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Some fixed point theorems for contractive multi-valued mappings induced by generalized distance in metric spaces

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Abstract

The purpose of this paper is to prove some existence theorems for fixed point problem by using a generalization of metric distance, namely *u*-distance. Consequently, some special cases are discussed and an interesting example is also provided. Presented results are generalizations of the important results due to Ume (Fixed Point Theory Appl 2010(397150), 21 pp, 2010) and Suzuki and Takahashi (Topol Methods Nonlinear Anal 8, 371-382, 1996).

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1. Introduction and preliminaries

Let (X, d) be a metric space. A mapping $T: X \rightarrow X$ is said to be *contraction* if there exists $r \in [0, 1)$ such that

$$d(T(x), T(y)) \leq rd(x, y), \quad \forall x, y \in X. \tag{1.1}$$

In 1922, Banach [1] proved that if (X, d) is a complete metric space and the mapping T satisfies (1.1), then T has a unique fixed point, that is $T(u) = u$ for some $u \in X$. Such a result is well known and called the Banach contraction mapping principle. Following the Banach contraction principle, Nadler Jr. [2] established the fixed point result for multi-valued contraction maps, which in turn is a generalization of the Banach contraction principle. Since then, there are several extensions and generalizations of these two important principles, see [3,4] and [5-11] for examples.

In 1996, Kada et al. [4] introduced the concept of *w*-distance on a metric space (X, d). By using such a *w*-distance concept, they improved some important theorems such as Caristi's fixed point theorem, Ekeland's variational principle and the nonconvex minimization theorem. Recently, Suzuki [7] introduced the concept of generalization metric distance, which is called *τ*-distance. By using concepts of *τ*-distance, he proved some results on fixed point problems and also showed that the class of *w*-distance is properly contained in the class of *τ*-distance. Most recently, Ume [11] introduced another concept of distance as the following.

Definition 1.1. [11]. Let (X, d) be a metric space. Then, a function $p: X \times X \to [0, \infty)$ is called *u-distance* on X if there exists a function $\theta: X \times X \times [0, \infty) \times [0, \infty) \to [0, \infty)$ such that

(u1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;

(u2) $\theta(x, y, 0, 0) = 0$ and $\theta(x, y, s, t) \geq \min\{s, t\}$ for all $x, y \in X$ and $s, t \in [0, \infty)$, and for any $x \in X$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|s - s_0| < \delta$, $|t - t_0| < \delta$, $s, s_0, t, t_0 \in [0, \infty)$ and $y \in X$ imply

$$| \theta(x, y, s, t) - \theta(x, y, s_0, t_0) | < \varepsilon; \tag{1.2}$$

(u3)

$$\lim_{n \to \infty} x_n = x,$$
$$\lim_{n \to \infty} \sup\{\theta(w_n, z_n, p(w_n, x_m), p(z_n, x_m)) : m \geq n\} = 0 \tag{1.3}$$

imply

$$p(y, x) \leq \liminf_{n \to \infty} p(y, x_n) \text{ for all } y \in X; \tag{1.4}$$

(u4)

$$\lim_{n \to \infty} \sup\{p(x_n, w_m) : m \geq n\} = 0,$$
$$\lim_{n \to \infty} \sup\{p(y_n, z_m) : m \geq n\} = 0,$$
$$\lim_{n \to \infty} \theta(x_n, w_n, s_n, t_n) = 0,$$
$$\lim_{n \to \infty} \theta(y_n, z_n, s_n, t_n) = 0 \tag{1.5}$$

imply

$$\lim_{n \to \infty} \theta(w_n, z_n, s_n, t_n) = 0 \tag{1.6}$$

or

$$\lim_{n \to \infty} \sup\{p(w_m, x_n) : m \geq n\} = 0,$$
$$\lim_{n \to \infty} \sup\{p(z_m, y_n) : m \geq n\} = 0,$$
$$\lim_{n \to \infty} \theta(x_n, w_n, s_n, t_n) = 0,$$
$$\lim_{n \to \infty} \theta(y_n, z_n, s_n, t_n) = 0 \tag{1.7}$$

imply

$$\lim_{n\to\infty} \theta(w_n,\, z_n,\, s_n,\, t_n) = 0; \tag{1.8}$$

($u5$)

