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Composite iterative schemes for maximal monotone operators in reflexive Banach spaces

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Abstract

In this article, we introduce composite iterative schemes for finding a zero point of a finite family of maximal monotone operators in a reflexive Banach space. Then, we prove strong convergence theorems by using a shrinking projection method. Moreover, we also apply our results to a system of convex minimization problems in reflexive Banach spaces.

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Introduction

Let E be a real Banach space and C a nonempty subset of E. Let E^* be the dual space of E. We denote the value of $x^* \in E^*$ at x 2 E by $\langle x^*, x\rangle$. Let $A : E \to 2^{E^*}$ be a set-valued mapping. We denote dom A by *domain* of A, that is, dom $A = \{x \in E : Ax \neq \varnothing\}$and also denote $G(A)$ by the *graph* of A, that is, $G(A) = f\,(x, x^*) \in E \times E^* : x^* \in Ax\}$. A set-valued mapping A is said to be *monotone* if $\langle x^* - y^*, x - y\rangle \geq 0$ whenever (x, x^*); $(y, y^*) \in G(A)$. It is said to be *maximal monotone* if its graph is not contained in the graph of any other monotone operator on E. It is known that if A is maximal monotone, then the set $A^{-1}(0^*) = \{z \in E : 0^* \in Az\}$ is closed and convex.

The problem of finding zero points for maximal monotone operators plays an important role in optimizations. This is because it can be reduced to a convex minimization problem and a variational inequality problem. Many authors have studied the convergence of such problems in several settings, (see [1-6]). Initiated by Martinet [7], in a Hilbert space, Rockafellar [8] introduced the following iterative schemes:

$$\begin{cases} x_1 = x \in E, \\ x_{n+1} = J_{\lambda_n} x_n, \quad \forall n \geq 1, \end{cases} \tag{1.1}$$

where $\{\lambda_n\} \subset (0, \infty)$ and J_λ is the resolvent of A defined by $J_\lambda = (I + \lambda A)^{-1}$ for all $\lambda > 0$, and A is a maximal monotone operator on E. Such an algorithm is called the *proximal point algorithm*. He proved that the sequence $\{x_n\}$ generated by (1.1) converges weakly to an element in $A^{-1}(0)$ provided $\liminf_{n\to\infty} \lambda_n > 0$. Later, Kamimura and Takahashi [9] introduced the following iteration in a Hilbert space:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{\lambda_n} x_n, \quad \forall n \geq 1, \tag{1.2}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$. The weak convergence theorems are also established in a Hilbert space under suitable conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$.

In 2005, Kohsaka and Takahashi [10] studied the above iteration process in a more general setting, reflexive Banach spaces. In fact, those authors proposed the following algorithm:

$$x_{n+1} = \nabla f^* \left(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(J_{\lambda_n} x_n)\right), \quad \forall n \geq 1, \tag{1.3}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\lambda_n\} \subset (0, \infty)$, $f : E \to \mathbb{R}$ is a Bregman function and $J_\lambda = (\nabla f + \lambda A)^{-1} \nabla f$ for all $\lambda > 0$. They also proved a weak convergence theorem of the proposed algorithm.

Very recently, in 2010, Reich and Sabach [11] proposed an algorithm for finding a zero point of maximal monotone operators $A_i : E \to 2^{E^*}$ $(i = 1, 2,..., N)$ in a general reflexive Banach space E as follows:

$$\begin{cases} x_0 \in E, \\ y_n^i = \mathrm{Res}^f_{\lambda_n^i A_i}(x_n + e_n^i), \\ C_n^i = \left\{z \in E : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\right\}, \\ C_n = \bigcap_{i=1}^N C_n^i, \\ Q_n = \left\{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\right\}, \\ x_{n+1} = P^f_{C_n \cap Q_n}(x_0), \quad \forall n \geq 0, \end{cases} \tag{1.4}$$

where $\{\lambda_n^i\}_{i=1}^N \subset (0, \infty)$, $\{e_n\}_{i=1}^N$ is an error sequence in E with $e_n^i \to 0$ and P_K^f the Bregman projection with respect to f from E onto a closed and convex subset K of E. Those authors showed that the sequence $\{x_n\}$ defined by (1.4) converges strongly to a common element in $\bigcap_{i=1}^N A_i^{-1}(0^*)$ under some mild conditions.

