$. *Let* $f : X \to X$ *be a nondecreasing mapping such that*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y))$$

for comparable $x, y \in X$, *where* ψ *and* φ *are altering distance functions. If there exists* $x_0 \preccurlyeq f(x_0)$, *then f has a fixed point.*

Altun and Simsek [3] introduced the concept of weakly increasing mappings as follows:

Definition 1.2. [3]*Let* $(X, \preccurlyeq)$ *be a partially ordered set. Two mappings* $f, g : X \to X$ *are said to be weakly increasing if* $fx \preccurlyeq g(fx)$ *and* $gx \preccurlyeq f(gx)$ *for all* $x \in X$.

Recently, Turkoglu [32] studied new common fixed point theorems for weakly compatible mappings on uniform spaces. While, Nashine and Samet [12] proved some new coincidence point theorems for a pair of weakly increasing mappings. Very recently, Shatanawi and Samet [33] proved some coincidence point theorems for a pair of weakly increasing mappings with respect to another map.

The aim of this article is to study new coincidence point theorems for a pair of weakly decreasing mappings satisfying (ψ, φ)-weakly contractive condition in an ordered metric space (X, d), where φ and ψ are altering distance functions.

2. Main Results

We start our study with the following definition:

Definition 2.1. *Let* $(X, \preccurlyeq)$ *be a partially ordered set and* $T, f : X \to X$ *be two mappings. We say that f is weakly decreasing with respect to T if the following conditions hold:*

(1) $fX \subseteq TX$.
(2) *For all* $x \in X$, *we have* $fy \preccurlyeq fx$ *for all* $y \in T^{-1}(fx)$.

We need the following definition in our arguments.

Definition 2.2. [34]*Let* (X, d) *be a metric space and* $f, g : X \to X$. *If* $w = fx = gx$ *for some* $x \in X$, *then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. The pair* $\{f, g\}$ *is said to be compatible if and only if*

$$\lim_{n \to +\infty} d(fgx_n, gfx_n) = 0$$

whenever (x_n) *is a sequence in X such that*

$$\lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} gx_n = t$$

for some $t \in X$.

Theorem 2.1. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T, f : X \to X$ be two maps such that for all $x, y \in X$ with Tx and Ty are comparable, we have*

$$\begin{aligned}\psi(d(fx,fy)) \le\ & \psi\left(\max\left\{d(Tx,Ty), d(fx,Tx), d(fy,Ty), \frac{1}{2}(d(fx,Ty)+d(fy,Tx))\right\}\right)\\ & -\phi\left(\max\left\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{1}{2}(d(fx,Ty)+d(fy,Tx))\right\}\right),\end{aligned} \tag{1}$$

where φ and ψ are altering distance functions. Assume that T and f satisfy the following hypotheses:

(i) *f is weakly decreasing with respect to T.*
(ii) *The pair $\{T, f\}$ is compatible.*
(iii) *f and T are continuous.*

Then, T and f have a coincidence point.

Proof. Let $x_0 \in X$. Since $fX \subseteq TX$, we choose $x_1 \in X$ such that $fx_0 = Tx_1$. Also, since $fX \subseteq TX$, we choose $x_2 \in X$ such that $fx_1 = Tx_2$. Continuing this process, we can construct a sequences (x_n) in X such that $Tx_{n+1} = fx_n$. Now, since $x_1 \in T^{-1}(fx_0)$ and $x_2 \in T^{-1}(fx_1)$, by using the assumption that f is weakly decreasing with respect to T, we obtain

$$fx_0 \succcurlyeq fx_1 \succcurlyeq fx_2.$$

By induction on n, we conclude that

$$fx_0 \succcurlyeq fx_1 \succcurlyeq \cdots \succcurlyeq fx_n \succcurlyeq fx_{n+1} \succcurlyeq \cdots .$$

Hence,

$$Tx_1 \succcurlyeq Tx_2 \succcurlyeq \cdots \succcurlyeq Tx_n \succcurlyeq Tx_{n+1} \succcurlyeq \cdots .$$

If $Tx_{n_0+1} = Tx_{n_0}$ for some $n_0 \in X$, then $fx_{n_0} = Tx_{n_0}$. Thus, x_{n_0} is a coincidence point of T and f. Hence, we may assume that $Tx_{n+1} \neq Tx_n$ for all $n \in \mathbb{N}$.

