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New Methods with Perturbations for Non-Expansive Mappings in Hilbert Spaces

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Abstract

In this paper, we suggest and analyze two iterative algorithms with perturbations for non-expansive mappings in Hilbert spaces. We prove that the proposed iterative algorithms converge strongly to a fixed point of some non-expansive mapping.

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that a mapping T: $C \to C$ is said to be non-expansive if

$$||Tx - Ty|| \le ||x - y|| \quad \text{for all } x, y \in C.$$

Denote by $Fix(T)$ the set of fixed points of T; that is, $Fix(T) = \{x \in C : Tx = x\}$.

Recently, iterative methods for finding fixed points of non-expansive mappings have received vast investigations due to its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery, and signal processing; see [1-34] and the references therein. There are perturbations always occurring in the iterative processes because the manipulations are inaccurate. It is no doubt that researching the convergent problems of iterative methods with perturbation members is a significant job.

It is our purpose in this paper that we suggest and analyze two iterative algorithms with errors for non-expansive mappings in Hilbert spaces. We prove that the proposed iterative algorithms converge strongly to a fixed point of some non-expansive mapping.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $|| \cdot ||$, respectively. Recall that the nearest point (or metric) projection from H onto a nonempty closed convex subset C of H is defined as follows: for each point $x \in H$, $P_C(x)$] is the unique point in C with the property:

$$||x - P_C(x)|| \le ||x - y||, \quad \forall y \in C.$$

A characterization for P_C is described below. Given $x \in H$ and $z \in C$. Then $z = P_C(x)$ if and only if there holds the inequality

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C. \tag{2.1}$$

It is known that P_C is non-expansive. The following well-known lemmas play an important role in our argument in the next sections.

Lemma 2.1. (Demiclosedness principle) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a non-expansive mapping with $Fix(T) \neq \emptyset$. Then, T is demiclosed on C, i.e., if $x_n \to x \in C$ weakly and $x_n - Tx_n \to y$ strongly, then $(I - T)x = y$.*

Lemma 2.2. (Suzuki's lemma) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}, \beta_n < 1$. *Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and* $\limsup_{n\to\infty} (||y_{n+1} - y_n ||- ||x_{n+1} - x_n||) \leq 0$. *Then,* $\lim_{n\to\infty} ||y_n - x_n|| = 0$.

Lemma 2.3. (Liu's lemma) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n + \sigma_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, *and $\{\delta_n\}$ and $\{\sigma_n\}$ are two sequences in R such that*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n\gamma_n| < \infty$;
(iii) $\sum_{n=0}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we introduce our algorithms with perturbations and state our main results.

Algorithm 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a non-expansive mapping. For given $x_0 \in C$, define a sequence $\{x_m\}$ by the following manner:*

$$x_m = P_C(\alpha_m u_m + (1 - \alpha_m)Tx_m), \quad m \geq 0, \tag{3.1}$$

where $\{\alpha_m\}$ is a sequence in $[0, 1]$, *and the sequence $\{u_m\} \subset H$ is a small perturbation for the m-step iteration satisfying $||u_m|| \to 0$ as $m \to \infty$.*

Remark 3.2. In this point, we want to point out that we permit the perturbation $\{u_m\}$ in the whole space H. If $\{u_m\} \subset C$, then (3.1) reduces to

$$x_m = \alpha_m u_m + (1 - \alpha_m)Tx_m, \quad m \geq 0. \tag{3.2}$$

Theorem 3.3. *Suppose $Fix(T) \neq \emptyset$. Then, as $\alpha_m \to 0$, the sequence $\{x_m\}$ generated by the implicit method (3.1) converges to $\tilde{x} \in Fix(T)$.*

Proof. We first show that $\{x_m\}$ is bounded. Indeed, take an $x^* \in Fix(T)$ to derive that

$$\begin{aligned}||x_m - x^*|| &= ||P_C(\alpha_m u_m + (1-\alpha_m)Tx_m) - x^*||\\ &\le \alpha_m||x^*|| + (1-\alpha_m)||Tx_m - x^*|| + \alpha_m||u_m||\\ &\le (1-\alpha_m)||x_m - x^*|| + \alpha_m||x^*|| + \alpha_m||u_m||.\end{aligned}$$

This implies that

$$||x_m - x^*|| \le ||x^*|| + ||u_m||.$$

Since $||u_m|| \to 0$, there exists a constant $M > 0$ such that $\sup_m\{||u_m||\} \le M$. Hence, $||x_m - x^*|| \le ||x^*|| + M$ for all $n \ge 0$. It follows that $\{x_m\}$ is bounded, so is the sequence $\{Tx_m\}$.