Motivated by the previous ones, we first introduce a composite iterative scheme which is different from (1.4) for finding a zero point of maximal monotone operators $A_i : E \to 2^{E^*}$ $(i = 1, 2,..., N)$ in reflexive Banach spaces. Using the shrinking projection technique, introduced by Takahashi et al. [12], we then prove that a sequence generated by the proposed algorithm converges strongly to an element in $\bigcap_{i=1}^N A_i^{-1}(0^*)$ under some appropriate control conditions. Finally, we also apply our result to a system of convex minimization problems.

Preliminaries and lemmas

Let E be a real reflexive Banach space with a norm $||\cdot||$ and E^* be the dual space of E. Throughout this article, $f : E \to (-\infty, +\infty]$ is a proper, lower semi-continuous, and convex function, and the Fenchel conjugate of f is the function $f^*: E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\left\{\langle x^*, x\rangle - f(x) : \ x \in E\right\}.$$

We denote by dom f the domain of f, that is, the set $\{x \in E : f(x) < +\infty\}$. For any $x \in$ int dom f and $y \in E$, the *right-hand derivative* of f at x in the direction y is defined by

$$f^o(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

The function f is said to be *Gâteaux differentiable* at x $\lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, $f^o(x, y)$ coincides with $\nabla f(x)$, the value of the *gradient* ∇f of f at x. The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in$ int dom f. The function f is said to be *Fréchet differentiable at x* if this limit is attained uniformly in $||y|| = 1$. Finally, f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit is attained uniformly for $x \in C$ and $||y|| = 1$.

Let E be a reflexive Banach space. The Legendre function is defined from a general Banach space E into $(-\infty, +\infty]$ (see [13]). According to [13], the function f is *Legendre* if and only if it satisfies the following conditions:

(L1) The interior of the domain of f (denoted by int dom f) is nonempty, f^* is Gâteaux differentiable on int dom f, and dom ∇f = int dom f ;

(L2) The interior of the domain f^*(denoted by int dom f^*) is nonempty, f^* is Gâteaux differentiable on int dom f^*, and dom ∇f^* = int dom f^*.

Since E is reflexive, we always have $(\partial f)^{-1} = \partial f^*$ (see [14]). This fact, when combined with the conditions (L1) and (L2), implies the following equalities [15]:

$$\begin{aligned} \nabla f &= (\nabla f^*)^{-1}, \\ \operatorname{ran} \nabla f &= \operatorname{dom} \nabla f^* = \operatorname{int} \operatorname{dom} f^*, \\ \operatorname{ran} \nabla f^* &= \operatorname{dom} \nabla f = \operatorname{int} \operatorname{dom} f. \end{aligned}$$

Also, the conditions (L1) and (L2), in conjunction with [13], imply that the functions f and f^* are strictly convex on the interior of their respective domains. Several interesting examples of the Legendre functions are presented in [13,16]. Especially, the functions $\frac{1}{s}||\cdot||^s$ with $s \in (1, \infty)$ are Legendre, where the Banach space E is smooth and strictly convex and, in particular, a Hilbert space. Throughout this article, we assume that the convex function $f : E \to (\infty, +\infty]$ is Legendre.

Lemma 2.1. [17]*If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E*.*

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function D_f : dom $f \times$ int dom $f \to [0, +\infty)$ is defined as follows:

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called the *Bregman distance* with respect to f [18].

Recall that the *Bregman projection* [19] of $x \in$ int dom f onto the nonempty, closed, and convex set $C \subset$ dom f is necessarily the unique vector $P_C^f(x) \in C$ satisfying

$$D_f\left(P_C^f(x), x\right) = \inf\left\{D_f(y,x) : y \in C\right\}.$$

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function f is said to be *totally convex* at $x \in$ int dom f if its modulus of total convexity at x, that is, the function ν_f : int dom $f \times [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(x,t) := \inf\left\{D_f(y,x) : \ y \in \operatorname{dom} f, ||y-x|| \ = t\right\}$$

is positive, whenever $t > 0$. The function f is said to be *totally convex* when it is totally convex at every point $x \in$ int dom f. In addition, the function f is said to be *totally convex on bounded sets* if $\nu_f(B, t)$ is positive for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function ν_f : int dom $f \times [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(B,t) := \inf\left\{\nu_f(x,t) : \ x \in B \cap \operatorname{dom} f\right\}.$$

Let C be a nonempty, closed, and convex subset of E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. It is known from [20] that $z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z\rangle \le 0$ for all $y \in C$. We also have

$$D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \le D_f(y, x), \quad \forall x \in E,\ y \in C. \tag{2.1}$$

Recall that the function f is said to be *sequentially consistent* [20] if, for any two sequences, $\{x_n\}$ and $\{y_n\}$, in E such that the first is bounded:

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \ \Rightarrow \lim_{n\to\infty} ||y_n - x_n|| = 0.$$

The following lemmas were proved by Reich and Sabach [11].