Since Tx_n and Tx_{n+1} are comparable, then by (1), we have

$$\begin{aligned}&\psi(d(Tx_{n+1}, Tx_{n+2}))\\
=\ &\psi(d(fx_n, fx_{n+1}))\\
\le\ &\psi\left(\max\left\{d(Tx_n, Tx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}),\right.\right.\\
&\left.\left.\frac{1}{2}(d(fx_n, Tx_{n+1}) + d(Tx_n, fx_{n+1}))\right\}\right)\\
&-\phi\left(\max\left\{d(Tx_n, Tx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}),\right.\right.\\
&\left.\left.\frac{1}{2}(d(fx_n, Tx_{n+1}) + d(Tx_n, fx_{n+1}))\right\}\right)\\
=\ &\psi\left(\max\left\{d(Tx_n, Tx_{n+1}), d(Tx_{n+2}, Tx_{n+1}), \frac{1}{2}d(Tx_n, Tx_{n+2})\right\}\right)\\
&-\phi\left(\max\left\{d(Tx_n, Tx_{n+1}), d(Tx_{n+1}, Tx_{n+2}), \frac{1}{2}d(Tx_n, Tx_{n+2})\right\}\right)\\
\le\ &\psi\left(\max\left\{d(Tx_n, Tx_{n+1}), d(Tx_{n+2}, Tx_{n+1}), \frac{1}{2}d(Tx_n, Tx_{n+2})\right\}\right)\\
&-\phi(\max\{d(Tx_n, Tx_{n+1}), d(Tx_{n+1}, Tx_{n+2})\})\\
\le\ &\psi(\max\{d(Tx_n, Tx_{n+1}), d(Tx_{n+1}, Tx_{n+2})\})\\
&-\phi(\max\{d(Tx_n, Tx_{n+1}), d(Tx_{n+1}, Tx_{n+2})\})\\
\le\ &\psi(\max\{d(Tx_n, Tx_{n+1}), d(Tx_{n+1}, Tx_{n+2})\}).\end{aligned}$$

If

$$\max\{d(Tx_n, Tx_{n+1}), d(Tx_{n+1}, Tx_{n+2})\} = d(Tx_{n+1}, Tx_{n+2}),$$

then

$$\psi(d(Tx_{n+1}, Tx_{n+2}) \leq \psi(d(Tx_{n+1}, Tx_{n+2})) - \phi(d(Tx_{n+1}, Tx_{n+2})).$$

So, $\varphi(d(Tx_{n+1}, Tx_{n+2})) = 0$ and hence $d(Tx_{n+1}, Tx_{n+2}) = 0$, a contradiction.
Thus,

$$\max\{d(Tx_n, Tx_{n+1}), d(Tx_{n+1}, Tx_{n+2})\} = d(Tx_n, Tx_{n+1}).$$

Therefore, we have

$$\psi(d(Tx_{n+1}, Tx_{n+2})) \leq \psi(d(Tx_n, Tx_{n+1})) - \phi(d(Tx_n, Tx_{n+1})) \leq \psi(d(Tx_n, Tx_{n+1})). \tag{2}$$

Since ψ is a nondecreasing function, we get that $\{d(Tx_{n+1}, Tx_n): n \in \mathbb{N}\}$ is a nonincreasing sequence. Hence, there is $r \geq 0$ such that

$$\lim_{n\to+\infty} d(Tx_n, Tx_{n+1}) = r.$$

Letting $n \to +\infty$ in (2) and using the continuity of ψ and φ, we get that

$$\psi(r) \leq \psi(r) - \phi(r).$$

Thus, $\varphi(r) = 0$ and hence $r = 0$. Therefore,

$$\lim_{n\to+\infty} d(Tx_n, Tx_{n+1}) = 0. \tag{3}$$

Now, we prove that (Tx_n) is a Cauchy sequence in X. Suppose to the contrary; that is, (Tx_n) is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find two subsequences of positive integers $(Tx_{m(i)})$ and $(Tx_{n(i)})$ such that $n(i)$ is the smallest index for which

$$n(i) > m(i) > i, \quad d(Tx_{m(i)}, Tx_{n(i)}) \geq \varepsilon. \tag{4}$$

This means that

$$d(Tx_{m(i)}, Tx_{n(i)-1}) < \varepsilon. \tag{5}$$

From (4), (5) and the triangular inequality, we have

$$\begin{aligned}\varepsilon &\leq d(Tx_{m(i)}, Tx_{n(i)}) \\ &\leq d(Tx_{m(i)}, Tx_{n(i)-1}) + d(Tx_{n(i)-1}, Tx_{n(i)}) \\ &< \varepsilon + d(Tx_{n(i)-1}, Tx_{n(i)}).\end{aligned}$$

On letting $i \to +\infty$ in above inequality and using (3), we have

$$\lim_{i\to+\infty} d(Tx_{m(i)}, Tx_{n(i)}) = \lim_{i\to+\infty} d(Tx_{m(i)}, Tx_{n(i)-1}) = \varepsilon. \tag{6}$$