Since $x_m \in C$ and also $Tx_m \in C$, we get

$$||x_m - Tx_m|| = ||P_C(\alpha_m u_m + (1-\alpha_m)Tx_m) - P_C(Tx_m)|| \le \alpha_m||u_m - Tx_m|| \to 0. \tag{3.3}$$

Setting $y_m = \alpha_m u_m + (1 - \alpha_m)Tx_m$ for all $n \ge 0$, we then have $x_m = P_C(y_m)$, and for any $x^* \in Fix(T)$,

$$\begin{aligned}x_m - x^* &= P_C(y_m) - y_m + y_m - x^*\\ &= P_C(y_m) - y_m + \alpha_m u_m + (1-\alpha_m)(Tx_m - x^*) - \alpha_m x^*.\end{aligned}$$

Noting that the fact by (2.1) that

$$\langle P_C(y_m) - y_m, P_C(y_m) - x^*\rangle \le 0.$$

Hence, we have

$$\begin{aligned}||x_m - x^*||^2 &= \langle P_C(y_m) - y_m, x_m - x^*\rangle + \alpha_m\langle u_m, x_m - x^*\rangle\\ &\quad + (1-\alpha_m)\langle Tx_m - x^*, x_m - x^*\rangle - \alpha_m\langle x^*, x_m - x^*\rangle\\ &\le (1-\alpha_m)||Tx_m - x^*||||x_m - x^*|| + \alpha_m||u_m||||x_m - x^*|| - \alpha_m\langle x^*, x_m - x^*\rangle\\ &\le (1-\alpha_m)||x_m - x^*||^2 + \alpha_m||u_m||||x_m - x^*|| - \alpha_m\langle x^*, x_m - x^*\rangle.\end{aligned}$$

It turns out that

$$||x_m - x^*||^2 \le \langle x^*, x^* - x_m\rangle + ||u_m||(||x^*|| + M), \quad x^* \in Fix(T). \tag{3.4}$$

Since $\{x_m\}$ is bounded, without loss of generality, we may assume that $\{x_m\}$ converges weakly to a point $\tilde{x} \in C$. Noticing (3.3), we can use Lemma 2.1 to get $\tilde{x} \in Fix(T)$. Therefore, we can substitute $\tilde{x}$ for x^* in (3.4) to get

$$||x_m - \tilde{x}||^2 \le \langle \tilde{x}, \tilde{x} - x_m\rangle + ||u_m||(||x^*|| + M).$$

Consequently, the weak convergence of $\{x_m\}$ to $\tilde{x}$ actually implies that $x_m \to \tilde{x}$ strongly. Finally, in order to complete the proof, we have to prove that the weak cluster points set $\omega_w(x_m)$ is singleton. As a matter of fact, if $x_{m_i} \rightharpoonup \hat{x} \in Fix(T)$ and $x_{m_k} \rightharpoonup \bar{x} \in Fix(T)$, then we have $x_{m_i} \to \hat{x} \in Fix(T)$ and $x_{m_k} \to \bar{x} \in Fix(T)$. From (3.4), we have

$$||x_{m_i} - \bar{x}||^2 \le \langle \bar{x}, \bar{x} - x_{m_i}\rangle + ||u_{m_i}||(||\bar{x}|| + M),$$

and

$$||x_{m_k} - \hat{x}||^2 \leq \langle \hat{x}, \hat{x} - x_{m_k} \rangle + ||u_{m_k}||(||\hat{x}|| + M).$$

Hence, we have $||\hat{x} - \bar{x}||^2 \leq \langle \bar{x}, \bar{x} - \hat{x} \rangle$ and $||\bar{x} - \hat{x}||^2 \leq \langle \hat{x}, \hat{x} - \bar{x} \rangle$. Therefore, we obtain

$$2||\hat{x} - \bar{x}||^2 \leq \langle \bar{x} - \hat{x}, \bar{x} - \hat{x} \rangle = ||\hat{x} - \bar{x}||^2.$$

We have immediately $\hat{x} = \bar{x}$. This completes the proof.

From Theorem 3.3, we have the following corollary.

