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Multiple-set split feasibility problems for total asymptotically strict pseudocontractions mappings

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Abstract

The purpose of this article is to propose and investigate an algorithm for solving the *multiple-set split feasibility problems for total asymptotically strict pseu-docontractions mappings in infinite-dimensional Hilbert spaces*. The results presented in this article improve and extend some recent results of A. Moudafi, H. K. Xu, Y. Censor, A. Segal, T. Elfving, N. Kopf, T. Bortfeld, X. A. Motova, Q. Yang, A. Gibali, S. Reich and others.
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1. Introduction and preliminaries

Throughout this article, we always assume that H_1, H_2 are real Hilbert spaces, "$\to$", "$\rightharpoonup$" are denoted by strong and weak convergence, respectively, and $F(T)$ is the fixed point set of a mapping T.

Let G be a nonempty closed convex subset of H_1 and $T : G \to G$ a mapping.

T is said to be a contraction if there exists a constant $\alpha \in (0,1)$ such that

$$\|T_x - T_y\| \le \alpha \|x - y\|, \quad \forall x, y \in G. \tag{1.1}$$

Banach contraction principle guarantees that every contractive mapping defined on complete metric spaces has a unique fixed point.

T is said to be a weak contraction if

$$\|T_x - T_y\| \le \|x - y\| - \psi\left(\|x - y\|\right), \quad \forall x, y \in G. \tag{1.2}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0, \infty)$, $\psi(0) = 0$, and $\lim_{t\to\infty} \psi(t) = \infty$. We remark that the class of weak contractions was introduced by Alber and Guerre-Delabriere [1]. In 2001, Rhoades [2] showed that every weak contraction defined on complete metric spaces has a unique fixed point.

T is said to be nonexpansive if

$$\|T_x - T_y\| \le \|x - y\|, \quad \forall x, y \in G. \tag{1.3}$$

T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\left\|T^n x - T^n y\right\| \leq \kappa_n \left\|x - y\right\|, \quad \forall n \geq 1, \quad x, y \in G. \tag{1.4}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [3] as a generalization of the class of nonexpansive mappings. They proved that if G is a nonempty closed convex bounded subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping on G, then T has a fixed point.

T is said to be total asymptotically nonexpansive if

$$\left\|T^n x - T^n y\right\| \leq \left\|x - y\right\| + \mu_n \phi\left(\left\|x - y\right\|\right) + \xi_n, \quad \forall n \geq 1, \quad x, y \in G. \tag{1.5}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\varphi(0) = 0$, and $\{\mu_n\}$ and $\{\xi_n\}$ are nonnegative real sequences such that $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$. The class of mapping was introduced by Alber et al. [4]. From the definition, we see that the class of total asymptotically nonexpansive mappings includes the class of asymptotically nonexpansive mappings as special cases, see [5,6] for more details.

T is said to be strictly pseudocontractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\left\|Tx - Ty\right\|^2 \leq \left\|x - y\right\|^2 + \kappa \left\|(I - T)x - (I - T)y\right\|^2, \quad \forall x, y \in G. \tag{1.6}$$

The class of strict pseudocontractions was introduced by Browder and Petryshyn [7] in a real Hilbert space. In 2007, Marino and Xu [8] obtained a weak convergence theorem for the class of strictly pseudocontractive mappings, see [8] for more details.

T is said to be an asymptotically strict pseudocontraction if there exist a constant $\kappa \in [0, 1)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\left\|T^n x - T^n y\right\|^2 \leq \kappa_n \left\|x - y\right\|^2 + \kappa \left\|\left(I - T^n\right)x - \left(I - T^n\right)y\right\|^2, \quad \forall_n \geq 1, \quad x, y \in G. \tag{1.7}$$

The class of asymptotically strict pseudocontractions was introduced by Qihou [9] in 1996. Kim and Xu [10] proved that the class of asymptotically strict pseudocontractions is demiclosed at the origin and also obtained a weak convergence theorem for the class of mappings; see [10] for more details.

In this article, we introduce the following mapping.

Definition 1.1 Let H be a real Hilbert space, and G be a nonempty closed convex subset of H. A mapping $T : G \to G$ is said to be $(\kappa, \{\mu_n\}, \{\xi_n\}, \varphi)$-*total asymptotically strict pseudocontractive*, if there exists a constant $\kappa \in [0, 1)$ and sequences $\{\mu_n\} \subset [0, \infty)$, $\{\xi_n\} \subset [0, \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$, and a continuous and strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$\left\|T^n x - T^n y\right\|^2 \leq \left\|x - y\right\|^2 + \kappa \left\|x - y - \left(T^n x - T^n y\right)\right\|^2 + \mu_n \phi\left(\left\|x - y\right\|\right) + \xi_n, \quad \forall_n \geq 1, \quad x, y \in G. \tag{1.8}$$

Now, we give an example of total asymptotically strict pseudocontractive mapping.

Let C be a unit ball in a real Hilbert space l^2 and let $T : C \to C$ be a mapping defined by

$$T : (x_1, x_2, \ldots,) \to \left(0, x_1^2, a_2 x_2, \ a_3 x_3, \ldots\right),$$

where $\{a_i\}$ is a sequence in (0, 1) such that $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$.