Also,

$$\begin{aligned}\varepsilon &\le d(Tx_{n(i)}, Tx_{m(i)})\\ &\le d(Tx_{n(i)}, Tx_{m(i)+1}) + d(Tx_{m(i)+1}, Tx_{m(i)})\\ &\le d(Tx_{n(i)}, Tx_{n(i)-1}) + d(Tx_{n(i)-1}, Tx_{m(i)+1}) + d(Tx_{m(i)+1}, Tx_{m(i)})\\ &\le d(Tx_{n(i)}, Tx_{n(i)-1}) + d(Tx_{n(i)-1}, Tx_{m(i)}) + 2d(Tx_{m(i)+1}, Tx_{m(i)})\\ &\le d(Tx_{n(i)}, Tx_{n(i)-1}) + \varepsilon + 2d(Tx_{m(i)+1}, Tx_{m(i)}).\end{aligned}$$

Letting $i \to +\infty$ in the above inequalities and using (3), we get that

$$\lim_{i\to+\infty} d(Tx_{n(i)-1}, Tx_{m(i)+1}) = \lim_{i\to+\infty} d(Tx_{n(i)}, Tx_{m(i)+1}) = \varepsilon. \tag{7}$$

Since $Tx_{n(i)-1}$ and $Tx_{m(i)}$ are comparable, by (1), we have

$$\begin{aligned} &\psi(d(Tx_{n(i)}, Tx_{m(i)+1}))\\ = \;&\psi(d(fx_{n(i)-1}, fx_{m(i)})\\ \le \;&\psi\Big(\max\Big\{d(Tx_{n(i)-1}, Tx_{m(i)}), d(fx_{n(i)-1}, Tx_{n(i)-1}), d(fx_{m(i)}, Tx_{m(i)}),\\ &\frac{1}{2}(d(fx_{n(i)-1}, Tx_{m(i)}) + d(Tx_{n(i)-1}, fx_{m(i)}))\Big\}\Big)\\ &-\phi\Big(\max\Big\{d(Tx_{n(i)-1}, Tx_{m(i)}), d(fx_{n(i)-1}, Tx_{n(i)-1}), d(fx_{m(i)}, Tx_{m(i)}),\\ &\frac{1}{2}(d(fx_{n(i)-1}, Tx_{m(i)}) + d(Tx_{n(i)-1}, fx_{m(i)}))\Big\}\Big)\\ = \;&\psi\Big(\max\Big\{d(Tx_{n(i)-1}, Tx_{m(i)}), d(Tx_{n(i)}, Tx_{n(i)-1}), d(Tx_{m(i)+1}, Tx_{m(i)}),\\ &\frac{1}{2}(d(Tx_{n(i)}, Tx_{m(i)}) + d(Tx_{n(i)-1}, Tx_{m(i)+1}))\Big\}\Big)\\ &-\phi\Big(\max\Big\{d(Tx_{n(i)-1}, Tx_{m(i)}), d(Tx_{n(i)}, Tx_{n(i)-1}), d(Tx_{m(i)+1}, Tx_{m(i)}),\\ &\frac{1}{2}(d(Tx_{n(i)}, Tx_{m(i)}) + d(Tx_{n(i)-1}, Tx_{m(i)+1}))\Big\}\Big).\end{aligned}$$

Letting $i \to +\infty$ in the above inequalities, and using (3), (6) and (7), we get that

$$\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon).$$

Therefore, $\varphi(\varepsilon) = 0$ and hence $\varepsilon = 0$, a contradiction. Thus, $\{Tx_n\}$ is a Cauchy sequence in the complete metric space X. Therefore, there exists $u \in X$ such that

$$\lim_{n\to+\infty} Tx_n = u.$$

By the continuity of T, we have

$$\lim_{n\to+\infty} T(Tx_n) = Tu.$$

Since $Tx_{n+1} = fx_n \to u$, $Tx_n \to u$, and the pair $\{T, f\}$ is compatible, we have

$$\lim_{n\to+\infty} d(f(Tx_n), T(fx_n)) = 0.$$

By the triangular inequality, we have

$$d(fu, Tu) \le d(fu, f(Tx_n)) + d(f(Tx_n), T(fx_n)) + d(T(fx_n), Tu).$$

Letting $n \to +\infty$ and using the fact that T and f are continuous, we get that $d(fu, Tu) = 0$. Hence, $fu = Tu$, that is, u is a coincidence point of T and f.