Corollary 3.4. *Suppose* $Fix(T) \neq \emptyset$. *Then, as* $\alpha_m \to 0$, *the sequence* $\{x_m\}$ *generated by the implicit method (3.2) converges to* $\tilde{x} \in Fix(T)$.

Next, we introduce an explicit algorithm.

Algorithm 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let* $T : C \to C$ *be a non-expansive mapping. For given* $x_0 \in C$, *define a sequence* $\{x_n\}$ *by the following manner:*

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C(\alpha_n u_n + (1 - \alpha_n)Tx_n), \quad n \geq 0, \tag{3.5}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* (0,1), *and the sequence* $\{u_n\} \subset H$ *is a perturbation for the n-step iteration.*

Remark 3.6. If $\{u_n\} \subset C$, then (3.5) reduces to

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n[\alpha_n u_n + (1 - \alpha_n)Tx_n], \quad n \geq 0, \tag{3.6}$$

Theorem 3.7. *Suppose* $Fix(T) \neq \emptyset$. *Assume the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\sum_{n=0}^{\infty} \alpha_n ||u_n|| < \infty$.

Then, the sequence $\{x_n\}$ *generated by the explicit iterative method (3.5) converges to* $\tilde{x} \in Fix(T)$.

Proof. First, we show that $\{x_n\}$ is bounded. Take an $x^* \in Fix(T)$ to derive that

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||(1 - \beta_n)(x_n - x^*) + \beta_n(P_C(\alpha_n u_n + (1 - \alpha_n)Tx_n) - x^*)|| \\ &\leq (1 - \beta_n)||x_n - x^*|| + \beta_n||\alpha_n u_n + (1 - \alpha_n)Tx_n - x^*|| \\ &\leq (1 - \beta_n)||x_n - x^*|| + \beta_n[\alpha_n||u_n|| + (1 - \alpha_n)||Tx_n - x^*|| + \alpha_n||x^*||] \\ &\leq (1 - \beta_n)||x_n - x^*|| + \beta_n[\alpha_n||u_n|| + (1 - \alpha_n)||x_n - x^*|| + \alpha_n||x^*||] \\ &= (1 - \beta_n\alpha_n)||x_n - x^*|| + \beta_n\alpha_n||x^*|| + \alpha_n||u_n|| \\ &\leq \max\{||x_n - x^*||, ||x^*||\} + \alpha_n||u_n||. \end{aligned}$$

By induction, we get

$$||x_n - x^*|| \leq \max\{||x_0 - x^*||, ||x^*||\} + \sum_{n=0}^{n-1} \alpha_n||u_n||.$$

Thus, $\{x_n\}$ is bounded, so is the sequence $\{Tx_n\}$. Next, we show that

$$||x_{n+1} - x_n|| \to 0. \tag{3.7}$$

Indeed, we write $x_{n+1} = (1 - \beta_n)x_n + \beta_n y_n$, $n \geq 0$. It is clear that $y_n = P_C(\alpha_n u_n + (1 - \alpha_n)Tx_n)$ for all $n \geq 0$. Then, we have

$$\begin{aligned}||y_{n+1} - y_n|| &\leq ||\alpha_{n+1}u_{n+1} + (1-\alpha_{n+1})Tx_{n+1} - \alpha_n u_n - (1-\alpha_n)Tx_n|| \\ &= ||\alpha_{n+1}u_{n+1} - \alpha_n u_n + (1-\alpha_{n+1})(Tx_{n+1} - Tx_n) + (\alpha_n - \alpha_{n+1})Tx_n|| \\ &\leq (1-\alpha_{n+1})||Tx_{n+1} - Tx_n|| + (\alpha_n + \alpha_{n+1})||Tx_n|| + \alpha_{n+1}||u_{n+1}|| + \alpha_n||u_n|| \\ &\leq (1-\alpha_{n+1})||x_{n+1} - x_n|| + (\alpha_n + \alpha_{n+1})||Tx_n|| + \alpha_{n+1}||u_{n+1}|| + \alpha_n||u_n||.\end{aligned}$$

It follows that

$$||y_{n+1} - y_n|| - ||x_{n+1} - x_n|| \leq \alpha_{n+1}||x_{n+1} - x_n|| + (\alpha_n + \alpha_{n+1})||Tx_n|| + \alpha_{n+1}||u_{n+1}|| + \alpha_n||u_n||.$$

