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Some new fixed point theorems for set-valued contractions in complete metric spaces

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Abstract

In this article, we obtain some new fixed point theorems for set-valued contractions in complete metric spaces. Our results generalize or improve many recent fixed point theorems in the literature.

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1 Introduction and preliminaries

Let (X, d) be a metric space, D a subset of X and $f : D \to X$ be a map. We say f is contractive if there exists $\alpha \in [0, 1)$ such that for all $x, y \in D$,

$$d(fx, fy) \leq \alpha \cdot d(x, y).$$

The well-known Banach's fixed point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X. It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, a mapping $f : X \to X$ is called a quasi-contraction if there exists $k < 1$ such that

$$d(fx, fy) \leq k \cdot \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}$$

for any $x, y \in X$. In 1974, C'iric' [2] introduced these maps and proved an existence and uniqueness fixed point theorem.

Throughout we denote the family of all nonempty closed and bounded subsets of X by $CB(X)$. The existence of fixed points for various multi-valued contractive mappings had been studied by many authors under different conditions. In 1969, Nadler [3] extended the famous Banach Contraction Principle from single-valued mapping to multi-valued mapping and proved the below fixed point theorem for multi-valued contraction.

Theorem 1 [3]*Let (X, d) be a complete metric space and $T : X \to CB(X)$. Assume that there exists $c \in [0, 1)$ such that*

$$\mathcal{H}(Tx, Ty) \leq cd(x, y) \quad \text{for all } x, y \in X,$$

where $\mathcal{H}$denotes the Hausdorff metric on $CB(X)$ induced by d, that is, $H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\}$, for all $A, B \in CB(X)$ and $D(x, B) = \inf_{z\in B} d(x, z)$. Then, T has a fixed point in X.

In 1989, Mizoguchi-Takahashi [4] proved the following fixed point theorem.

Theorem 2 [4]*Let* (X, d) *be a complete metric space and* $T : X \to CB(X)$. *Assume that*

$$\mathcal{H}(Tx, Ty) \le \xi(d(x, y)) \cdot d(x, y)$$

for all $x, y \in X$, *where* $\xi : [0, \infty) \to [0, 1)$ *satisfies* $\limsup_{s\to t^+}\xi(s) < 1$*for all* $t \in [0, \infty)$. *Then,* T *has a fixed point in* X.

In the recent, Amini-Harandi [5] gave the following fixed point theorem for set-valued quasi-contraction maps in metric spaces.

Theorem 3 [5]*Let* (X, d) *be a complete metric space. Let* $T : X \to CB(X)$ *be a* k*-set-valued quasi-contraction with* $k < \frac{1}{2}$, *that is,*

$$\mathcal{H}(Tx, Ty) \le k \cdot \max\{(x, y), D(x, Tx), D(y, Ty), D(x, Ty)), D(y, Tx)\}$$

for any $x, y \in X$. *Then,* T *has a fixed point in* X.

2 Fixed point theorem (I)

In this section, we assume that the function $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ satisfies the following conditions:

(C1) ψ is a strictly increasing, continuous function in each coordinate, and

(C2) for all $t \in \mathbb{R}^+$, $\psi(t, t, t, 0, 2t) < t$, $\psi(t, t, t, 2t, 0) < t$, $\psi(0, 0, t, t, 0) < t$ and $\psi(t, 0, 0, t, t) < t$.

Definition 1 *Let* (X, d) *be a metric space. The set-valued map* $T : X \to X$ *is said to be a set-valued* ψ*-contraction, if*

$$\mathcal{H}(Tx, Ty) \le \psi(d(x, y), D(x, Tx), D(y, Ty), D(x, Ty)), D(y, Tx))$$

for all $x, y \in X$.

We now state the main fixed point theorem for a set-valued ψ-contraction in metric spaces, as follows:

Theorem 4 *Let* (X, d) *be a complete metric space. Let* $T : X \to CB(X)$ *be a set-valued* ψ*-contraction. Then,* T *has a fixed point in* X.