Theorem 2.2. *Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose that there exists a metric d on X. Let* $T, f : X \to X$ *be two maps such that for all* $x, y \in X$ *with Tx and Ty are comparable, we have*

$$\begin{aligned}\psi(d(fx,fy)) \le \psi\left(\max\left\{d(Tx,Ty), d(fx,Tx), d(fy,Ty), \frac{1}{2}(d(fx,Ty)+d(fy,Tx))\right\}\right)\\ -\phi\left(\max\left\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{1}{2}(d(fx,Ty)+d(fy,Tx))\right\}\right),\end{aligned} \tag{8}$$

where φ *and* ψ *are altering distance functions. Suppose that the following hypotheses are satisfied:*

(i) *If* (x_n) *is a nonincreasing sequence in X with respect to* $\preccurlyeq$ *such that* $x_n \to x \in X$ *as* $n \to +\infty$, *then* $x_n \succcurlyeq x$ *for all* $n \in \mathbb{N}$.
(ii) *f is weakly decreasing with respect to T.*
(iii) *TX is a complete subspace of X.*

Then, T and f have a coincidence point.

Proof. Following the proof of Theorem 2.1, we have (Tx_n) is a Cauchy sequence in (TX, d). Since TX is complete, there is $v \in X$ such that

$$\lim_{n\to+\infty} Tx_n = Tv = u.$$

Since $\{Tx_n\}$ is a nonincreasing sequence in X. By hypotheses, we have $Tx_n \succcurlyeq Tv$ for all $n \in \mathbb{N}$. Thus, by (8), we have

$$\begin{aligned}&\psi(d(Tx_{n+1},fv)) = \psi(fx_n,fv)\\ \le\ &\psi\left(\max\left\{d(Tx_n,Tv), d(fx_n,Tx_n), d(fv,Tv), \frac{1}{2}(d(fx_n,Tv)+d(fv,Tx_n))\right\}\right)\\ &-\phi\left(\max\left\{d(Tx_n,Tv), d(fx_n,Tx_n), d(fv,Tv), \frac{1}{2}(d(fx_n,Tv)+d(fv,Tx_n))\right\}\right)\\ =\ &\psi\left(\max\left\{d(Tx_n,Tv), d(Tx_{n+1},Tx_n), d(fv,Tv), \frac{1}{2}(d(Tx_{n+1},Tv)+d(fv,Tx_n))\right\}\right)\\ &-\phi\left(\max\left\{d(Tx_n,Tv), d(Tx_{n+1},Tx_n), d(fv,Tv), \frac{1}{2}(d(Tx_{n+1},Tv)+d(fv,Tx_n))\right\}\right).\end{aligned}$$

Letting $n \to +\infty$ in the above inequalities, we get that

$$\psi(d(Tv,fv)) \le \psi(d(Tv,fv)) - \phi(d(Tv,fv)).$$

Hence, $\varphi(d(Tv, fv)) = 0$. Since φ is an altering distance function, we get that $d(Tv, fv) = 0$. Therefore, $Tv = fv$. Thus, v is a coincidence point of T and f.

By taking $\psi(t) = t$ and $\varphi(t) = (1 - k)t$, $k \in [0, 1)$ in Theorems 2.1 and 2.2, we have the following two results.

Corollary 2.1. *Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let* $T, f : X \to X$ *be two maps such that for all* $x, y \in X$ *with Tx and Ty are comparable, we have*

$$d(fx,fy) \le k\max\left\{d(Tx,Ty), d(fx,Tx), d(fy,Ty), \frac{1}{2}(d(fx,Ty)+d(fy,Tx))\right\}.$$

Assume that T and f satisfy the following hypotheses:

(i) *f is weakly decreasing with respect to T.*
(ii) *The pair {T, f} is compatible.*
(iii) *f and T are continuous.*

If $k \in [0, 1)$, *then T and f have a coincidence point.*

Corollary 2.2. *Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose that there exists a metric d on X. Let* $T, f : X \to X$ *be two maps such that for all* $x, y \in X$ *with Tx and Ty are comparable, we have*

$$d(fx, fy) \le k \max \left\{ d(Tx, Ty), d(fx, Tx), d(fy, Ty), \frac{1}{2}(d(fx, Ty) + d(fy, Tx)) \right\}.$$

Suppose that the following hypotheses are satisfied:

(i) *If* (x_n) *is a nonincreasing sequence in X with respect to* $\preccurlyeq$ *such that* $x_n \to x \in X$ *as* $n \to +\infty$, *then* $x_n \succcurlyeq x$ *for all* $n \in \mathbb{N}$.
(ii) *f is weakly decreasing with respect to T.*
(iii) *TX is a complete subspace of X.*