It is proven in Goebal and Kirk [3] that

(i) $\|Tx - Ty\| \le 2\,\|x - y\|, \quad \forall x, y \in C;$
(ii) $\|T^n x - T^n y\| \le 2\prod_{j=2}^{n} a_j\,\|x - y\|, \quad \forall x, y \in C, \quad \forall n \ge 2.$.

Denote by $\kappa_1^{\frac{1}{2}} = 2,\ \kappa_n^{\frac{1}{2}} = 2\prod_{j=2}^{n} a_j, \quad n \ge 2$, then

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty}\left(2\prod_{j=2}^{n} a_j\right)^2 = 1.$$

Letting $\mu_n = (\kappa_n - 1), \quad \forall n \ge 1, \quad \phi(t) = t^2, \quad \forall t \ge 0,\ \kappa = 0$ and $\{\xi_n\}$ be a nonnegative real sequence with $\xi_n \to 0$, then $\forall x, y \in C, \quad n \ge 1$, we have

$$\|T^n x - T^n y\|^2 \le \|x - y\|^2 + \mu_n \phi\left(\|x - y\|\right) + \kappa|\,\|x - y - \left(T^n x - T^n y\right)\|^2 + \xi_n.$$

Remark 1.2 If $\varphi(\lambda) = \lambda^2$ and $\xi_n = 0$, then total asymptotically strict pseudocontractive mapping is asymptotically strict pseudocontraction mapping.

It is easy to see the following proposition holds.

Proposition 1.3 Let $T : G \to G$ be a (κ, $\{\mu_n\}$, $\{\xi_n\}$, φ)-total asymptotically strict pseudocontractive mapping. If $F(T) \ne \emptyset$, then for each $q \in F(T)$ and for each $x \in G$, the following inequalities hold and are equivalent:

$$\langle x - q, \quad T^n x - q\rangle \le \frac{\kappa + 1}{2k}\|x - q\|^2 + \frac{\kappa - 1}{2k}\|T^n x - q\|^2 + \frac{\mu_n}{2\kappa}\phi\left(\|x - q\|\right) + \frac{\xi_n}{2\kappa}; \tag{1.9}$$

$$\langle x - T^n x, \quad x - q\rangle \ge \frac{1 - \kappa}{2}\|T^n x - x\|^2 - \frac{\mu_n}{2}\phi\left(\|x - q\|\right) - \frac{\xi_n}{2}; \tag{1.10}$$

$$\langle x - T^n x, \quad q - T^n x\rangle \le \frac{\kappa + 1}{2}\|T^n x - x\|^2 + \frac{\mu_n}{2}\phi\left(\|x - q\|\right) + \frac{\xi_n}{2}. \tag{1.11}$$

The *split feasibility problem* (SFP) in finite-dimensional spaces was first introduced by Censor and Elfving [11] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [12]. Recently, it has been found that the SFP can also be used in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [13-15].

The *SFP* in an infinite-dimensional Hilbert space can be found in [12,14,16-18].

The purpose of this article is to introduce and study the following *multiple-set SFP* (MSSFP) *for total asymptotically strict pseudocontraction* in the framework of infinite-dimensional Hilbert spaces:

$$\textit{find} \quad x^* \in C \quad \textit{such that} \quad Ax^* \in Q, \tag{1.12}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $S_i : H_1 \to H_1$ and $T_i : H_2 \to H_2$, i = 1, 2, ..., N are mappings, $C : \bigcap_{i=1}^{N} F(S_i)$ and $Q : \bigcap_{i=1}^{N} F(T_i)$. In the sequel, we use Γ to denote the set of solutions of (MSSFP)–(1.12), i.e.,

$$\Gamma = \{x \in C, \quad Ax \in Q\}. \tag{1.13}$$

To prove our main results, we first recall some definitions, notations, and conclusions.

Let E be a Banach space. A mapping $T : E \to E$ is said to be *demi-closed at origin*, if for any sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x^*$ and $||(I - T)x_n|| \to 0$, then $x^* = Tx^*$.

A Banach space E is said to have *the Opial property*, if for any sequence $\{x_n\}$ with $x_n \rightharpoonup x^*$, then

$$\liminf_{n\to\infty} \|x_n - x^*\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \quad \text{with} \quad y \neq x^*.$$

Remark 1.4 It is well known that each Hilbert space possesses the Opial property.

Definition 1.5 Let H bea real Hilbert space.

(1) A mapping $T : H \to H$ is said to be uniformly L-Lipschitzian, if there exists a constant $L > 0$, such that

$$\|T^n x - T^n y\| \leq L \|x - y\|, \quad \forall x, y \in H \quad \text{and } n \geq 1.$$

(2) A mapping $T : H \to H$ *is* said to be *semi-compact*, if for any bounded sequence $\{x_n\} \subset H$ with $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that x_{n_i} converges strongly to some point $x^* \in H$.