Proof. Note that for each $A, B \in CB(X)$, $a \in A$ and $\gamma > 0$ with $\mathcal{H}(A, B) < \gamma$, there exists $b \in B$ such that $d(a, b) < \gamma$. Since $T : X \to CB(X)$ is a set-valued ψ-contraction, we have

$$\mathcal{H}(Tx, Ty) \le \psi(d(x, y), D(x, Tx), D(y, Ty), D(x, Ty)), D(y, Tx))$$

for all $x, y \in X$. Suppose that $x_0 \in X$ and that $x_1 \in X$. Then, by induction and by the above observation, we can find a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and for each $n \in \mathbb{N}$,

$$\begin{aligned} d(x_{n+1}, x_n) &\le \psi(d(x_n, x_{n-1}), D(x_n, Tx_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_{n-1}), D(x_{n-1}, Tx_n)) \\ &\le \psi(d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n), d(x_{n-1}, x_{n+1})) \\ &\le \psi(d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), 0, d(x_{n-1}, x_n) + d(x_n, x_{n+1})), \end{aligned}$$

and hence, we can deduce that for each $n \in \mathbb{N}$,

$$d(x_{n+1}, x_n) \le d(x_n, x_{n-1}).$$

Let we denote $c_m = d(x_{m+1}, x_m)$. Then, c_m is a non-increasing sequence and bounded below. Thus, it must converges to some $c \ge 0$. If $c > 0$, then by the above inequalities, we have

$$c \leq c_{n+1} \leq \psi(c_n, c_n, c_n, 0, 2c_n).$$

Passing to the limit, as $n \to \infty$, we have

$$c \leq c \leq \psi(c, c, c, 0, 2c) < c,$$

which is a contradiction. Hence, $c = 0$.

We next claim that the following result holds:

for each $\gamma > 0$, there is $n_0(\gamma) \in \mathbb{N}$ such that for all $m > n > n_0(\gamma)$,

$$d(x_m, x_n) < \gamma. \quad (*)$$

We shall prove (*) by contradiction. Suppose that (*)is false. Then, there exists some $\gamma > 0$ such that for all $k \in \mathbb{N}$, there exist m_k, $n_k \in \mathbb{N}$ with $m_k > n_k \geq k$ satisfying:

(1) m_k is even and n_k is odd;
(2) $d(x_{m_k}, x_{n_k}) \geq \gamma$;
(3) m_k is the smallest even number such that the conditions (1), (2) hold.

Since $c_m \searrow 0$, by (2), we have $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \gamma$ and

$$\begin{aligned}\gamma &\leq d(x_{m_k}, x_{n_k}) \leq \mathcal{H}(Tx_{m_k-1}, Tx_{n_k-1}) \\ &\leq \psi(d(x_{m_k-1}, x_{n_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), d(x_{m_k-1}, x_{n_k}), d(x_{n_k-1}, x_{m_k})) \\ &\leq \psi(c_{m_k-1} + d(x_{m_k}, x_{n_k}) + c_{n_k-1}, c_{m_k-1}, c_{n_k-1}, c_{m_k-1} + d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{n_k}) + c_{n_k-1})).\end{aligned}$$

Letting $k \to \infty$. Then, we get

$$\gamma \leq \psi(\gamma, 0, 0, \gamma, \gamma) < \gamma,$$

a contradiction. It follows from (*) that the sequence $\{x_n\}$ must be a Cauchy sequence.

Similarly, we also conclude that for each $n \in \mathbb{N}$,

$$\begin{aligned}d(x_n, x_{n+1}) &\leq \psi(d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n), D(x_{n-1}, Tx_n), D(x_n, Tx_{n-1})) \\ &\leq \psi(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)) \\ &\leq \psi(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n) + d(x_n, x_{n+1}), 0),\end{aligned}$$

and hence, we have that for each $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).$$

Let we denote $b_m = d(x_m, x_{m+1})$. Then, b_m is a non-increasing sequence and bounded below. Thus, it must converges to some $b \geq 0$. If $b > 0$, then by the above inequalities, we have

$$b \leq b_{n+1} \leq \psi(b_n, b_n, b_n, 2b_n, 0).$$

Passing to the limit, as $n \to \infty$, we have

$$b \leq b \leq \psi(b, b, b, 2b, 0) < b,$$

which is a contradiction. Hence, $b = 0$. By the above argument, we also conclude that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $\mu \in X$ such that $\lim_{n\to\infty} x_n = \mu$. Therefore,