If $k \in [0, 1)$, *then T and f have a coincidence point.*

Corollary 2.3. *Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose that there exists a metric d on X such that* (X, d) *is a complete metric space. Let* $T, f : X \to X$ *be two maps such that for all* $x, y \in X$ *with Tx and Ty are comparable, we have*

$$d(fx, fy) \le a_1 d(Tx, Ty) + a_2 d(fx, Tx) + a_3 d(fy, Ty) + \frac{a_4}{2}(d(fx, Ty) + d(fy, Tx)).$$

Assume that T and f satisfy the following hypotheses:

(i) *f is weakly decreasing with respect to T.*
(ii) *The pair {T, f} is compatible.*
(iii) *f and T are continuous.*

If $a_1 + a_2 + a_3 + a_4 \in [0, 1)$, *then T and f have a coincidence point.*

Proof. Follows from Corollary 2.1 by noting that

$$\begin{aligned} & a_1 d(Tx, Ty) + a_2 d(fx, Tx) + a_3 d(fy, Ty) + \frac{a_4}{2}(d(fx, Ty) + d(fy, Tx)) \\ \le\ & (a_1 + a_2 + a_3 + a_4) \max \left\{ d(Tx, Ty), d(fx, Tx), d(fy, Ty), \frac{1}{2}(d(fx, Ty) + d(fy, Tx)) \right\}. \end{aligned}$$

□

Corollary 2.4. *Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose that there exists a metric d on X such that* (X, d) *is a complete metric space. Let* $f : X \to X$ *be a map such that for all comparable* $x, y \in X$, *we have*

$$\psi(d(fx,fy)) \le \psi\left(\max\left\{d(x,y), d(fx,x), d(fy,y), \frac{1}{2}(d(fx,y)+d(fy,x))\right\}\right) - \phi\left(\max\left\{d(x,y), d(fx,x), d(fy,y), \frac{1}{2}(d(fx,y)+d(fy,x))\right\}\right),$$

where φ and ψ are altering distance functions. Assume that f satisfies the following hypotheses:

(i) $f(fx) \preccurlyeq fx$ *for all* $x \in X$.
(ii) *f is continuous.*

Then, f has a fixed point.

Proof. Follows from Theorem 2.1 by taking $T = i_X$ (the identity map).

Corollary 2.5. *Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is complete. Let* $f : X \to X$ *be a map such that for all comparable* $x, y \in X$, *we have*

$$\psi(d(fx,fy)) \le \psi\left(\max\left\{d(x,y), d(fx,x), d(fy,y), \frac{1}{2}(d(fx,y)+d(fy,x))\right\}\right) - \phi\left(\max\left\{d(x,y), d(fx,x), d(fy,y), \frac{1}{2}(d(fx,y)+d(fy,x))\right\}\right),$$

where φ and ψ are altering distance functions. Suppose that the following hypotheses are satisfied:

(i) *If* (x_n) *is a nonincreasing sequence in X with respect to* $\preccurlyeq$ *such that* $x_n \to x \in X$ *as* $n \to +\infty$, *then* $x_n \succcurlyeq x$ *for all* $n \in \mathbb{N}$.
(ii) $f(fx) \preccurlyeq fx$ *for all* $x \in X$.

Then, f has a fixed point.

Proof. Follows from Theorem 2.2 by taking $T = i_X$ (the identity map).

By taking $\psi(t) = t$ and $\varphi(t) = (1 - k)t$, $k \in [0, 1)$ in Corollaries 2.4 and 2.5, we have the following results.

Corollary 2.6. *Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let* $f : X \to X$ *be a map such that for all comparable* $x, y \in X$, *we have*

$$d(fx,fy) \le k\max\left\{d(x,y), d(fx,x), d(fy,y), \frac{1}{2}(d(fx,y)+d(fy,x))\right\}.$$

Assume f satisfies the following hypotheses:

(i) $f(fx) \preccurlyeq fx$ *for all* $x \in X$.
(ii) *f is continuous.*

If $k \in [0, 1)$, *then f has a fixed point.*

Corollary 2.7. *Let* $(X, \preccurlyeq)$ *be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is complete. Let* $f : X \to X$ *be a map such that for all comparable* $x, y \in X$, *we have*

$$d(fx, fy) \leq k \max\left\{d(x, y), d(fx, x), d(fy, y), \frac{1}{2}(d(fx, y) + d(fy, x))\right\}.$$

Suppose that the following hypotheses are satisfied:

(i) *If* (x_n) *is a nonincreasing sequence in* X *with respect to* $\preccurlyeq$ *such that* $x_n \to x \in X$ *as* $n \to +\infty$, *then* $x_n \succcurlyeq x$ *for all* $n \in \mathbb{N}$.
(ii) $f(fx) \preccurlyeq fx$ *for all* $x \in X$.