Lemma 1.6 [10] Let H be a real Hilbert space. If $\{x_n\}$ is a sequence in H weakly convergent to z, then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2 \quad \forall y \in H.$$

Proposition 1.7 Assume that G is a closed convex subset of a real Hilbert space H and let $T : G \to G$ be a (κ, $\{\mu_n\}$, $\{\xi_n\}$, φ)*-total asymptotically strict pseudocon-traction mapping* and uniformly L-Lipschitzian. Then the demiclosedness principle holds for $I - T$ in the sense that if $\{x_n\}$ is a sequence in G such that $x_n \rightharpoonup x^*$, and $\limsup_{m\to\infty} \limsup_{n\to\infty} ||x_n - T^m x_n|| = 0$ then $(I - T)x^* = 0$. In particular, $x_n \rightharpoonup x^*$, and $(I - T)x_n \to 0 \Rightarrow (I - T)x^* = 0$, i.e., T is demiclosed at origin.

Proof Since $\{x_n\}$ is bounded, we can define a function f on H by

$$f(x) = \limsup_{n\to\infty} \|x_n - x\|^2, \quad \forall x \in H.$$

By Lemma 1.6, the weak convergence $x_n \rightharpoonup x^*$ implies that

$$f(x) = f(x^*) + \|x - x^*\|^2, \quad \forall x \in H.$$

In particular, for each $m \geq 1$,

$$f(T^m x^*) = f(x^*) + \|T^m x^* - x^*\|^2. \tag{1.14}$$

On the other hand, since T is a $(\kappa, \{\mu_n\}, \{\xi_n\})$-total asymptotically strict pseudocontraction mapping, by (1.8), we get

$$\begin{aligned} f(T^m x^*) &= \limsup_{n\to\infty} \|x_n - T^m x^*\|^2 \\ &= \limsup_{n\to\infty} \|x_n - T^m x_n + T^m x_n - T^m x^*\|^2 \\ &= \limsup_{n\to\infty} \left(\|x_n - T^m x_n\|^2 + 2\langle x_n - T^m x_n, T^m x_n - T^m x^*\rangle + \|T^m x_n - T^m x^*\|^2\right) \\ &\le \limsup_{n\to\infty} \|x_n - T^m x_n\| \left(\|x_n - T^m x_n\| + 2L\|x_n - x^*\|\right) \\ &+ \limsup_{n\to\infty} \left(\|x_n - x^*\|^2 + k\|x_n - T^m x_n - (x^* - T^m x^*)\|^2 + \mu_m \phi(\|x_n - x^*\|) + \xi_m\right) \end{aligned}$$

Taking $\limsup_{m\to\infty}$ on both sides and observing the facts that $\lim_{m\to\infty} \mu_m = 0$, $\lim_{m\to\infty} \xi_m = 0$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} ||x_n - T^m x_n|| = 0$, we derive that

$$\limsup_{m\to\infty} f(T^m x^*) \le \limsup_{n\to\infty} \|x_n - x^*\|^2 + k \limsup_{m\to\infty} \|x^* - T^m x^*\|^2 \tag{1.15}$$

Since $\limsup_{m\to\infty} f(T^m x^*) = f(x^*) + \limsup_{m\to\infty} ||T^m x^* - x^*||^2$, and $f(x^*) = \limsup_{n\to\infty} ||x_n - x^*||^2$, it follows from (1.15) that $\limsup_{m\to\infty} ||x^* - T^m x^*||^2 = 0$. That is, $T^m x^* \to x^*$; hence $Tx^* = x^*$.

Lemma 1.8 [19] Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \forall n \ge 1.$$

If $\sum_{i=1}^{\infty} \delta_n < \infty$ and $\sum_{i=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exists.

2. Multiple-set split feasibility problem

For solving the multiple-set split feasibility problem (1.12), let us assume that the following conditions are satisfied:

1. H_1 and H_2 are two real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator;
2. Let G, $\tilde{G}$ be a nonempty closed convex subset of H_1 and H_2 respectively, $S_i : G \to G$, $i = 1, 2,\ldots,N$, is a uniformly L_i-Lipschitzian and $(\beta_i, \{\mu_{i,n}\}, \{\xi_{i,n}\}, \varphi_i)$-total asymptotically strictly pseudocontractive mapping and $T_i : \tilde{G} \to \tilde{G}, \quad i = 1, 2, \ldots, N$, is a uniformly $\tilde{L}_i$-Lipschitzian and $\left(k_i \{\tilde{\mu}_{i,n}\}, \left\{\tilde{\xi}_{i,n}\right\}, \tilde{\phi}_i\right)$-total asymptotically strictly pseudocontractive mapping which satisfy the following conditions:
   (i) $C : \bigcap_{i=1}^{N} F(S_i) \neq \emptyset, \quad Q := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset;$
   (ii) $\beta = \max_{1\le i\le N} \beta_i < 1, \quad \kappa = \max_{1\le i\le N} \kappa_i < 1$;
   (iii) $L := \max_{1\le i\le N} L_i < \infty, \quad \tilde{L} := \max_{1\le i\le N} \tilde{L}_i < \infty$;
   (iv) $\mu_n = \max_{1\le i\le N} \{\mu_{i,n}, \tilde{\mu}_{i,n}\}, \quad \xi_n = \max_{1\le i\le N} \left\{\xi_{i,n}, \tilde{\xi}_{i,n}\right\}$ and $\sum_{i=1}^{\infty} \mu_n < \infty$, $\sum_{i=1}^{\infty} \xi_n < \infty$..
   (v) $\phi = \max_{1\le i\le N} \left\{\phi_i, \left\{\tilde{\phi}_i\right\}\right.$