$$\begin{aligned} D(\mu, T\mu) &= \lim_{n\to\infty} D(x_{n+1}, T\mu) \\ &\le \lim_{n\to\infty} \mathcal{H}(Tx_n, T\mu) \\ &\le \lim_{n\to\infty} \psi(d(x_n, \mu), D(x_n, Tx_n), D(\mu, T\mu), D(x_n, T\mu), D(\mu, Tx_n)) \\ &\le \lim_{n\to\infty} \psi(d(x_n, \mu), d(x_n, x_{n+1}), D(\mu, T\mu), D(x_n, T\mu), d(\mu, x_{n+1})) \\ &\le \psi(0, 0, D(\mu, T\mu), D(\mu, T\mu), 0) \\ &< D(\mu, T\mu), \end{aligned}$$

and hence, $D(\mu, T\mu) = 0$, that is, $\mu \in T\mu$, since $T\mu$ is closed.

3 Fixed point theorem (II)

In 1972, Chatterjea [6] introduced the following definition.

Definition 2 *Let* (X, d) *be a metric space. A mapping* $f : X \to X$ *is said to be a* $\mathcal{C}$*-contraction if there exists* $\alpha \in (0, \frac{1}{2})$ *such that for all* $x, y \in X$*, the following inequality holds:*

$$d(fx, fy) \le \alpha \cdot (d(x, fy) + d(y, fx)).$$

Choudhury [7] introduced a generalization of $\mathcal{C}$-contraction, as follows:

Definition 3 *Let* (X, d) *be a metric space. A mapping* $f : X \to X$ *is said to be a weakly* $\mathcal{C}$*-contraction if for all* $x, y \in X$*,*

$$d(fx, fy) \le \frac{1}{2}(d(x, fy) + d(y, fx) - \phi(d(x, fy), d(y, fx))),$$

where $\varphi : \mathbb{R}^{+2} \to \mathbb{R}^{+}$ *is a continuous function such that* $\psi(x, y) = 0$ *if and only if* $x = y = 0$.

In [6,7], the authors proved some fixed point results for the $\mathcal{C}$-contractions. In this section, we present some fixed point results for the weakly ψ-$\mathcal{C}$-contraction in complete metric spaces.

Definition 4 *Let* (X, d) *be a metric space. The set-valued map* $T : X \to X$ *is said to be a set-valued weakly* ψ-$\mathcal{C}$*-contraction, if for all* $x, y \in X$

$$\mathcal{H}(Tx, Ty) \le \psi([D(x, Ty) + D(y, Tx) - \phi(D(x, Ty), D(y, Tx))]),$$

where

(1) $\psi : \mathbb{R}^{+} \to \mathbb{R}^{+}$ *is a strictly increasing, continuous function with* $\psi(t) \le \frac{1}{2}t$ *for all* $t > 0$ *and* $\psi(0) = 0$;

(2) $\varphi : \mathbb{R}^{+2} \to \mathbb{R}^{+}$ *is a strictly decreasing, continuous function in each coordinate, such that* $\varphi(x, y) = 0$ *if and only if* $x = y = 0$ and $\varphi(x, y) \le x + y$ for all $x, y \in \mathbb{R}^{+}$.

Theorem 5 *Let* (X, d) *be a complete metric space. Let* $T : X \to CB(X)$ *be a set-valued weakly* $\mathcal{C}$*-contraction. Then,* T *has a fixed point in* X.

Proof. Note that for each $A, B \in CB(X)$, $a \in A$ and $\gamma > 0$ with $\mathcal{H}(A, B) < \gamma$, there exists $b \in B$ such that $d(a, b) < \gamma$. Since $T : X \to CB(X)$ be a set-valued weakly ψ-$\mathcal{C}$-contraction, we have that

$$\mathcal{H}(Tx, Ty) \le \psi([D(x, Ty) + D(y, Tx) - \phi(D(x, Ty), D(y, Tx))])$$

for all $x, y \in X$. Suppose that $x_0 \in X$ and that $x_1 \in X$. Then, by induction and by the above observation, we can find a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and for each