If $k \in [0, 1)$, *then* f *has a fixed point.*

Now, we introduce an example to support our results.

Example 2.1. *Let* $X = [0, +\infty)$. *Define* $d : X \times X \to \mathbb{R}$ *by* $d(x, y) = |x - y|$. *Define* f, T : $X \to X$ *by*

$$f(x) = \begin{cases} \frac{1}{16}x^4, \ 0 \leq x \leq 1; \\ \frac{1}{16\sqrt{x}}, \ x > 1 \end{cases}$$

and

$$T(x) = \begin{cases} x^2, \ 0 \leq x \leq 1; \\ x, \quad x > 1. \end{cases}$$

Then,

(1) $fX \subseteq TX$.
(2) f *and* T *are continuous.*
(3) *The pair* $\{f, T\}$ *is compatible.*
(4) f *is weakly decreasing with respect to* T.
(5) *For all* $x, y \in X$, *we have*

$$d(fx, fy) \leq \frac{1}{4} \max\left\{d(Tx, Ty), \frac{1}{2}(d(fx, Ty) + d(fy, Tx))\right\}.$$

Proof. The proof of (1) and (2) is clear.

To prove (3), let (x_n) be any sequence in X such that

$$\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} Tx_n = t$$

for some $t \in X$. Since $0 \leq fx_n \leq \frac{1}{16}$, we have $0 \leq t \leq \frac{1}{16}$. Since $Tx_n \to t$ as $n \to +\infty$, we have (x_n) has at most only finitely many elements greater than 1. Thus, $fx_n = \frac{1}{16}x_n^4$ and $Tx_n = x_n^2$ for all $n \in \mathbb{N}$ except at most for finitely many elements. Thus, we have $x_n \to 2\sqrt[4]{t}$ and $x_n \to \sqrt{t}$ as $n \to +\infty$. By uniqueness of limit, we get that $\sqrt{t} = 2\sqrt[4]{t}$ and hence $t = 0$. Thus, $x_n \to 0$ as $n \to +\infty$. Since f and T are continuous, we have $fx_n \to f0 = 0$ and $Tx_n \to T0 = 0$ as $n \to +\infty$. Therefore,

$$\lim_{n\to+\infty} d(T(fx_n), f(Tx_n)) = d(T0, f0) = d(0, 0) = 0.$$

Thus, the pair $\{f, T\}$ is compatible.

To prove f is weakly decreasing with respect to T, let $x, y \in X$ be such that $y \in T^{-1}(fx)$. If $x \in [0, 1]$, then

$$Ty = \frac{1}{16}x^4 \in \left[0, \frac{1}{16}\right].$$

In this case, we must have $Ty = y^2$. Thus, $y^2 = \frac{1}{16}x^4$. Hence, $y = \frac{1}{4}x^2$. Therefore,

$$fy = f\left(\frac{1}{4}x^2\right) = \frac{1}{16}\left(\frac{1}{4}x^2\right)^4 \leq \frac{1}{16}x^4 = fx.$$

If $x > 1$, then $fx = \frac{1}{16\sqrt{x}} \in \left(0, \frac{1}{16}\right)$. Thus, $Ty = fx \in \left(0, \frac{1}{16}\right)$. In this case, we have $Ty = y^2$. Thus,

$$y^2 = \frac{1}{16\sqrt{x}}.$$

So,

$$y = \frac{1}{4\sqrt[4]{x}}.$$

Therefore,

$$fy = f\left(\frac{1}{4\sqrt[4]{x}}\right) = \frac{1}{16}\left(\frac{1}{256x}\right) \leq \frac{1}{16x} \leq \frac{1}{16\sqrt{x}} = fx.$$

Therefore, f is weakly decreasing with respect to T.

To prove (5), let $x, y \in X$.

Case 1: If $x, y \in [0, 1]$, then

$$\begin{aligned}
|fx - fy| &= \left|\frac{1}{16}x^4 - \frac{1}{16}y^4\right| \\
&= \frac{1}{16}|x^2 + y^2||x^2 - y^2| \\
&\leq \frac{1}{8}|Tx - Ty| \\
&= \frac{1}{8}d(Tx, Ty) \\
&\leq \frac{1}{4}\max\left\{d(Tx, Ty), \frac{1}{2}(d(fx, Ty) + d(fy, Tx))\right\}.
\end{aligned}$$