We are now in a position to give the following result:

Theorem 2.1 Let H_1, H_2, G, $\tilde{G}$, A, $\{S_i\}$, $\{T_i\}$, C, Q, β, κ, L, $\tilde{L}$, $\{\mu_n\}$, $\{\xi_n\}$ and φ be the same as above. In addition, there exist positive constants M and M^* such that $\varphi(\lambda) \leq M^*\lambda^2$ for all $\lambda \geq M$. Let $\{x_n\}$ be the sequence generated by:

$$\begin{cases} x_1 \in G \quad \text{chosen arbitrarily} \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n S_n^n(u_n), \\ u_n = x_n + \gamma A^*(T_n^n - I)Ax_n, \quad \forall n \geq 1, \end{cases} \tag{2.1}$$

where $S_n^n = S_{n(\bmod N)}^n$, $T_n^n = T_{n(\bmod N)}^n$ $\forall n \geq 1$, $\{\alpha_n\}$ is a sequence in [0, 1] and $\gamma > 0$ is a constant satisfying the following conditions:

(vi) $\alpha_n \in (\delta, 1-\beta)$, $\forall n \geq 1$ and $\gamma \in \left(0, \dfrac{1-\kappa}{\|A\|^2}\right)$, where $\delta \in (0, 1-\beta)$ is a positive constant.

(I) If $\Gamma \neq \emptyset$ (where Γ is the set of solutions to (MSSFP)–(1.12)), then $\{x_n\}$ converges weakly to a point $x^* \in \Gamma$.

(II) In addition, if there exists a positive integer j such that S_j is semi-compact, then $\{x_n\}$ and $\{u_n\}$ both converge strongly to $x^* \in \Gamma$.

The proof of conclusion (I)

(1) First we prove that for each $p \in \Gamma$, the following limits exist

$$\lim_{n\to\infty} \|x_n - p\| \quad \text{and} \quad \lim_{n\to\infty} \|u_n - p\|. \tag{2.2}$$

In fact, since φ is an increasing function, it results that $\varphi(\lambda) \leq \varphi(M)$, if $\lambda \leq M$ and $\varphi(\lambda) \leq M^*\lambda^2$, if $\lambda \geq M$. In either case, we can obtain that

$$\phi(\lambda) \leq \phi(M) + M^*\lambda^2, \quad \forall \lambda \geq 0. \tag{2.3}$$

Since $p \in \Gamma$, then $p \in C := \bigcap_{i=1}^N F(S_i)$ and $Ap \in Q := \bigcap_{i=1}^N F(T_i)$. From (2.1) and (1.10) we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|u_n - p - \alpha_n(u_n - S_n^n u_n)\|^2 \\ &= \|u_n - p\|^2 - 2\alpha_n\langle u_n - p, u_n - S_n^n u_n\rangle + \alpha_n^2\|u_n - S_n^n u_n\|^2 \\ &\leq \|u_n - p\|^2 - \alpha_n(1-\beta)\|u_n - S_n^n u_n\|^2 \\ &\quad + \alpha_n\mu_n\phi(\|u_n - p\|) + \alpha_n\xi_n + \alpha_n^2\|u_n - S_n^n u_n\|^2 (by\ (1.10)) \\ &\leq \|u_n - p\|^2 - \alpha_n(1-\beta-\alpha_n)\|u_n - S_n^n u_n\|^2 \\ &\quad + \alpha_n\mu_n\left(\phi(M) + M^*(\|u_n - p\|)^2\right) + \alpha_n\xi_n \\ &= (1 + \alpha_n\mu_n M^*)\|u_n - p\|^2 - \alpha_n(1-\beta-\alpha_n)\|u_n - S_n^n u_n\|^2 \\ &\quad + \alpha_n\mu_n\phi(M) + \alpha_n\xi_n \end{aligned} \tag{2.4}$$

On the other hand, since

$$\begin{aligned} \|u_n - p\|^2 &= \|x_n - p + \gamma A^*(T_n^n - I)Ax_n\|^2 \\ &= \|x_n - p\|^2 + \gamma^2\|A^*(T_n^n - I)Ax_n\|^2 + 2\gamma\langle x_n - p, A^*(T_n^n - I)Ax_n\rangle, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\left\|A^*\left(T_n^n-I\right)Ax_n\right\|^2 &= \left\langle A^*\left(T_n^n-I\right)Ax_n,\ \ A^*\left(T_n^n-I\right)Ax_n\right\rangle \\ &= \left\langle AA^*\left(T_n^n-I\right)Ax_n,\ \ \left(T_n^n-I\right)Ax_n\right\rangle \\ &\leq \|A\|^2\left\|T_n^nAx_n-Ax_n\right\|^2,\end{aligned} \tag{2.6}$$