$n \in \mathbb{N}$,

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \mathcal{H}(Tx_n, Tx_{n-1}) \\ &\leq \psi([D(x_n, Tx_{n-1}) + D(x_{n-1}, Tx_n) - \phi(D(x_n, Tx_{n-1}), D(x_{n-1}, Tx_n))]) \\ &\leq \psi([d(x_n, x_n) + d(x_{n-1}, x_{n+1}) - \phi(d(x_n, x_n), d(x_{n-1}, x_{n+1}))]) \\ &= \psi([0 + d(x_{n-1}, x_{n+1}) - \phi(0, d(x_{n-1}, x_{n+1}))]) \\ &\leq \psi([d(x_{n-1}, x_n) + d(x_n, x_{n+1})]) \\ &\leq \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \end{aligned}$$

and hence, we deduce that for each $n \in \mathbb{N}$,

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}).$$

Thus, $\{d(x_{n+1}, x_n)\}$ is non-increasing sequence and bounded below and hence it is convergent. Let $\lim_{n\to\infty} d(x_{n+1}, x_n) = \xi$. Letting $n \to \infty$ in (**), we have

$$\begin{aligned} \xi = \lim_{n\to\infty} d(x_{n+1}, x_n) &\leq \lim_{n\to\infty} \psi([d(x_{n-1}, x_{n+1})]) \\ &\leq \lim_{n\to\infty} \frac{1}{2}[d(x_{n-1}, x_{n+1})] \\ &\leq \lim_{n\to\infty} \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\leq \frac{1}{2}[\xi + \xi] = \xi, \end{aligned}$$

that is,

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2\xi.$$

By the continuity of ψ and φ, letting $n \to \infty$ in (**), we have

$$\xi \leq \psi(2\xi - \phi(0, 2\xi)) \leq \xi - \frac{1}{2} \cdot \phi(0, 2\xi) \leq \xi.$$

Hence, we have $\varphi(0, 2\xi) = 0$, that is, $\xi = 0$. Thus, $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

We next claim that the following result holds:

for each $\gamma > 0$, there is $n_0(\gamma) \in \mathbb{N}$ such that for all $m > n > n_0(\gamma)$,

$$d(x_m, x_n) < \gamma. \quad (***)$$

We shall prove (***) by contradiction. Suppose that (***) is false. Then, there exists some $\gamma > 0$ such that for all $k \in \mathbb{N}$, there exist $m_k, n_k \in \mathbb{N}$ with $m_k > n_k \geq k$ satisfying:

(1) m_k is even and n_k is odd;
(2) $d(x_{m_k}, x_{n_k}) \geq \gamma$;
(3) m_k is the smallest even number such that the conditions (1), (2) hold.

Since $d(x_{n+1}, x_n) \searrow 0$, by (2), we have $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \gamma$ and

$$\begin{aligned} \gamma &\leq d(x_{m_k}, x_{n_k}) \leq \mathcal{H}(Tx_{m_k-1}, Tx_{n_k-1}) \\ &\leq \psi([D(x_{m_k-1}, Tx_{n_k-1}) + D(x_{n_k-1}, Tx_{m_k-1}) - \phi(D(x_{m_k-1}, Tx_{n_k-1}), D(x_{n_k-1}, Tx_{m_k-1}))]) \\ &\leq \psi([d(x_{m_k-1}, x_{n_k}) + d(x_{n_k-1}, x_{m_k}) - \phi(d(x_{m_k-1}, x_{n_k}), d(x_{n_k-1}, Tx_{m_k}))]). \end{aligned}$$

Since

$$d(x_{m_k-1}, x_{n_k}) + d(x_{n_k-1}, x_{m_k}) \leq d(x_{m_k-1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{n_k-1}, x_{n_k}),$$

letting $k \to \infty$, then we get

$$\gamma \le \psi(2\gamma - \phi(\gamma, \gamma)) \le \gamma,$$

and hence, $\varphi(\gamma, \gamma)) = 0$. By the definition of φ, we get $\gamma = 0$, a contradiction. This proves that the sequence $\{x_n\}$ must be a Cauchy sequence.

Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Therefore,

$$\begin{aligned} D(z, Tz) &= \lim_{n\to\infty} D(x_{n+1}, Tz) \\ &\le \lim_{n\to\infty} \mathcal{H}(Tx_n, Tz) \\ &\le \lim_{n\to\infty} \psi([D(x_n, Tz) + D(z, Tx_n) - \phi(D(x_n, Tz), D(z, Tx_n))]) \\ &\le \lim_{n\to\infty} \psi([D(x_n, Tz) + d(z, x_{n+1}) - \phi(D(x_n, Tz), d(z, x_{n+1}))]) \\ &\le \frac{1}{2} D(z, Tz) \end{aligned}$$

and hence, $D(z, Tz) = 0$, that is, $z \in Tz$, since Tz is closed.