Lemma 2.2. [11]*Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\left\{D_f(x_n, x_0)\right\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ is also bounded.*

We know that the resolvent of A, denoted by $\mathrm{Res}_A^f : E \to 2^E$, is defined as follows [21]:

$$\mathrm{Res}_A^f(x) = (\nabla f + A)^{-1} \circ \nabla f(x).$$

It is known that $F(\mathrm{Res}_A^f) = A^{-1}(0^*)$, and Res_A^f is single-valued (see [21]). If f is a Legendre function which is bounded, uniformly Fréchet differentiable on bounded, subsets of E, then $\hat{F}(\mathrm{Res}_A^f) = F(\mathrm{Res}_A^f)$ (see [22]). The *Yosida approximation* $A\lambda : E \to E$, $\lambda > 0$, is also defined by

$$A_\lambda(x) = \frac{1}{\lambda}\left(\nabla f(x) - \nabla f\left(\mathrm{Res}_{\lambda A}^f(x)\right)\right)$$

for all $x \in E$. From Proposition 2.7 in [11], we know that $(\mathrm{Res}_{\lambda A}^f(x), A_\lambda(x)) \in G(A)$ and $0^* \in Ax$ if and only if $0^* \in A_\lambda x$ for all $x \in E$ and $\lambda > 0$.

Lemma 2.3. [11]*Let $A : E \to 2^{E^*}$ be a maximal monotone operator such that $A^{-1}(0^*) \neq \emptyset$. Then,*

$$D_f(p, \mathrm{Res}_{\lambda A}^f(x)) + D_f(\mathrm{Res}_{\lambda A}^f(x), x) \le D_f(p, x)$$

for all $\lambda > 0$, $p \in A^{-1}(0^*)$ and $x \in E$.

Strong convergence theorems

Now, in this section, we prove our main results of this article.

Theorem 3.1. *Let E be a real reflexive Banach space and $f : E \to \mathbb{R}$ a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $A_i : E \to 2^{E^*}$ ($i = 1, 2,..., N$) be maximal monotone operators such that $F := \bigcap_{i=1}^{N} A_i^{-1}(0^*) \neq \emptyset$. Let $\{e_n\}_{n=1}^{\infty} \subset E$ be such that* $\lim_{n\to\infty} e_n = 0$. *Define a sequence $\{x_n\}_{n=1}^{\infty}$ in E as follows:*

$$\begin{cases} x_1 \in E, \\ C_1 = E, \\ y_n = \mathrm{Res}^f_{\lambda_n^N A_N} \circ \mathrm{Res}^f_{\lambda_n^{N-1} A_{N-1}} \circ \cdots \circ \mathrm{Res}^f_{\lambda_n^1 A_1}(x_n + e_n), \\ C_{n+1} = \left\{ z \in C_n : D_f(z, y_n) \le D_f(z, x_n + e_n) \right\}, \\ x_{n+1} = P^f_{C_{n+1}}(x_1), \quad \forall n \ge 1. \end{cases} \tag{3.1}$$

If $\liminf_{n\to\infty} \lambda_n^i > 0$ *for each* $i = 1, 2,..., N$, *then the sequence* $\{x_n\}$ *converges strongly to a point* $P^f_F(x_1)$

Proof. We divide our proof into six steps as follows:

Step 1. $F \subset C_n$ for all $n \geq 1$.