It follows from (1.11) we have

$$\begin{aligned}&\left\langle x_n-p,\ \ A^*\left(T_n^n-I\right)Ax_n\right\rangle = \left\langle Ax_n-Ap,\ \ \left(T_n^n-I\right)Ax_n\right\rangle \\ &= \left\langle \left(Ax_n-Ap\right)+\left(T_n^n-I\right)Ax_n-\left(T_n^n-I\right)Ax_n,\ \ \left(T_n^n-I\right)Ax_n\right\rangle \\ &= \left\langle T_n^nAx_n-Ap,\ \ T_n^nAx_n-Ax_n\right\rangle-\left\|\left(T_n^n-I\right)Ax_n\right\|^2. \\ &\leq \frac{1+\kappa}{2}\left\|\left(T_n^n-I\right)Ax_n\right\|^2+\frac{\mu_n}{2}\phi\left(\left\|Ax_n-Ap\right\|\right)+\frac{\xi n}{2}-\left\|\left(T_n^n-I\right)Ax_n\right\|^2. \\ &\leq \frac{\kappa-1}{2}\left\|\left(T_n^n-I\right)Ax_n\right\|^2+\frac{\mu_n}{2}\left(\phi(M)+M^*\left\|Ax_n-Ap\right\|^2\right)+\frac{\xi_n}{2}. \\ &\leq \frac{\kappa-1}{2}\left\|\left(T_n^n-I\right)Ax_n\right\|^2+\frac{\mu_n}{2}M^*\left\|Ax_n-Ap\right\|^2+\frac{\mu_n}{2}\phi(M)+\frac{\xi_n}{2}.\end{aligned} \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5) and simplifying it, we have

$$\begin{aligned}\left\|u_n-p\right\|^2 &\leq \left\|x_n-p\right\|^2+\gamma^2\|A\|^2\left\|T_n^nAx_n-Ax_n\right\|^2+\gamma\left(\kappa-1\right)\left\|\left(T_n^n-I\right)Ax_n\right\|^2 \\ &\quad +\gamma\mu_nM^*\left\|Ax_n-Ap\right\|^2+\gamma\mu_n\phi(M)+\gamma\xi_n \\ &= \left\|x_n-p\right\|^2-\gamma\left(1-\kappa-\gamma\|A\|^2\right)\left\|T_n^nAx_n-Ax_n\right\|^2 \\ &\quad +\gamma\mu_nM^*\left\|Ax_n-Ap\right\|^2+\gamma\mu_n\phi(M)+\gamma\xi_n \\ &\leq \left(1+\gamma\mu_nM^*\|A\|^2\right)\left\|x_n-p\right\|^2-\gamma\left(1-\kappa-\gamma\|A\|^2\right)\left\|T_n^nAx_n-Ax_n\right\|^2 \\ &\quad +\gamma\mu_n\phi(M)+\gamma\xi_n\end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.4) and after simplifying we have

$$\begin{aligned}\left\|x_{n+1}-p\right\|^2 &\leq \left(1+\alpha_n\mu_nM^*\right)\Big\{\left(1+\gamma\mu_nM^*\|A\|^2\right)\left\|x_n-p\right\|^2 \\ &\quad -\gamma\left(1-\kappa-\gamma\|A\|^2\right)\left\|T_n^nAx_n-Ax_n\right\|^2+\gamma\mu_n\phi(M)+\gamma\xi_n\Big\} \\ &\quad -\alpha_n\left(1-\beta-\alpha_n\right)\left\|u_n-S_n^nu_n\right\|^2+\alpha_n\mu_n\phi(M)+\alpha_n\xi_n \\ &\leq \left(1+\delta_n\right)\left\|x_n-p\right\|^2-\gamma\left(1-\kappa-\gamma\|A\|^2\right)\left\|T_n^nAx_n-Ax_n\right\|^2 \\ &\quad -\alpha_n\left(1-\beta-\alpha_n\right)\left\|u_n-S_n^nu_n\right\|^2+b_n\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}\delta_n &= \alpha_n\mu_nM^*+\gamma\mu_nM^*\|A\|^2+\gamma\|A\|^2\alpha_n\mu_n^2(M^*)^2 \\ b_n &= \left(\left(1+\alpha_n\mu_nM^*\right)\gamma+\alpha_n\right)\mu_n\phi(M)+\left(\left(1+\alpha_n\mu_nM^*\right)\gamma+\alpha_n\right)\xi_n\end{aligned}$$

By condition (vi) we have

$$\left\|x_{n+1}-p\right\|^2 \leq \left(1+\delta_n\right)\left\|x_n-p\right\|^2+b_n$$