This together with (i) and (iii) implies that

$$\limsup_{n\to\infty} \left(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||\right) \leq 0.$$

Hence, by Lemma 2.2, we get

$$\lim_{n\to\infty} ||y_n - x_n|| = 0 \tag{3.8}$$

Consequently, $\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} \beta_n ||y_n - x_n|| = 0$. We now show that

$$||x_n - Tx_n|| \to 0.$$

Notice that

$$\begin{aligned}||x_n - Tx_n|| &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - Tx_n|| \\ &\leq ||x_n - x_{n+1}|| + (1-\beta_n)||x_n - Tx_n|| + \beta_n||y_n - P_C(Tx_n)|| \\ &\leq ||x_n - x_{n+1}|| + (1-\beta_n)||x_n - Tx_n|| + \beta_n[\alpha_n||u_n|| + \alpha_n||Tx_n||].\end{aligned}$$

Hence,

$$||x_n - Tx_n|| \leq \frac{1}{\beta_n}||x_n - x_{n+1}|| + \alpha_n||u_n|| + \alpha_n||Tx_n||.$$

Therefore,

$$\lim_{n\to\infty} ||x_n - Tx_n|| = 0. \tag{3.9}$$

We next show that

$$\limsup_{n\to\infty} \langle \tilde{x}, \tilde{x} - x_n \rangle \leq 0,$$

where $\tilde{x} = \lim_{m\to\infty} y_m$ and $\{y_m\}$ be the sequence defined by the implicit method (3.1). Since $x_n \in C$ and $\langle y_m - [\alpha_m u_m + (1 - \alpha_m)Ty_m], y_m - x_n \rangle \leq 0$, we have

$$\begin{aligned}||y_m - x_n||^2 &= \langle y_m - x_n, y_m - x_n \rangle \\ &= \langle y_m - [\alpha_m u_m + (1-\alpha_m)Ty_m], y_m - x_n \rangle + \langle \alpha_m u_m + (1-\alpha_m)Ty_m - x_n, y_m - x_n \rangle \\ &\leq \langle \alpha_m u_m + (1-\alpha_m)Ty_m - x_n, y_m - x_n \rangle \\ &= \alpha_m \langle u_m, y_m - x_n \rangle + (1-\alpha_m)\langle Ty_m - x_n, y_m - x_n \rangle - \alpha_m \langle x_n, y_m - x_n \rangle \\ &= \alpha_m \langle u_m, y_m - x_n \rangle + (1-\alpha_m)\langle Ty_m - Tx_n, y_m - x_n \rangle \\ &\quad + (1-\alpha_m)\langle Tx_n - x_n, y_m - x_n \rangle - \alpha_m \langle x_n - y_m, y_m - x_n \rangle - \alpha_m \langle y_m, y_m - x_n \rangle \\ &\leq \alpha_m||u_m||||y_m - x_n|| + (1-\alpha_m)||Ty_m - Tx_n||||y_m - x_n|| \\ &\quad + (1-\alpha_m)||Tx_n - x_n||||y_m - x_n|| + \alpha_m||y_m - x_n||^2 - \alpha_m \langle y_m, y_m - x_n \rangle \\ &\leq \alpha_m||u_m||M_1 + ||y_m - x_n||^2 + ||Tx_n - x_n||M_1 - \alpha_m \langle y_m, y_m - x_n \rangle,\end{aligned}$$

where $M_1 > 0$ such that $\sup\{||y_m - x_n||, m, n \geq 0\} \leq M_1$. It follows that

$$\langle y_m, y_m - x_n \rangle \leq ||u_m||M_1 + \frac{||Tx_n - x_n||M_1}{\alpha_m}.$$

Therefore,

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \langle y_m, y_m - x_n \rangle \leq 0. \tag{3.10}$$

We note that

$$\begin{aligned}\langle \tilde{x}, \tilde{x} - x_n \rangle &= \langle \tilde{x}, \tilde{x} - y_m \rangle + \langle \tilde{x} - y_m, y_m - x_n \rangle + \langle y_m, y_m - x_n \rangle \\ &\leq \langle \tilde{x}, \tilde{x} - y_m \rangle + ||\tilde{x} - y_m||||y_m - x_n|| + \langle y_m, y_m - x_n \rangle \\ &\leq \langle \tilde{x}, \tilde{x} - y_m \rangle + ||\tilde{x} - y_m||M + \langle y_m, y_m - x_n \rangle .\end{aligned}$$