Case 2: If $x, y \in (1, +\infty)$, then

$$\begin{aligned}
|fx - fy| &= \left|\frac{1}{16\sqrt{x}} - \frac{1}{16\sqrt{y}}\right| \\
&= \frac{1}{16}\left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}}\right| \\
&= \frac{1}{16}\left|\frac{\sqrt{y} - \sqrt{x}}{\sqrt{x}\sqrt{y}}\right| \\
&= \frac{1}{16}\left|\frac{y - x}{\sqrt{x}\sqrt{y}(\sqrt{y} + \sqrt{x})}\right| \\
&\leq \frac{1}{32}|y - x| \\
&= \frac{1}{32}d(Tx, Ty) \\
&\leq \frac{1}{4}\max\left\{d(Tx, Ty), \frac{1}{2}(d(fx, Ty) + d(fy, Tx))\right\}.
\end{aligned}$$

Case 3: ($x \in [0, 1]$ and $y \in (1, +\infty)$) or ($y \in [0, 1]$ and $x \in (1, +\infty)$).

Without loss of generality, we may assume that $x \in [0, 1]$ and $y \in (1, +\infty)$. Then,

$$\begin{aligned}|fx - fy| &= \frac{1}{16}\left|x^4 - \frac{1}{\sqrt{y}}\right| \\ &= \frac{1}{16}\left|x^2 - \frac{1}{\sqrt[4]{y}}\right|\left|x^2 + \frac{1}{\sqrt[4]{y}}\right| \\ &\leq \frac{1}{8}\left|x^2 - \frac{1}{\sqrt[4]{y}}\right|.\end{aligned}$$

If

$$\frac{1}{\sqrt[4]{y}} \geq x^2,$$

then

$$\begin{aligned}|fx - fy| &\leq \frac{1}{8}\left(\frac{1}{\sqrt[4]{y}} - x^2\right) \\ &\leq \frac{1}{8}(y - x^2) \\ &= \frac{1}{8}d(Ty, Tx) \\ &\leq \frac{1}{4}\max\left\{d(Tx, Ty), \frac{1}{2}(d(fx, Ty) + d(fy, Tx))\right\}.\end{aligned}$$

If

$$x^2 \geq \frac{1}{\sqrt[4]{y}},$$

then

$$\begin{aligned}|fx - fy| &\leq \frac{1}{8}\left(x^2 - \frac{1}{\sqrt[4]{y}}\right) \\ &\leq \frac{1}{8}\left(x^2 - \frac{1}{16\sqrt[4]{y}}\right) \\ &\leq \frac{1}{8}\left(x^2 - \frac{1}{16\sqrt{y}}\right) \\ &= \frac{1}{8}d(Tx, fy) \\ &\leq \frac{1}{4}\left(\frac{1}{2}(d(fx, Ty) + d(fy, Tx))\right) \\ &\leq \frac{1}{4}\max\left\{d(Tx, Ty), \frac{1}{2}(d(fx, Ty) + d(fy, Tx))\right\}.\end{aligned}$$

Thus, f and T satisfy all the hypotheses of Corollary 2.1. Therefore, T and f have a coincidence point. Here (0, 0) is the coincidence point of f and T.

3. Applications

Denote by Λ the set of functions $\lambda : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(1) λ is a Lebesgue-integrable mapping on each compact of $[0, +\infty)$.
(2) For every $\varepsilon > 0$, we have $\int_0^{\varepsilon} \lambda(s)ds > 0$.

It is an easy matter to see that the mapping $\psi : [0, +\infty) \to [0, +\infty)$ defined by

$$\psi(t) = \int_0^t \lambda(s)ds$$

is an altering distance function. Now, we have the following results:

Theorem 3.1. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $T, f : X \to X$ be two maps such that for all $x, y \in X$ with Tx and Ty are comparable, we have*

$$\int_0^{d(fx,fy)} \lambda(s)ds \le \int_0^{\max\left\{d(Tx,Ty),d(fx,Tx),d(fy,Ty),\frac{1}{2}(d(fx,Ty)+d(fy,Tx))\right\}} \lambda(s)ds - \int_0^{\max\left\{d(fx,fy),d(fx,Tx),d(fy,Ty),\frac{1}{2}(d(fx,Ty)+d(fy,Tx))\right\}} \mu(s)ds,$$

where $\lambda, \mu \in \Lambda$. Assume that T and f satisfy the following hypotheses:

(1) *f is weakly decreasing with respect to T.*
(2) *The pair $\{T, f\}$ is compatible.*
(3) *f and T are continuous.*

Then, T and f have a coincidence point.