By condition (iv), $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}b_n<\infty$. Hence, from Lemma 1.8 we know that the following limit exists

$$\lim_{n\to\infty}\left\|x_n-p\right\|. \tag{2.10}$$

Consequently, from (2.9) and (2.10) we have that

$$\gamma\left(1-\kappa-\gamma\|A\|^2\right)\left\|\left(T_n^n-I\right)Ax_n\right\|^2+\alpha_n\left(1-\beta-\alpha_n\right)\left\|u_n-S_n^nu_n\right\|^2 \\ \leq \left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2+\delta_n\left\|x_n-p\right\|^2+b_n\to 0\ \ (as\ n\to\infty).$$

This together with the condition (vi) implies that

$$\lim_{n\to\infty}\left\|u_n-S_n^nu_n\right\|=0; \tag{2.11}$$

and

$$\lim_{n\to\infty}\left\|\left(T_n^n-I\right)Ax_n\right\|=0. \tag{2.12}$$

It follows from (2.5), (2.10) and (2.12) that the limit $||u_n - p||$ exists.
The conclusion (1) is proved.

(2) Next we prove that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0 \text{ and } \lim_{n\to\infty}\|u_{n+1}-u_n\|=0. \tag{2.13}$$

In fact, it follows from (2.1) that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \left\|(1-\alpha_n)u_n+\alpha_nS_n^n(u_n)-x_n\right\| \\ &= \left\|(1-\alpha_n)\left(x_n+\gamma A^*\left(T_n^n-I\right)Ax_n\right)+\alpha_nS_n^n(u_n)-x_n\right\| \\ &= \left\|(1-\alpha_n)\gamma A^*\left(T_n^n-I\right)Ax_n+\alpha_n\left(S_n^n(u_n)-x_n\right)\right\| \\ &= \left\|(1-\alpha_n)\gamma A^*\left(T_n^n-I\right)Ax_n+\alpha_n\left(S_n^n(u_n)-u_n\right)+\alpha_n\left(u_n-x_n\right)\right\| \\ &= \left\|(1-\alpha_n)\gamma A^*\left(T_n^n-I\right)Ax_n+\alpha_n\left(S_n^n(u_n)-u_n\right)+\alpha_n\gamma A^*\left(T_n^n-I\right)Ax_n\right\| \\ &= \left\|\gamma A^*\left(T_n^n-I\right)Ax_n+\alpha_n\left(S_n^n(u_n)-u_n\right)\right\|.\end{aligned}$$

In view of (2.11) and (2.12) we have that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.14}$$

Similarly, it follows from (2.1), (2.12), and (2.14) that

$$\begin{aligned}\|u_{n+1}-u_n\| &= \left\|x_{n+1}+\gamma A^*\left(T_{n+1}^{n+1}-I\right)Ax_{n+1}-\left(x_n+\gamma A^*\left(T_n^n-I\right)Ax_n\right)\right\| \\ &\leq \|x_{n+1}-x_n\|+\gamma\left\|A^*\left(T_{n+1}^{n+1}-I\right)Ax_{n+1}\right\| \\ &\quad +\gamma\left\|A^*\left(T_n^n-I\right)Ax_n\right\|\to 0\ (\text{as } n\to\infty).\end{aligned} \tag{2.15}$$

The conclusion (2.13) is proved.

(3) Next we prove that for each j = 1, 2,..., N - 1,

$$\left\|u_{iN+j}-S_ju_{iN+j}\right\|\to 0 \ \text{ and }\ \left\|Ax_{iN+j}-T_jAx_{iN+j}\right\|\to 0\ \ (as\ i\to\infty), \tag{2.16}$$

In fact, from (2.11) we have

$$\eta_{iN+j}:=\left\|u_{iN+j}-S_j^{iN+j}u_{iN+j}\right\|\to 0\ \ (as\ i\to\infty). \tag{2.17}$$

Since S_j is uniformly L_j-Lipschitzian continuous, it follows from (2.13) and (2.17) that

$$\begin{aligned}
\left\|u_{iN+j} - S_j u_{iN+j}\right\| &= \left\|u_{iN+j} - S_j^{iN+j} u_{iN+j}\right\| + \left\|S_j^{iN+j} u_{iN+j} - S_j u_{iN+j}\right\| \\
&\leq \eta_{iN+j} + L_j \left\|S_j^{iN+j-1} u_{iN+j} - u_{iN+j}\right\| \\
&\leq \eta_{iN+j} + L_j \left\{\left\|S_j^{iN+j-1} u_{iN+j} - S_j^{iN+j-1} u_{iN+j-1}\right\| \right. \\
&\quad \left. + L_j \left\|S_j^{iN+j-1} u_{iN+j-1} - u_{iN+j}\right\|\right\} \\
&\leq \eta_{iN+j} + L_j^2 \left\|u_{iN+j} - u_{iN+j-1}\right\| \\
&\quad + L_j \left\|S_j^{iN+j-1} u_{iN+j-1} - u_{iN+j-1} + u_{iN+j-1} - u_{iN+j}\right\| \\
&\leq \eta_{iN+j} + L_j \left(1 + L_j\right) \left\|u_{iN+j} - u_{iN+j-1}\right\| + L_j \eta_{iN+j-1} \to 0 \ \ (as\ i \to \infty)
\end{aligned}$$