Since $A_i^{-1}(0^*)$ is closed and convex for each $i = 1, 2,..., N$, we get that $F := \bigcap_{i=1}^N A_i^{-1}(0^*)$ is a nonempty, closed and convex subset of E. It is easy to see that C_n is closed and convex for all $n \geq 1$. Indeed, for each $z \in C_n$, it follows that $D_f(z, y_n) \leq D_f(z, x_n + e_n)$ is equivalent to

$$\left\langle \nabla f(x_n + e_n) - \nabla f(y_n), z \right\rangle \le f(y_n) - f(x_n + e_n) + \left\langle \nabla f(x_n + e_n), x_n + e_n \right\rangle - \left\langle \nabla f(y_n), y_n \right\rangle.$$

This shows that C_n is closed and convex for all $n \geq 1$. It is obvious that $F \subset C1 = E$. Now, suppose that $F \subset C_k$ for some $k \in \mathbb{N}$. For any $p \in F$, by Lemma 2.3, we have

$$\begin{aligned} D_f(p, y_k) &= D_f\left(p, \mathrm{Res}^f_{\lambda_k^N A_N} \circ \mathrm{Res}^f_{\lambda_k^{N-1} A_{N-1}} \circ \cdots \circ \mathrm{Res}^f_{\lambda_k^1 A_1}(x_k + e_k)\right) \\ &\le D_f\left(p, \mathrm{Res}^f_{\lambda_k^{N-1} A_{N-1}} \circ \mathrm{Res}^f_{\lambda_k^{N-2} A_{N-2}} \circ \cdots \circ \mathrm{Res}^f_{\lambda_k^1 A_1}(x_k + e_k)\right) \\ &\quad \cdots \\ &\le D_f\left(p, \mathrm{Res}^f_{\lambda_k^2 A_2} \circ \mathrm{Res}^f_{\lambda_k^1 A_1}(x_k + e_k)\right) \\ &\le D_f\left(p, \mathrm{Res}^f_{\lambda_k^1 A_1}(x_k + e_k)\right) \\ &\le D_f(p, x_k + e_k). \end{aligned} \tag{3.2}$$

This implies that $F \subset C_{k+1}$. By induction, we can conclude that $F \subset C_n$ for all $n \geq 1$.

Step 2. $\lim_{n\to\infty} D_f(x_n, x_0)$ exists.

From $x_n = P^f_{C_n}(x_1)$ and $x_{n+1} = P^f_{C_{n+1}}(x_1) \in C_{n+1} \subset C_n$ we have

$$D_f(x_n, x_1) \le D_f(x_{n+1}, x_1), \quad \forall n \ge 1. \tag{3.3}$$

By (2.1), for any $p \in F \subset C_n$, we have

$$D_f(x_n, x_1) = D_f(P^f_{C_n}(x_1), x_1) \le D_f(p, x_1) - D_f(p, x_n) \le D_f(p, x_1). \tag{3.4}$$

Combining (3.3) and (3.4), we know that $\lim_{n\to\infty} D_f(x_n, x_1)$ exists.

Step 3. $\lim_{n\to\infty} ||\nabla f(y_n) - \nabla f(x_n + e_n)|| = 0$

Since $x_m = P^f_{C_m}(x_1) \in C_m \subset C_n$ for $m > n \geq 1$, by (2.1), it follows that

$$\begin{aligned} D_f(x_m, x_n) = D_f\left(x_m, P^f_{C_n}(x_1)\right) &\le D_f\left(x_m, x_1) - D_f(P^f_{C_n}(x_1), x_1\right) \\ &= D_f(x_m, x_1) - D_f(x_n, x_1). \end{aligned}$$

Letting $m, n \to \infty$, we have $D_f(x_m, x_n) \to 0$. Since f is totally convex on bounded subsets of E, f is sequentially consistent by Butnariu and Resmerita [20]. It follows that $||x_m - x_n|| \to 0$ as $m, n \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence. By the completeness of the space E, we can assume that $x_n \to q \in E$ as $n \to \infty$. In particular, we obtain

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0.$$

Since $e_n \to 0$, we also obtain

$$\lim_{n\to\infty} ||x_{n+1} - (x_n + e_n)|| = 0. \tag{3.5}$$

Since $x_{n+1} = P^f_{C_{n+1}}(x_1) \in C_{n+1}$

$$\begin{aligned} D_f(x_{n+1}, y_n) &\leq D_f(x_{n+1}, x_n + e_n) \\ &= f(x_{n+1}) - f(x_n + e_n) - \left\langle \nabla f(x_n + e_n), x_{n+1} - (x_n + e_n) \right\rangle. \end{aligned}$$