4 Fixed point theorem (III)

In this section, we recall the notion of the Meir-Keeler type function (see [8]). A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Meir-Keeler type function, if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in \mathbb{R}^+$ with $\eta \le t < \eta + \delta$, we have $\phi(t) < \eta$. We now introduce the new notions of the weaker Meir-Keeler type function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ in a metric space and the ϕ-function using the weaker Meir-Keeler type function, as follow:

Definition 5 *Let (X, d) be a metric space. We call $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ a weaker Meir-Keeler type function, if for each $\eta > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\eta \le d(x, y) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(d(x, y)) < \gamma_\eta$.*

Definition 6 *Let (X, d) be a metric space. A weaker Meir-Keeler type function ϕ ; $\mathbb{R}^+ \to \mathbb{R}^+$ is called a ϕ-function, if the following conditions hold:*

(ϕ_1) $\phi(0) = 0$, $0 < \phi(t) < t$ for all $t > 0$;

(ϕ_2) ϕ is a strictly increasing function;

(ϕ_3) for each $t \in \mathbb{R}^+$, $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;

(ϕ_4) for each $t_n \in \mathbb{R}^+ \setminus \{0\}$, if $\lim_{n\to\infty} t_n = \gamma > 0$, then $\lim_{n\to\infty} \phi(t_n) < \gamma$;

(ϕ_5) for each $t_n \in \mathbb{R}^+$, if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} \phi(t_n) = 0$.

Definition 7 *Let (X, d) be a metric space. The set-valued map $T : X \to X$ is said to be a set-valued weaker Meir-Keeler type ϕ-contraction, if*

$$\mathcal{H}(Tx, Ty) \le \varphi\left(\frac{1}{2}[D(x, Ty) + D(y, Tx)]\right)$$

for all $x, y \in X$.

We now state the main fixed point theorem for a set-valued weaker Meir-Keeler type ψ-contraction in metric spaces, as follows:

Theorem 6 *Let (X, d) be a complete metric space. Let $T : CB(X)$ be a set-valued weaker Meir-Keeler type ψ-contraction. Then, T has a fixed point in X.*

Proof. Note that for each $A, B \in CB(X)$, $a \in A$ and $\gamma > 0$ with $\mathcal{H}(A, B) < \gamma$, there exists $b \in B$ such that $d(a, b) < \gamma$. Since $T : X \to CB(X)$ be a set-valued ψ-contraction, we have that

$$\mathcal{H}(Tx, Ty) \le \varphi\left(\frac{1}{2}[D(x, Ty) + D(y, Tx)]\right)$$

for all x, $y \in X$. Suppose that $x_0 \in X$ and that $x_1 \in X$. Then, by induction and by the above observation, we can find a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and for each $n \in \mathbb{N}$,

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \varphi\left(\frac{1}{2}[D(x_n, Tx_{n-1}) + D(x_{n-1}, Tx_n)]\right) \\ &\leq \varphi\left(\frac{1}{2}[d(x_n, x_n) + d(x_{n-1}, x_{n+1})]\right) \\ &\leq \varphi\left(\frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\right), \end{aligned}$$

and by the conditions (ϕ_1) and (ϕ_2), we can deduce that for each $n \in \mathbb{N}$,

$$d(x_{n+1}, x_n) \leq \varphi(d(x_n, x_{n-1})) < d(x_n, x_{n-1})$$

and

$$d(x_{n+1}, x_n) \leq \varphi(d(x_n, x_{n-1})) \leq \cdots \leq \varphi^n(d(x_1, x_0)).$$

By the condition (ϕ_3), $\{\phi^n(d(x_0, x_1))\}_{n\in \mathbb{N}}$ is decreasing, it must converges to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then, by the definition of the weaker Meir-Keeler type function, there exists $\delta > 0$ such that for x_0, $x_1 \in X$ with $\eta \leq d(x_0, x_1) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(d(x_0, x_1)) < \eta$. Since $\lim_{n\to\infty}\phi^n(d(x_0, x_1)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \leq \phi^m(d(x_0, x_1)) < \delta + \eta$, for all $m \geq m_0$. Thus, we conclude that $\varphi^{m_0+n_0}(d(x_0, x_1)) < \eta$. Hence, we get a contradiction. Hence, $\lim_{n\to\infty}\phi^n(d(x_0, x_1)) = 0$, and hence, $\lim_{n\to\infty}d(x_n, x_{n+1}) = 0$.