$$\lim_{n\to\infty} \theta(w_n,\, z_n,\, p(w_n,\, x_n),\, p(z_n,\, x_n)) = 0,$$
$$\lim_{n\to\infty} \theta(w_n,\, z_n,\, p(w_n,\, y_n),\, p(z_n,\, y_n)) = 0 \tag{1.9}$$

imply

$$\lim_{n\to\infty} d(x_n,\, y_n) = 0 \tag{1.10}$$

or

$$\lim_{n\to\infty} \theta(a_n,\, b_n,\, p(x_n,\, a_n),\, p(x_n,\, b_n)) = 0,$$
$$\lim_{n\to\infty} \theta(a_n,\, b_n,\, p(y_n,\, a_n),\, p(y_n,\, b_n)) = 0 \tag{1.11}$$

imply

$$\lim_{n\to\infty} d(x_n,\, y_n) = 0. \tag{1.12}$$

We give the following remark and example, which can be found in [11].

Remark 1.2. Suppose that $\theta\colon X \times X \times [0, \infty) \times [0, \infty) \to [0, \infty)$ is a mapping satisfying ($u2$) ~ ($u5$). Then, there exists a mapping $\eta\colon X \times X \times [0, \infty) \times [0, \infty) \to [0, \infty)$ such that η is nondecreasing in its third and fourth variable, respectively, satisfying ($u2$)η ~ ($u5$)η, where ($u2$)η ~ ($u5$)η stand for substituting η for θ in ($u2$) ~ ($u5$), respectively.

Example 1.3. Let p be a τ-distance on metric space (X, d), then p is also a u-distance on X. On the other hand, let $(X, ||\cdot||)$ be a normed space then a function $p\colon X \times X \to [0, \infty)$ defined by $p(x, y) = ||x||$ for every $x, y \in X$ is a u-distance on X but not a τ-distance. These imply that the class of τ-distance is properly contained in the class of u-distance.

In this paper, we will prove some fixed point theorems in metric spaces by using such a u-distance concept. Consequently, as shown by Example 1.3, our results generalize many of the existing results presented in metric spaces. Indeed, it provides more choices of tool implements to check whether a fixed point of considered mapping exists.

Our main results are concerned with the following class of mappings.

Definition 1.4. Let (X, d) be a metric space and 2^X be a set of all nonempty subset of X. A multi-valued mapping $T\colon X \to 2^X$ is called *p-contractive* if there exist a u-

distance p on X and $r \in [0, 1)$ such that for any $x_1, x_2 \in X$ and $y_1 \in T(x_1)$ there is $y_2 \in T(x_2)$ such that

$$p(y_1, y_2) \le rp(x_1, x_2). \qquad (1.13)$$

Remark 1.5. Definition 1.4 was introduced and its fixed point theorems were proved in [9], but by using the concept of w-distance. Note that in the case $p = d$, the mapping T is called a contraction.

We now recall some basic concepts and well-known results.

Definition 1.6. [11]. Let (X, d) be a metric space and p be a u-distance on X. Then, a sequence $\{x_n\}$ of X is called *p-Cauchy sequence* if there exists a sequence $\{z_n\}$ of X such that

$$\lim_{n \to \infty} \sup\{\theta(z_n, z_n, p(z_n, x_m), p(z_n, x_m)) : m \ge n\} = 0, \qquad (1.14)$$

or

$$\lim_{n \to \infty} \sup\{\theta(z_n, z_n, p(x_m, z_n), p(x_m, z_n)) : m \ge n\} = 0. \qquad (1.15)$$

where $\theta: X \times X \times [0, \infty) \times [0, \infty) \to [0, \infty)$ is a function satisfying $(u2) \sim (u5)$ for a u-distance p.

Lemma 1.7. [11]. Let (X, d) be a metric space and let p be a u-distance on X. Suppose that a sequence $\{x_n\}$ of X satisfies

$$\lim_{n \to \infty} \sup\{p(x_n, x_m) : m \ge n\} = 0. \qquad (1.16)$$

or

$$\lim_{n \to \infty} \sup\{p(x_m, x_n) : m \ge n\} = 0. \qquad (1.17)$$

Then, $\{x_n\}$ is a p-Cauchy sequence.

Lemma 1.8. [11]. Let (X, d) be a metric space and let p be a u-distance on X. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence.