Similarly, for each j = 1, 2,..., N - 1, from (2.13) we have

$$\varsigma_{iN+j} := \left\|Ax_{iN+j} - T_j^{iN+j} Ax_{iN+j}\right\| \to 0 \ \ (\text{as } i \to \infty). \tag{2.18}$$

Since T_j is uniformly $\tilde{L}_j$-Lipschitzian continuous, by the same way as above, from (2.13) and (2.18), we can also prove that

$$\left\|Ax_{iN+j} - T_j Ax_{iN+j}\right\| \to 0 \ \ (as\ i \to \infty). \tag{2.19}$$

(4) Finally we prove that $x_n \rightharpoonup x^*$ and $u_n \rightharpoonup x^*$ which is a solution of (MSSFP)–(1.12).

Since $\{u_n\}$ is bounded. There exists a subsequence $\left\{u_{n_i}\right\} \subset \{u_n\}$ such that $u_{n_i} \rightharpoonup x^*$ (some point in H_1). Hence, for any positive integer j = 1, 2,..., N, there exists a subsequence $\{n_i(j)\} \subset \{n_i\}$ with $n_i(j)(modN) = j$ such that $u_{n_i(j)} \rightharpoonup x^*$. Again from (2.16) we have

$$\left\|u_{n_i(j)} - S_j u_{n_i(j)}\right\| \to \ 0 \ \ (\text{as } n_i(j) \to \infty) \tag{2.20}$$

Since S_j is demiclosed at zero (see Proposition 1.7), it gets that $x^* \in F(S_j)$. By the arbitrariness of j = 1, 2,..., N, we have $x^* \in C := \bigcap_{j=1}^N F\left(S_j\right)$.

Moreover, from (2.1) and (2.12) we have

$$x_{n_i} = u_{n_i} - \gamma A^* \left(T_{n_i}^{n_i} - I\right) Ax_{n_i} \rightharpoonup x^*.$$

Since A is a linear bounded operator, it gets $Ax_{n_i} \rightharpoonup Ax^*$. For any positive integer k = 1, 2,..., N, there exists a subsequence $\{n_i(k)\} \subset \{n_i\}$ with $n_i(k)(modN) = k$ such that $Ax_{n_i(k)} \rightharpoonup Ax^*$. In view of (2.16) we have

$$\left\|Ax_{n_i(k)} - T_k Ax_{n_i(k)}\right\| \to 0 \ \ (\text{as } n_i(k) \to \infty).$$

Since T_k is demiclosed at zero, we have $Ax^* \in F(T_k)$. By the arbitrariness of k = 1, 2,..., N, it yields $Ax^* \in Q := \bigcap_{k=1}^N F(T_k)$. This together with $x^* \in C$ shows that $x^* \in \Gamma$, i.e., x^* is a solution to the (MSSFP)–(1.12).

Now we prove that $x_n \rightharpoonup x^*$ and $u_n \rightharpoonup x^*$.

In fact, if there exists another subsequence $\{u_{n_i}\} \subset \{u_n\}$ such that $u_{n_i(j)} \rightharpoonup y^* \in \Gamma$ with $y^* \neq x^*$. Consequently, by virtue of (2.2) and the Opial property of Hilbert space, we have

$$\begin{aligned}\liminf_{n_i \to \infty} \|u_{n_i} - x^*\| &< \liminf_{n_i \to \infty} \|u_{n_i} - y^*\| = \lim_{n \to \infty} \|u_n - y^*\| \\ &= \liminf_{n_i \to \infty} \|u_{n_i} - y^*\| < \lim_{n_j \to \infty} \|u_{n_j} - x^*\| \\ &= \liminf_{n \to \infty} \|u_n - x^*\| = \lim_{n_i \to \infty} \|u_{n_i} - x^*\|.\end{aligned}$$

This is a contradiction. Therefore, $u_n \rightharpoonup x^*$. By using (2.1) and (2.12), we have

$$x_n = u_n - \lambda A^* \left(T_n^n - I\right) A x_n \rightharpoonup x^*.$$

The proof of conclusion (II).

Without loss of generality, we can assume that S_1 is semi-compact. It follows from (2.20) that

$$\|u_{n_i(1)} - S_1 u_{n_i(1)}\| \to 0 \ \ (\text{as } n_i(1) \to \infty) \tag{2.21}$$

Therefore, there exists a subsequence of $\{u_{n_i(1)}\}$ (for the sake of convenience we still denote it by $\{u_{n_i(1)}\}$ such that $u_{n_i(1)} \to u^* \in H_1$ (some point in H_1). Since $u_{n_i(1)} \rightharpoonup x^*$. This implies that $x^* = u^*$, and so $u_{n_i(1)} \to x^* \in \Gamma$. By virtue of (2.2) we know that $\lim_{n\to\infty} ||u_n - x^*|| = 0$ and $\lim_{n\to\infty} ||x_n - x^*|| = 0$, i.e., $\{u_n\}$ and $\{x_n\}$ both converge strongly to $x^* \in \Gamma$.