We know from [23] that, if f is bounded on bounded subsets of E, then ∇f is also bounded on bounded subsets of E. Moreover, if f is uniformly Fréchet differentiable on bounded subsets of E, then f is uniformly continuous on bounded subsets of E (see [24]). Using (3.5), we have

$$\lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0.$$

Also, we have

$$\lim_{n\to\infty} ||x_{n+1} - y_n|| = 0$$

and hence,

$$\lim_{n\to\infty} ||y_n - x_n|| = 0$$

and, since $e_n \to 0$,

$$\lim_{n\to\infty} ||y_n - (x_n + e_n)|| = 0. \tag{3.6}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, ∇f is norm-to-norm uniformly continuous on bounded subsets of E by Lemma 2.1. Hence, we have

$$\lim_{n\to\infty} ||\nabla f(y_n) - \nabla f(x_n + e_n)|| = 0. \tag{3.7}$$

Step 4. $\lim_{n\to\infty} \left\| \nabla f\left(\Theta^i_n(x_n + e_n)\right) - \nabla f\left(\Theta^{i-1}_n(x_n + e_n)\right)\right\| = 0 \ \forall i = 1, 2, \ldots, N.$

Denote $\Theta^i_n = \mathrm{Res}^f_{\lambda^i_n A_i} \circ \mathrm{Res}^f_{\lambda^{i-1}_n A_{i-1}} \circ \cdots \circ \mathrm{Res}^f_{\lambda^1_n A_1}$ for each $i \in \{1, 2,..., N\}$ and $\Theta^0_n = I$ for each $n \geq 1$. We note that $y_n = \Theta^N_n(x_n + e_n)$ for each $n \geq 1$. For any $p \in F$, by (3.2), it follows that

$$\begin{aligned} D_f(p, \Theta^{N-1}_n(x_n + e_n)) &\leq D_f(p, \Theta^{N-2}_n(x_n + e_n)) \\ &\leq D_f(p, \Theta^{N-3}_n(x_n + e_n)) \\ &\cdots \\ &\leq D_f(p, (x_n + e_n)). \end{aligned} \tag{3.8}$$

Since $p \in A_N^{-1}(0^*)$, by Lemma 2.3 and (3.8), it follows that

$$
\begin{aligned}
&D_f(y_n, \Theta_n^{N-1}(x_n + e_n)) \\
&\quad \le D_f(p, \Theta_n^{N-1}(x_n + e_n)) - D_f(p, y_n) \\
&\quad \le D_f(p, (x_n + e_n)) - D_f(p, y_n) \\
&\quad = f(y_n) - f(x_n + e_n) - \langle \nabla f(x_n + e_n), p - (x_n + e_n)\rangle + \langle \nabla f(y_n), p - y_n\rangle \\
&\quad = f(y_n) - f(x_n + e_n) - \langle \nabla f(x_n + e_n), p - y_n\rangle \\
&\qquad + \langle \nabla f(x_n + e_n), (x_n + e_n) - y_n\rangle + \langle \nabla f(y_n), p - y_n\rangle \\
&\quad = f(y_n) - f(x_n + e_n) + \langle \nabla f(y_n) - \nabla f(x_n + e_n), p - y_n\rangle \\
&\qquad + \langle \nabla f(x_n + e_n), (x_n + e_n) - y_n\rangle.
\end{aligned}
$$

From (3.6) and (3.7), we get that $\lim_{n\to\infty} D_f(y_n, \Theta_n^{N-1}(x_n + e_n)) = 0$. Since f is sequentially consistent,

$$\lim_{n\to\infty} ||y_n - \Theta_n^{N-1}(x_n + e_n)|| = 0. \tag{3.9}$$

Thus, from (3.6) and (3.9), it follows that

$$\lim_{n\to\infty} ||(x_n + e_n) - \Theta_n^{N-1}(x_n + e_n)|| = 0 \tag{3.10}$$

and hence,

$$\lim_{n\to\infty} \left\| \nabla f(x_n + e_n) - \nabla f(\Theta_n^{N-1}(x_n + e_n)) \right\| = 0. \tag{3.11}$$

Again, since $p \in A_N^{-1}(0^*)$, by Lemma 2.3 and (3.8), we know that

$$
\begin{aligned}
&D_f(\Theta_n^{N-1}(x_n + e_n), \Theta_n^{N-2}(x_n + e_n)) \\
&\quad \le D_f(p, \Theta_n^{N-2}(x_n + e_n)) - D_f(p, \Theta_n^{N-1}(x_n + e_n)) \\
&\quad \le D_f(p, (x_n + e_n)) - D_f(p, \Theta_n^{N-1}(x_n + e_n)).
\end{aligned}
$$