Lemma 1.9. [11]. Let (X, d) be a metric space and let p be a u-distance on X. Let $x, y \in X$. If there exists $z \in X$ such that $p(z, x) = 0$ and $p(z, y) = 0$, then $x = y$.

Definition 1.10. Let (X, d) be a metric space and $T: X \to 2^X$ be a mapping. For any fixed $x_0 \in X$, a sequence $\{x_n\} = \{x_0, x_1, x_2, ...\} \subset X$ such that $x_{n+1} \in T(x_n)$ is called an *orbit* of x_0 with respect to mapping T. We will denote by $\mathcal{O}(T, x_0)$ the set of all orbital sequences of x_0 with respect to mapping T.

2. Main results

From now on, in view of Remark 1.2, if $\theta: X \times X \times [0, \infty) \times [0, \infty) \to [0, \infty)$ is a mapping satisfying $(u2) \sim (u5)$ for the considered u-distance, we will always understand that θ is a nondecreasing function in its third and fourth variables.

Inspired by an idea presented by Suzuki [8], we have an important tool for proving our main result.

Lemma 2.1. *Let (X, d) be a metric space and let p be a u-distance on X. If $\{x_n\}$ is a p-Cauchy sequence and $\{y_n\}$ is a sequence satisfying*

$$\lim_{n \to \infty} \sup\{p(x_n, y_m) : m \ge n\} = 0, \qquad (2.1)$$

then $\{y_n\}$ is also a p-Cauchy sequence and $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.

Proof. Let $\theta: X \times X \times [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying $(u2) \sim (u5)$ for a u-distance p. Since $\{x_n\}$ is a p-Cauchy sequence, there exists a sequence $\{z_n\}$ of X such that

$$\lim_{n\to\infty} \sup\{\theta(z_n, z_n, p(z_n, x_m), p(z_n, x_m) : m \geq n\} = 0. \tag{2.2}$$

Now, let $\{y_n\}$ be a sequence satisfying $\lim\limits_{n\to\infty} \sup\{p(x_n, y_m) : m \geq n\} = 0$. Let us put

$$\alpha_n = \sup\{p(z_n, x_m) : m \geq n\},$$

and

$$\beta_n = \sup\{p(x_i, y_j) : j > i \geq n\}.$$

We note that $\{\beta_n\}$ is a nonincreasing sequence and converge to 0. Thus, from $(u2)$ we can define a strictly increasing function f from $\mathbb{N}$ into itself such that

$$\theta(z_n, z_n, \alpha_n + \beta_{f(n)}, \alpha_n + \beta_{f(n)}) \leq \theta(z_n, z_n, \alpha_n, \alpha_n) + \frac{1}{n}, \tag{2.3}$$

for all $n \in \mathbb{N}$. Using such a function f, we now define a function $g: \mathbb{N} \to \mathbb{N}$ by

$$g(n) = \begin{cases} 1, & \text{if } n < f(1), \\ k, & \text{if } f(k) \leq n < f(k+1) \text{ for some } k \in \mathbb{N}. \end{cases}$$

Then, we can see that

- $g(n) \leq f(g(n)) \leq n$ for all $n \in \mathbb{N}$ with $g(n) \geq 2$.
- $\theta(z_{g(n)}, z_{g(n)}, \alpha_{g(n)} + \beta_n, \alpha_{g(n)} + \beta_n) \leq \theta(z_{g(n)}, z_{g(n)}, \alpha_{g(n)}, \alpha_{g(n)}) + \frac{1}{g(n)}$.
- $\lim\limits_{n\to\infty} g(n) = \infty$.

Now we consider

$$\limsup_{n\to\infty} \sup\{\theta(z_{g(n)}, z_{g(n)}, p(z_{g(n)}, y_m), p(z_{g(n)}, y_m)) : m \geq n\}$$

$$\leq \limsup_{n\to\infty} \sup\{\theta(z_{g(n)}, z_{g(n)}, p(z_{g(n)}, x_n) + p(x_n, y_m), p(z_{g(n)}, x_n) + p(x_n, y_m)) : m \geq n\}$$

$$\leq \limsup_{n\to\infty} \theta(z_{g(n)}, z_{g(n)}, \alpha_{g(n)} + \beta_n, \alpha_{g(n)} + \beta_n)$$