This together with $y_m \to \tilde{x}$ and (3.10) implies that

$$\limsup_{n\to\infty} \langle \tilde{x}, \tilde{x} - x_n \rangle = \limsup_{m\to\infty} \limsup_{n\to\infty} \langle \tilde{x}, \tilde{x} - x_n \rangle \leq \limsup_{m\to\infty} \limsup_{n\to\infty} \langle y_m, y_m - x_n \rangle \leq 0 \tag{3.11}$$

From (3.8), (3.9) and (3.11), we have

$$\limsup_{n\to\infty} \langle \tilde{x}, \tilde{x} - y_n \rangle \leq 0 \text{ and } \limsup_{n\to\infty} \langle \tilde{x}, \tilde{x} - Tx_n \rangle \leq 0. \tag{3.12}$$

Finally, we show that $x_n \to \tilde{x}$. Set $z_n = \alpha_n u_n + (1 - \alpha_n)Tx_n, n \geq 0$. Since $\tilde{x} \in C$ and $y_n = P_C(z_n)$. Hence $\langle y_n - z_n, y_n - \tilde{x} \rangle \leq 0$. From (3.5), we have

$$\begin{aligned}||x_{n+1} - \tilde{x}||^2 &= ||(1 - \beta_n)(x_n - \tilde{x}) + \beta_n(y_n - \tilde{x})||^2 \\ &\leq (1 - \beta_n)||x_n - \tilde{x}||^2 + \beta_n||y_n - \tilde{x}||^2 \\ &= (1 - \beta_n)||x_n - \tilde{x}||^2 + \beta_n \langle y_n - z_n, y_n - \tilde{x} \rangle + \beta_n \langle z_n - \tilde{x}, y_n - \tilde{x} \rangle \\ &\leq (1 - \beta_n)||x_n - \tilde{x}||^2 + \beta_n \langle z_n - \tilde{x}, y_n - \tilde{x} \rangle \\ &= (1 - \beta_n)||x_n - \tilde{x}||^2 + \beta_n(1 - \alpha_n) \langle Tx_n - \tilde{x}, y_n - \tilde{x} \rangle \\ &+ \beta_n\alpha_n \langle \tilde{x}, \tilde{x} - y_n \rangle + \beta_n\alpha_n \langle u_n, y_n - \tilde{x} \rangle .\end{aligned}$$

Note that

$$\langle Tx_n - \tilde{x}, y_n - \tilde{x} \rangle \leq ||Tx_n - \tilde{x}||||y_n - \tilde{x}|| \leq \frac{1}{2}(||x_n - \tilde{x}||^2 + ||y_n - \tilde{x}||^2) \leq \frac{1}{2}(||x_n - \tilde{x}||^2 + ||z_n - \tilde{x}||^2),$$

and

$$\begin{aligned}||z_n - \tilde{x}||^2 &= ||(1 - \alpha_n)(Tx_n - \tilde{x})^2 - \alpha_n\tilde{x} + \alpha_n u_n||^2 \\ &\leq ||(1 - \alpha_n)(Tx_n - \tilde{x})^2 - \alpha_n\tilde{x}||^2 + \alpha_n||u_n||||(1 - \alpha_n)(Tx_n - \tilde{x})^2 - \alpha_n\tilde{x}|| + \alpha_n^2||u_n||^2 \\ &\leq (1 - \alpha_n)||Tx_n - \tilde{x}||^2 - 2\alpha_n(1 - \alpha_n) \langle \tilde{x}, Tx_n - \tilde{x} \rangle + \alpha_n^2||\tilde{x}||^2 \\ &+ \alpha_n||u_n||||(1 - \alpha_n)(Tx_n - \tilde{x})^2 - \alpha_n\tilde{x}|| + \alpha_n^2||u_n||^2 \\ &\leq (1 - \alpha_n)||x_n - \tilde{x}||^2 - 2\alpha_n(1 - \alpha_n) \langle \tilde{x}, Tx_n - \tilde{x} \rangle + \alpha_n^2||\tilde{x}||^2 + \alpha_n||u_n||M_1,\end{aligned}$$

where M_2 is a constant such that $\sup_n\{||(1-\alpha_n)(Tx_n-\tilde{x})^2-\alpha_n\tilde{x}||+\alpha_n||u_n||\}\le M_2$. Hence, we have