This completes the proof of Theorem 2.1.

Remark 2.2 Since the class of total asymptotically strict pseudocontractive mappings includes the class of asymptotically strict pseudocontractions mappings and the class of strict pseudocontractions mappings as special cases, Theorem 2.1 improves and extend the corresponding results of Censor et al. [14,15], Yang [17], Moudafi [20], Xu [21], Censor and Segal [22], Censor et al. [23] and others.

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References

1. Alber, YaI, Guerre-Delabriere, S: On the projection methods for fixed point problems. Analysis. **21**, 17–39 (2001)
2. Rhoades, BE: Some theorems on weakly contractive maps. Nonlinear Anal: Theory Methods Appl. **47**, 2683–2693 (2001). doi:10.1016/S0362-546X(01)00388-1
3. Goebel, K, Kirk, WA: A fixed point theorem for asymptotically nonexpansive mappings. Proc Am Math Soc. **35**, 171–174 (1972). doi:10.1090/S0002-9939-1972-0298500-3
4. Alber, YaI, Chidume, CE, Zegeye, H: Approximating fixed points of total asymptotically nonex-pansive mappings. Fixed Point Theory Appl. **10673**, 20 (2006)

5. Chidume, CE, Ofoedu, EU: A new iteration process for approximation of common fixed points for finite families of total asymptotically nonexpansive mappings. Int J Math Math Sci. **615107**, 17 (2009)
6. Chidume, CE, Ofoedu, EU: Approximation of common fixed points for finite families of total asymptotically nonexpansive mappings. J Math Anal Appl. **333**, 128–141 (2007). doi:10.1016/j.jmaa.2006.09.023
7. Browder, FE, Petryshyn, WV: Construction of fixed points of nonlinear mappings in Hilbert space. J Math Anal Appl. **20**, 197–228 (1967). doi:10.1016/0022-247X(67)90085-6
8. Marino, G, Xu, HK: Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. J Math Anal Appl. **329**, 336–346 (2007). doi:10.1016/j.jmaa.2006.06.055
9. Qihou, L: Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings. Nonlinear Anal: Theory Methods Appl. **26**, 1835–1842 (1996). doi:10.1016/0362-546X(94)00351-H
10. Kim, TH, Xu, HK: Convergence of the modified Mann's iteration method for asymptotically strict pseudo-contractions. Nonlinear Anal: Theory Methods Appl. **68**, 2828–2836 (2008). doi:10.1016/j.na.2007.02.029
11. Censor, Y, Elfving, T: A multi-projection algorithm using Bregman projections in a product space. Numer Algorithms. **8**, 221–239 (1994). doi:10.1007/BF02142692
12. Byrne, C: Iterative oblique projection onto convex subsets and the split feasibility problem. Inverse Problem. **18**, 441–453 (2002). doi:10.1088/0266-5611/18/2/310
13. Censor, Y, Bortfeld, T, Martin, B, Trofimov, A: A unified approach for inversion problem in intensity-modulated radiation therapy. Phys Med Biol. **51**, 2353–2365 (2006). doi:10.1088/0031-9155/51/10/001
14. Censor, Y, Elfving, T, Kopf, N, Bortfeld, T: The multiple-sets split feasibility problem and its applications. Inverse Problem. **21**, 2071–2084 (2005). doi:10.1088/0266-5611/21/6/017
15. Censor, Y, Motova, XA, Segal, A: Pertured projections and subgradient projections for the multiple-setssplit feasibility problem. J Math Anal Appl. **327**, 1244–1256 (2007). doi:10.1016/j.jmaa.2006.05.010
16. Xu, HK: A variable Krasnosel'skii-Mann algorithm and the multiple-sets split feasibility problem. Inverse Problem. **22**, 2021–2034 (2006). doi:10.1088/0266-5611/22/6/007
17. Yang, Q: The relaxed CQ algorithm for solving the split feasibility problem. Inverse Problem. **20**, 1261–1266 (2004). doi:10.1088/0266-5611/20/4/014
18. Zhao, J, Yang, Q: Several solution methods for the split feasibility problem. Inverse Problem. **21**, 1791–1799 (2005). doi:10.1088/0266-5611/21/5/017
19. Aoyama, K, Kimura, W, Takahashi, W, Toyoda, M: Approximation of common fixed points of acountable family of nonexpansive mappings on a Banach space. Nonlinear Anal: Theory Methods Appl. **67**(8), 2350–2360 (2007). doi:10.1016/j.na.2006.08.032
20. Moudafi, A: The split common fixed point problem for demi-contractive mappings. Inverse Prob-lem. **26**(055007), 6 (2010)
21. Xu, HK: Iterative methods for split feasibility problem in infinite-dimensional Hilbert spaces. Inverse Problem. **26**(105018), 17 (2010)
22. Censor, Y, Segal, A: The split common fixed point problem for directed operators. J Convex Anal. **16**, 587–600 (2009)
23. Censor, Y, Gibali, A, Reich, S: Algorithms for the split variational inequality problem. Numer Algorithm (accepted)

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