$$\leq \lim_{n\to\infty} \left[\theta(z_{g(n)}, z_{g(n)}, \alpha_{g(n)}, \alpha_{g(n)}) + \frac{1}{g(n)} \right]$$

$$= 0.$$

This means $\{y_n\}$ is a p-Cauchy sequence. Furthermore, since

$$\limsup_{n\to\infty} \theta(z_{g(n)}, z_{g(n)}, p(z_{g(n)}, x_n), p(z_{g(n)}, x_n)) \leq \lim_{n\to\infty} \theta(z_{g(n)}, z_{g(n)}, \alpha_{g(n)}, \alpha_{g(n)}) = 0,$$

we have $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$, by $(u5)$. This completes the proof. $\square$

Now we present our main results, which are related to p-contractive mapping.

Lemma 2.2. Let (X, d) be a metric space and let $T: X \to 2^X$ be a p-contractive mapping. Then, for each $u_0 \in X$, there exists an orbit $\{u_n\} \in \mathcal{O}(T, u_0)$ such that

$$\lim_{n \to \infty} \sup\{p(u_n, u_m) : m \geq n\} = 0. \tag{2.4}$$

Consequently, $\{u_n\}$ is a p-Cauchy sequence.

Proof. Let $u_0 \in X$ be arbitrary and $u_1 \in T(u_0)$ be chosen. Then, by T as a p-contractive mapping, there exists $u_2 \in T(u_1)$ such that

$$p(u_1, u_2) \leq rp(u_0, u_1).$$

For this $u_2 \in T(u_1)$, again by T as a p-contractive, we can find $u_3 \in T(u_2)$ such that

$$p(u_2, u_3) \leq rp(u_1, u_2).$$

Continuing this process, we obtain a sequence $\{u_n\}$ in X such that $u_{n+1} \in T(u_n)$ and

$$p(u_n, u_{n+1}) \leq rp(u_{n-1}, u_n), \text{ for all } n \in \mathbb{N}.$$

Notice that we have

$$p(u_n, u_{n+1}) \leq rp(u_{n-1}, u_n) \leq r^2 p(u_{n-2}, u_{n-1}) \leq \cdots \leq r^n p(u_0, u_1),$$

for each $n \in \mathbb{N}$. This gives,

$$
\begin{aligned}
p(u_n, u_m) &\leq & p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \cdots + p(u_{m-1}, u_m) \\
&\leq & r^n p(u_0, u_1) + r^{n+1} p(u_0, u_1) + \cdots + r^{m-1} p(u_0, u_1) \\
&\leq & \frac{r^n}{1-r} p(u_0, u_1),
\end{aligned}
\tag{2.5}
$$

where $n, m \in \mathbb{N}$ with $m \geq n$. Consequently,

$$0 \leq \lim_{n \to \infty} \sup\{p(u_n, u_m) : m \geq n\} \leq \lim_{n \to \infty} \frac{r^n}{1-r} p(u_0, u_1) = 0. \tag{2.6}$$

This proves the first part of this lemma. Furthermore, the second part is followed from (2.6) and Lemma 1.7. □

Lemma 2.3. *Let (X, d) be a complete metric space and $T: X \to 2^X$ be a p-contractive mapping. Then, there exist a sequence $\{w_n\}$ and v_0 in $\times$ such that $\{w_n\} \subset T(v_0)$ and $\{w_n\}$ converges to v_0.*

Proof. Let $\theta: X \times X \times [0, \infty) \times [0, \infty) \to [0, \infty)$ be a mapping satisfying $(u2) \sim (u5)$ for this u-distance p.