$$\begin{aligned}||x_{n+1}-\tilde{x}||^2 &\le (1-\beta_n)||x_n-\tilde{x}||^2+\frac{\beta_n(1-\alpha_n)}{2}(||x_n-\tilde{x}||^2+||z_n-\tilde{x}||^2)\\&\quad+\beta_n\alpha_n\langle\tilde{x},\tilde{x}-y_n\rangle+\beta_n\alpha_nM_2||u_n||\\&\le(1-\beta_n)||x_n-\tilde{x}||^2+\frac{\beta_n(1-\alpha_n)}{2}||x_n-\tilde{x}||^2\\&\quad+\frac{\beta_n(1-\alpha_n)}{2}\big((1-\alpha_n)||x_n-\tilde{x}||^2-2\alpha_n(1-\alpha_n)\langle\tilde{x},Tx_n-\tilde{x}\rangle\\&\quad+\alpha_n^2||\tilde{x}||^2+\alpha_n||u_n||M_2\big)+\beta_n\alpha_n\langle\tilde{x},\tilde{x}-y_n\rangle+\beta_n\alpha_nM_2||u_n||\\&\le(1-\beta_n\alpha_n)||x_n-\tilde{x}||^2+\beta_n\alpha_n(1-\alpha_n)^2\langle\tilde{x},\tilde{x}-Tx_n\rangle\\&\quad+\beta_n\alpha_n\langle\tilde{x},\tilde{x}-y_n\rangle+\beta_n\alpha_n^2||\tilde{x}||^2+2M_2\alpha_n||u_n||\\&=(1-\beta_n\alpha_n)||x_n-\tilde{x}||^2+\beta_n\alpha_n\Big\{(1-\alpha_n)^2\langle\tilde{x},\tilde{x}-Tx_n\rangle\\&\quad+\beta_n\alpha_n\langle\tilde{x},\tilde{x}-y_n\rangle+\alpha_n||\tilde{x}||^2\Big\}+2M_2\alpha_n||u_n||\\&=(1-\gamma_n)||x_n-\tilde{x}||^2+\gamma_n\delta_n+\sigma_n,\end{aligned}$$

where $\gamma_n=\beta_n\alpha_n, \delta_n=(1-\alpha_n)^2\langle\tilde{x},\tilde{x}-Tx_n\rangle+\beta_n\alpha_n\langle\tilde{x},\tilde{x}-y_n\rangle+\alpha_n||\tilde{x}||^2$ and $\sigma_n=2M_2\alpha_n||u_n||$. Now, applying Lemma 2.3 to the last inequality, we conclude that $x_n\to\tilde{x}$. This completes the proof.

Corollary 3.8. *Suppose* $Fix(T)\ne\emptyset$. *Assume the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$;

(ii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty},\beta_n<1$;

(iii) $\sum_{n=0}^{\infty}\alpha_n||u_n||<\infty$.

Then, the sequence $\{x_n\}$ *generated by the explicit iterative method (3.6) converges to* $\tilde{x}\in Fix(T)$.

Remark 3.9. We would like to point out that our algorithms (3.1) and (3.5) converge strongly to the minimum-norm fixed point $\tilde{x}$ of *T*. As a matter of fact, from (3.4), as $m\to\infty$, we deduce

$$||\tilde{x}-x^*||^2\le\langle x^*,x^*-\tilde{x}\rangle,\quad\forall x^*\in Fix(T),$$

which is equivalent to

$$||\tilde{x}||^2\le\langle x^*,\tilde{x}\rangle\le||x^*||||\tilde{x}||,\quad\forall x^*\in Fix(T).$$

Therefore,

$$||\tilde{x}||\le||x^*||,\quad\forall x^*\in Fix(T).$$

That is, $\tilde{x}$ is the minimum-norm fixed point of *T*.

Minimum-norm solutions are important in applied problems, e.g., defining the pseudoinverse of a bounded linear operator, and many other problems in signal processing. Therefore, using iterative methods to find the minimum-norm solution of a given nonlinear problem is of significant value. Finding the minimum-norm solution of a nonlinear problem has recently been received a lot of attention, and for some related works, please see [35-37]. Our paper provides such iterative methods (an implicit and an explicit) for finding minimum-norm solutions of nonlinear operator equations governed by non-expansive mappings.

4. Competing interests

The authors declare that they have no competing interests.

5. Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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