Proof. Follows from Theorem 2.1 by taking $\psi(t) = \int_0^t \lambda(s)ds$ and $\phi(t) = \int_0^t \mu(s)ds$. □

Theorem 3.2. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X. Let $T, f : X \to X$ be two maps such that for all $x, y \in X$ with Tx and Ty are comparable, we have*

$$\int_0^{d(fx,fy)} \lambda(s)ds \le \int_0^{\max\left\{d(Tx,Ty),d(fx,Tx),d(fy,Ty),\frac{1}{2}(d(fx,Ty)+d(fy,Tx))\right\}} \lambda(s)ds - \int_0^{\max\left\{d(fx,fy),d(fx,Tx),d(fy,Ty),\frac{1}{2}(d(fx,Ty)+d(fy,Tx))\right\}} \mu(s)ds,$$

where $\lambda, \mu \in \Lambda$. Suppose that the following hypotheses are satisfied:

(1) *If (x_n) is a nonincreasing sequence in X with respect to $\preccurlyeq$ such that $x_n \to x \in X$ as $n \to +\infty$, then $x_n \succcurlyeq x$ for all $n \in \mathbb{N}$.*
(2) *f is weakly decreasing with respect to T.*
(3) *TX is a complete subspace of X.*

Then, T and f have a coincidence point.

Proof. Follows from Theorem 2.2 by taking $\psi(t) = \int_0^t \lambda(s)ds$ and $\phi(t) = \int_0^t \mu(s)ds$. □

Now, our aim is to give an existence theorem for a solution of the following integral equation:

$$u(t) = \int_0^T K(t, s, u(s))ds + g(t), \quad t \in [0, T], \tag{9}$$

where $T > 0$. Let $X = C([0, T])$ be the set of all continuous functions defined on $[0, T]$. Define

$$d : X \times X \to \mathbb{R}^+$$

by

$$d(x, y) = \sup_{t \in [0,T]} |x(t) - y(t)|.$$

Then, (X, d) is a complete metric space. Define an ordered relation $\leq$ on X by

$$x \leq y \quad \text{iff } x(t) \leq y(t), \quad \forall t \in [0, T].$$

Then, $(X, \leq)$ is a partially ordered set. Now, we prove the following result.

Theorem 3.3. *Suppose the following hypotheses hold:*

(1) $K : [0, T] \times [0, T] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous.*

(2) *For each* $t, s \in [0, T]$, *we have*

$$K(t, s, \int_0^T K(s, \tau, u(\tau))d\tau + g(s)) \leq K(t, s, u(s)).$$

(3) *There exist a continuous function* $G : [0, T] \times [0, T] \to [0, +\infty]$ *such that*

$$|K(t, s, u) - K(t, s, v)| \leq G(t, s)|u - v|$$

for each comparable $u, v \in \mathbb{R}$ *and each* $t, s \in [0, T]$.

(4) $sup_{t \in [0,T]} \int_0^T G(t, s)ds \leq r$ *for some* $r < 1$.

Then, the integral equation (9) has a solution $u \in C([0, T])$.

Proof. Define $f : C([0, T]) \to C([0, T])$ by

$$fx(t) = \int_0^T K(t, s, x(s))ds + g(t), \quad t \in [0, T].$$

Now, we have

$$\begin{aligned} f(fx(t)) &= \int_0^T K(t, s, fx(s))ds + g(t) \\ &= \int_0^T K\left(t, s, \int_0^T K(s, \tau, x(\tau))d\tau + g(s)\right) ds + g(t) \\ &\leq \int_0^T K(t, s, x(s))ds + g(t) \\ &= fx(t). \end{aligned}$$

Thus, we have $f(fx) \leq fx$ for all $x \in C([0, T])$.

For $x, y \in C([0, T])$ with $x \preccurlyeq y$, we have

$$\begin{aligned} d(fx, fy) &= \sup_{t\in[0,T]} |fx(t) - fy(t)| \\ &= \sup_{t\in[0,T]} \left| \int_0^T K(t, s, x(s)) - K(t, s, y(s))ds \right| \\ &\le \sup_{t\in[0,T]} \int_0^T |K(t, s, x(s)) - K(t, s, y(s))|ds \\ &\le \sup_{t\in[0,T]} \int_0^T G(t, s)|x(s) - y(s)|ds \\ &\le \sup_{t\in[0,T]} |x(t) - y(t)| \sup_{t\in[0,T]} \int_0^T G(t, s)ds \\ &= d(x, y) \sup_{t\in[0,T]} \int_0^T G(t, s)ds \\ &\le rd(x, y). \end{aligned}$$

Moreover, if (f_n) is a nonincreasing sequence in $C([0, T])$ such that $f_n \to f$ as $n \to +\infty$, then $f_n \ge f$ for all $n \in \mathbb{N}$ (see [9]). Thus, all the required hypotheses of Corollary 2.7 are satisfied. Thus, there exist a solution $u \in C([0, T])$ of the integral equation (9).

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All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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