Let $u_0 \in X$ be chosen. By Lemma 2.2, we know that there exists $\{u_n\} \in \mathcal{O}(T, u_0)$ such that $\{u_n\}$ is a p-Cauchy sequence. Moreover, it satisfies

$$p(u_n, u_m) \leq \frac{r^n}{1-r} p(u_0, u_1), \tag{2.7}$$

where $n, m \in \mathbb{N}$ with $m \geq n$. Since $\{u_n\}$ is a p-Cauchy sequence in a metric complete space (X, d), it is a convergent sequence, say $\lim_{n \to \infty} u_n = v_0$, for some $v_0 \in X$. Consequently, by (u_3) and (2.7), we have

$$p(u_n, v_0) \leq \liminf_{m \to \infty} p(u_n, u_m) \leq \frac{r^n}{1-r} p(u_0, u_1). \tag{2.8}$$

For this $v_0 \in X$, by using the p-contractiveness of mapping T, we can find a sequence $\{w_n\}$ in $T(v_0)$ such that

$$p(u_n, w_n) \le rp(u_{n-1}, v_0).$$

It follows that

$$p(u_n, w_n) \le rp(u_{n-1}, v_0) \le \frac{r^n}{1-r} p(u_0, u_1), \tag{2.9}$$

for any $n \in \mathbb{N}$.

Now we show that $\{w_n\}$ converges to v_0. In fact, since θ is a nondecreasing function in its third and fourth variables, then (2.8) and (2.9) imply

$$\lim_{n \to \infty} \theta(u_n, u_n, p(u_n, v_0), p(u_n, v_0)) = 0,$$

and

$$\lim_{n \to \infty} \theta(u_n, u_n, p(u_n, w_n), p(u_n, w_n)) = 0.$$

Hence, by ($u5$), we conclude that $\lim_{n \to \infty} d(v_0, w_n) = 0$. This means that $\{w_n\}$ converges to v_0, and the proof is completed. □

For a metric space (X, d), we will denote by $Cl(X)$ the set of all closed subsets of X. In view of proving Lemma 2.3, we can obtain a fixed point theorem in the general metric space setting.

Theorem 2.4. *Let (X, d) be a metric space and $T: X \to 2^X$ be a p-contractive mapping. If there exist $u_0, v_0 \in X$ and $\{u_n\} \in \mathcal{O}(T, u_0)$ such that*

(i) $\lim_{n \to \infty} p(u_n, v_0) = 0$;

(ii) $T(v_0) \in Cl(X)$.

Then, $F(T) \ne \emptyset$. Furthermore, $v_0 \in F(T)$.

Next, we provide some fixed point theorems for p-contractive mapping in a complete metric space. □

Theorem 2.5. *Let (X, d) be a complete metric space and $T: X \to Cl(X)$ be a p-contractive mapping. Then, there exists $v_0 \in X$ such that $v_0 \in T(v_0)$ and $p(v_0, v_0) = 0$.*

Proof. Let $u_0 \in X$ be chosen. From Lemma 2.2 and Theorem 2.4, we know that there exist a p-Cauchy sequence $\{u_n\} \in \mathcal{O}(T, u_0)$ and $v_0 \in F(T)$ such that $\{u_n\}$ converges to v_0,

$$\lim_{n \to \infty} \sup\{p(u_n, u_m) : m \ge n\} = 0, \tag{2.10}$$

and

$$p(u_n, v_0) \le \frac{r^n}{1-r} p(u_0, u_1). \tag{2.11}$$

We now show that $p(v_0, v_0) = 0$. Observe that, since T is a p-contractive mapping and $v_0 \in T(v_0)$, we can find $v_1 \in T(v_0)$ such that

$$p(v_0, v_1) \le rp(v_0, v_0).$$

In fact, by using this process, we can obtain a sequence $\{v_n\}$ in X such that $v_{n+1} \in T(v_n)$ and

$$p(v_0, v_{n+1}) \leq rp(v_0, v_n), \quad \text{for all } n \in \mathbb{N}.$$

It follows that

$$p(v_0, v_n) \leq rp(v_0, v_{n-1}) \leq \cdots \leq r^n p(v_0, v_0). \tag{2.12}$$

By using (2.11) and (2.12), we have

$$\lim_{n \to \infty} p(u_n, v_n) \leq \lim_{n \to \infty} [p(u_n, v_0) + p(v_0, v_n)]$$
$$= 0.$$

Consequently, by using this one together with (2.10), we get

$$\limsup_{n \to \infty} \sup\{p(u_n, v_m) : m \geq n\} \leq \lim_{n \to \infty} [\sup\{p(u_n, u_m) : m \geq n\} + \sup\{p(u_m, v_m) : m \geq n\}] = 0. \tag{2.13}$$