From (3.10) and (3.11), we have

$$\lim_{n\to\infty} D_f(\Theta_n^{N-1}(x_n + e_n), \Theta_n^{N-2}(x_n + e_n)) = 0.$$

Since f is sequentially consistent, it follows that

$$\lim_{n\to\infty} \left\| \Theta_n^{N-1}(x_n + e_n) - \Theta_n^{N-2}(x_n + e_n) \right\| = 0. \tag{3.12}$$

From (3.10) and (3.12), we have

$$\lim_{n\to\infty} \left\| (x_n + e_n) - \Theta_n^{N-2}(x_n + e_n) \right\| = 0,$$

and hence,

$$\lim_{n\to\infty} \left\| \nabla f(x_n + e_n) - \nabla f(\Theta_n^{N-2}(x_n + e_n)) \right\| = 0.$$

In a similar way, we can show that
$\lim_{n\to\infty} ||\Theta_n^{N-2}(x_n+e_n) - \Theta_n^{N-3}(x_n+e_n)|| = \cdots = \lim_{n\to\infty} ||\Theta_n^{1}(x_n+e_n) - (x_n+e_n)|| = 0$,
$\lim_{n\to\infty} ||(x_n + e_n) - \Theta_n^{N-3}(x_n + e_n)|| = \cdots = \lim_{n\to\infty} ||(x_n + e_n) - \Theta_n^{1}(x_n + e_n)|| = 0$ and

$$\begin{aligned}
&\lim_{n\to\infty} \left\| \nabla f(x_n + e_n) - \nabla f(\Theta_n^{N-3}(x_n + e_n)) \right\| \\
&\quad = \lim_{n\to\infty} \left\| \nabla f(x_n + e_n) - \nabla f(\Theta_n^{N-4}(x_n + e_n)) \right\| \\
&\qquad \cdots \\
&\quad = \lim_{n\to\infty} \left\| \nabla f(x_n + e_n) - \nabla f(\Theta_n^{1}(x_n + e_n)) \right\| \\
&\quad = 0.
\end{aligned}$$

Hence, we can conclude that

$$\lim_{n\to\infty} \left\| \nabla f(\Theta_n^{i}(x_n + e_n)) - \nabla f(\Theta_n^{i-1}(x_n + e_n)) \right\| = 0 \tag{3.13}$$

for each $i = 1,2,..., N$.

Step 5. $q \in \bigcap_{i=1}^{N} A_i^{-1}(0^*)$.

For each $i = 1, 2,..., N$, we note that $\Theta_n^i(x_n + e_n) = \mathrm{Res}^f_{\lambda_n^i A_i} \Theta_n^{i-1}(x_n + e_n)$ and so

$$\left\| A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n) \right\| = \frac{1}{\lambda_n^i} \left\| \nabla f(\Theta_n^{i-1}(x_n + e_n)) - \nabla f(\Theta_n^{i}(x_n + e_n)) \right\| .$$

From (3.13) and $\liminf_{n\to\infty} \lambda_n^i > 0$, we have

$$\lim_{n\to\infty} ||A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n)|| = 0. \tag{3.14}$$

We note that $(\Theta_n^i(x_n + e_n), A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n)) \in G(A_i)$ for each $i = 1, 2,..., N$. If $(w, w^*) \in G(Ai)$ for each $i = 1, 2,..., N$, then it follows from the monotonicity of A_i that

$$\left\langle w^* - A_{\lambda_n^i} \Theta_n^{i-1}(x_n + e_n), w - \Theta_n^i(x_n + e_n) \right\rangle \geq 0.$$

Since $x_n \to q$ and $e_n \to 0$, $x_n + e_n \to q$. Therefore, $\Theta_n^i(x_n + e_n) \to q$ for each $i = 1, 2,..., N$. Thus, from (3.14), we have

$$\langle w^*, w - q \rangle \geq 0.$$

By the maximality of A_i, we have $q \in A_i^{-1}(0^*)$ for each $i = 1, 2,..., N$. Hence, $q \in F := \bigcap_{i=1}^{N} A_i^{-1}(0^*)$.

Step 6. $q = P_F^f(x_1)$.