Next, we let $c_m = d(x_m, x_{m+1})$, and we claim that the following result holds:

for each $\varepsilon > 0$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that for all m , $n \geq n_0(\varepsilon)$,

$$d(x_m, x_{m+1}) < \varepsilon. \quad (****)$$

We shall prove (****) by contradiction. Suppose that (****) is false. Then, there exists some $\varepsilon > 0$ such that for all $p \in N$, there are m_p, $n_p \in \mathbb{N}$ with $m_p > n_p \geq p$ satisfying:

(i) m_p is even and n_p is odd,
(ii) $d(x_{m_p}, x_{n_p}) \geq \varepsilon$, and
(iii) m_p is the smallest even number such that the conditions (i), (ii) hold.

Since $c_m \searrow 0$, by (*ii*), we have $\lim_{p\to\infty}d(x_{m_p}, x_{n_p}) = \varepsilon$, and

$$\begin{aligned} \varepsilon &\leq d(x_{m_p}, x_{n_p}) \\ &\leq \mathcal{H}(Tx_{m_p-1}, Tx_{n_p-1}) \\ &\leq \varphi\left(\frac{1}{2}[D(x_{m_p-1}, Tx_{n_p-1}) + D(x_{n_p-1}, Tx_{m_p-1})]\right) \\ &\leq \varphi\left(\frac{1}{2}[d(x_{m_p-1}, x_{n_p}) + d(x_{n_p-1}, x_{m_p})]\right) \\ &\leq \varphi\left(\frac{1}{2}[d(x_{m_p-1}, x_{m_p}) + 2d(x_{n_p}, x_{m_p}) + d(x_{n_p-1}, x_{n_p})]\right). \end{aligned}$$

Letting $p \to \infty$. By the condition (ϕ_4), we have

$$\varepsilon \leq \lim_{p\to\infty} \varphi\left(\frac{1}{2}[d(x_{m_p-1}, x_{m_p}) + 2d(x_{n_p}, x_{m_p}) + d(x_{n_p-1}, x_{n_p})]\right) < \varepsilon,$$

a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $\mu \in X$ such that $\lim n{\to}\infty x_{n+1} = \mu$. Therefore,

$$\begin{aligned} D(\mu, T\mu) &= \lim_{n\to\infty} D(x_{n+1}, T\mu) \\ &\leq \lim_{n\to\infty} \mathcal{H}(Tx_n, T\mu) \\ &\leq \lim_{n\to\infty} \varphi\left(\frac{1}{2}[D(x_n, T\mu)) + D(\mu, Tx_n)\right) \\ &\leq \lim_{n\to\infty} \varphi\left(\frac{1}{2}[D(x_n, T\mu)) + d(\mu, x_{n+1})\right) \\ &\leq \frac{1}{2} D(\mu, T\mu), \end{aligned}$$

and hence, $D(\mu, T\mu) = 0$, that is, $\mu \in T\mu$, since $T\mu$ is closed.

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Competing interests
The author declares he has no competing interests

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References
1. Banach, S: Sur les operations dans les ensembles abstraits et leur application aux equations integerales. Fund Math. **3**, 133–181 (1922)
2. C'iric', LB: A generalization of Banach's contraction principle. Proc Am Math Soc. **45**(2), 45–181 (1974)
3. Nadler, SB Jr: Multi-valued contraction mappings. Pacific J Math. **30**, 475–488 (1969)
4. Mizoguchi, N, Takahashi, W: Fixed point theorems for multi-valued mappings on complete metric spaces. J Math Anal Appl. **141**, 177–188 (1989). doi:10.1016/0022-247X(89)90214-X
5. Amini-Harandi, A: Fixed point theory for set-valued quasi-contraction maps in metric spaces. Appl Math Lett. **24**(2), 24–1794 (2011)
6. Chatterjea, SK: Fixed point theorems. C.R Acad Bulgare Sci. **25**, 727–730 (1972)
7. Choudhury, BS: Unique fixed point theorem for weakly $\mathcal{C}$-contractive mappings. Kathmandu Uni J Sci Eng Technol. **5**(2), 5–13 (2009)
8. Meir, A, Keeler, E: A theorem on contraction mappings. J Math Anal Appl. **28**, 326–329 (1969). doi:10.1016/0022-247X(69)90031-6

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