Thus, since $\{u_n\}$ is a p-Cauchy sequence, we know from (2.13) and Lemma 2.1 that $\{v_n\}$ is a p-Cauchy sequence and $\lim_{n \to \infty} d(u_n, v_n) = 0$. Thus, since (X, d) is a complete metric space, there exists $x_0 \in X$ such that $\lim_{n \to \infty} v_n = x_0$. Consequently, by using (u3), we obtain

$$p(v_0, x_0) \leq \liminf_{n \to \infty} p(v_0, v_n) \leq 0.$$

This implies

$$p(v_0, x_0) = 0. \tag{2.14}$$

On the other hand, since $\lim_{n \to \infty} u_n = v_0$, $\lim_{n \to \infty} v_n = x_0$ and $\lim_{n \to \infty} d(u_n, v_n) = 0$, we know that $x_0 = v_0$. Hence, from (2.14), we conclude that $p(v_0, v_0) = 0$. This completes the proof. □

Remark 2.6. Theorem 2.5 extends a result presented by Suzuki and Takahashi [9], from the concept of w-distance to the concept of u-distance.

By using Theorem 2.5, we can obtain the following result.

Corollary 2.7. *Let (X, d) be a complete metric space, and let $T: X \to X$ be a p-contractive mapping. Then, T has a unique fixed point $v_0 \in X$. Further, such v_0 satisfies $p(v_0, v_0) = 0$.*

Proof. It follows from Theorem 2.5 that there exists $v_0 \in X$ such that $T(v_0) = v_0$ and $p(v_0, v_0) = 0$. Now if $y_0 = T(y_0)$, we see that

$$p(v_0, y_0) = p(T(v_0), T(y_0)) \leq rp(v_0, y_0),$$

where $r \in [0, 1)$ satisfies the condition of p-contractive mapping T. Consequently, since $r \in [0, 1)$, we have $p(v_0, y_0) = 0$. Hence, by $p(v_0, v_0) = 0$ and Lemma 1.9, we conclude that $v_0 = y_0$. This completes the proof. □

Obviously, our Corollary 2.7 is a generalization of Banach contraction principle. Now we provide an interesting example.

Example 2.8. Let $a \in (1, \infty)$ be a fixed real number. Let c and d be two positive real numbers such that $c \in \left(1, \frac{2a+1}{a}\right)$ and $d \in \left(ac - a, ac - \frac{a}{2}\right)$, respectively. Let $X = [0, a]$ and $d: X \times X \to [0, \infty)$ be a usual metric. Let us consider a mapping $T: X \to X$, which is defined by

$$Tx = \begin{cases} \frac{x}{2} & \text{if } x \in [0, \frac{d}{c}), \\ cx - d & \text{if } x \in [\frac{d}{c}, \ a]. \end{cases}$$

Observe that for each $x, \ y \in [\frac{d}{c}, \ a]$, we have

$$d(Tx, \ Ty) = c \ | \ x - y \ | \ > \ | \ x - y \ | = d(x, \ y),$$

since $c > 1$. This fact implies that the Banach contraction principle cannot be used for guaranteeing the existence of fixed point of our considered mapping T.

On the other hand, define now a function $p: X \times X \to [0, \infty)$ by

$$p(x, \ y) = | \ x \ |, \quad \text{for all} \quad x, \ y \in X.$$

It follows that p is a u-distance, see [11].

Let us choose $r = \frac{ac-d}{a}$. Notice that, by the choice of d, we have $r \in (\frac{1}{2}, 1)$. We will show that T satisfies all hypotheses of our Corollary 2.7, with respect to this real number r and u-distance p. To do this, we consider the following cases:

Case 1: If $x \in [0, \ \frac{d}{c})$. We have

$$p(Tx, \ Ty) = | \ Tx \ | = \frac{x}{2} < rx = rp(x, \ y).$$

Case 2: If $x \in [\frac{d}{c}, \ a]$. We have

$$p(Tx, \ Ty) = | \ cx - d \ | \le \left(\frac{ac - d}{a} \right) x = rp(x, \ y).$$

By using above facts, we can show that all assumptions of Corollary 2.7 are satisfied. In fact, we can check that $F(T) = \{0\}$.

Remark 2.9. Example 2.8 shows that Corollary 2.7 is a genuine generalization of the Banach contraction principle.

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Authors' contributions
Both authors contributed equally in this paper. They read and approved the final manuscript.

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