From $x_n = P_{C_n}^f(x_1)$, we have

$$\left\langle \nabla f(x_1) - \nabla f(x_n), x_n - z \right\rangle \geq 0, \quad \forall z \in C_n.$$

Since $F \subset C_n$, we also have

$$\left\langle \nabla f(x_1) - \nabla f(x_n), x_n - z \right\rangle \geq 0, \quad \forall z \in F. \tag{3.15}$$

Letting $n \to \infty$ in (3.15), we obtain

$$\left\langle \nabla f(x_1) - \nabla f(q), q - z \right\rangle \geq 0, \quad \forall z \in F.$$

Hence, we have $q = P_F^f(x_1)$. This completes the proof.

As a direct consequence of Theorem 3.1, we also obtain the following result concerning a system of convex minimization problems in reflexive Banach spaces:

Theorem 3.2. *Let E be a real reflexive Banach space and* $f : E \to \mathbb{R}$ *a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let* $g_i : E \to (-\infty, \infty]$ $(i = 1, 2,..., N)$ *be proper lower semi-continuous convex functions such that* $F := \bigcap_{i=1}^{N} (\partial g_i^{-1})(0) \neq \emptyset$. *Let* $\{e_n\}_{n=1}^{\infty} \subset E$ *be a sequence in E such that* $\lim_{n\to\infty} e_n = 0$. *Define a sequence* $\{x_n\}_{n=1}^{\infty}$ *in E as follows:*

$$\begin{cases} x_1 \in E, \\ C_1 = E, \\ z_n^1 = \arg\min_{y\in E}\left\{g1(y) + \dfrac{1}{\lambda_n^1}D_f(y, x_n + e_n)\right\}, \\ \quad \cdots \\ z_n^{N-1} = \arg\min_{y\in E}\left\{gN-1(y) + \dfrac{1}{\lambda_n^{N-1}}D_f(y, z_n^{N-2})\right\}, \\ y_n = \arg\min_{y\in E}\left\{gN(y) + \dfrac{1}{\lambda_n^N}D_f(y, z_n^{N-1})\right\}, \\ C_{n+1} = \left\{z \in C_n : D_f(z, y_n) \leq D_f(z, x_n + e_n)\right\}, \\ x_{n+1} = P_{C_{n+1}}^f(x_1), \quad \forall n \geq 1. \end{cases} \tag{3.16}$$

If $\liminf_{n\to\infty}\lambda_n^i > 0$ *for each* $i = 1, 2,..., N$, *then the sequence* $\{x_n\}$ *converges strongly to a point* $P_F^f(x_1)$.

Proof. By Rockafellar's theorem [25,26], ∂g_i are maximal monotone operators for each $i = 1, 2,..., N$. Let $\lambda^i > 0$ for each $i = 1, 2,..., N$. Then $z^i = \mathrm{Res}_{\lambda^i\partial g_i}^f(x)$ if and only if

$$\begin{aligned} 0 &\in \partial g_i(z^i) + \frac{1}{\lambda^i}\left(\nabla f(z^i) - \nabla f(x)\right) \\ &= \partial\left(g_i + \frac{1}{\lambda^i}(f - \nabla f(x))\right)(z^i), \end{aligned}$$

which is equivalent to

$$\begin{aligned} z^i &= \arg\min_{y\in E}\left\{g_i(y) + \frac{1}{\lambda^i}(f(y) - \langle y, \nabla f(x)\rangle)\right\} \\ &= \arg\min_{y\in E}\left\{g_i(y) + \frac{1}{\lambda^i}D_f(y, x)\right\}. \end{aligned}$$

Using Theorem 3.1, we can complete the proof.

Remark 3.3. By means of the composite iterative scheme together with the shrinking projection method, we can construct the proximal point algorithms for finding a common element in the set $\bigcap_{i=1}^{N} A_i^{-1}(0^*)$. Moreover, our algorithm is different from that of Reich and Sabach [11] which is based on a finite intersection of sets.

Remark 3.4. Theorems 3.1 and 3.2 also hold in a uniformly convex and uniformly smooth Banach space with the generalized duality mapping.

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Authors' contributions

PC designed of the study, performed the nonlinear and convex analysis and also wrote the article. YJC participated in the design of the study, carried out the materials and helped to check the manuscript. SS conceived of the study, participated in its design and also helped to draft the manuscript. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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