Balooee et al. Fixed Point Theory and Applications 2011, **2011**:86
http://www.fixedpointtheoryandapplications.com/content/2011/1/86

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

The Wiener-Hopf Equation Technique for Solving General Nonlinear Regularized Nonconvex Variational Inequalities

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Abstract

In this paper, we introduce and study some new classes of extended general nonlinear regularized non-convex variational inequalities and the extended general nonconvex Wiener-Hopf equations, and by the projection operator technique, we establish the equivalence between the extended general nonlinear regularized nonconvex variational inequalities and the fixed point problems as well as the extended general nonconvex Wiener-Hopf equations. Then by using this equivalent formulation, we discuss the existence and uniqueness of solution of the problem of extended general nonlinear regularized nonconvex variational inequalities. We apply the equivalent alternative formulation and a nearly uniformly Lipschitzian mapping S for constructing some new p-step projection iterative algorithms with mixed errors for finding an element of set of the fixed points of nearly uniformly Lipschitzian mapping S which is unique solution of the problem of extended general nonlinear regularized nonconvex variational inequalities. We also consider the convergence analysis of the suggested iterative schemes under some suitable conditions.

Mathematics Subject Classification (2010)
Primary 47H05; Secondary 47J20, 49J40

Keywords: variational inequalities, fixed point problems, prox-regularity, nearly uniformly Lipschitzian mappings, p-step projection iterative algorithms, extended general nonconvex Wiener-Hopf equations, convergence analysis

1 Introduction

The theory of variational inequalities, which was initially introduced by Stampacchia [1] in 1964, is a branch of the mathematical sciences dealing with general equilibrium problems. It has a wide range of applications in economics, optimizations research, industry, physics, and engineering sciences. Many research papers have been written lately, both on the theory and applications of this field. Important connections with main areas of pure and applied sciences have been made, see for example [2,3] and the references cited therein. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solution to important classes of problems; on the other hand, it also enables us to develop highly efficient and powerful new numerical methods to solve, for example, obstacle, unilateral, free, moving and the complex equilibrium problems. One of the most interesting and important problems in variational inequality theory is the development of an efficient numerical method. There is a substantial number of numerical methods including projection method and its variant forms, Wiener-Holf (normal) equations, auxiliary principle, and descent framework for solving variational inequalities and complementarity problems. For the applications on physical formulations, numerical methods and other aspects of variational inequalities, see [1-52] and the references therein.

Projection method and its variant forms represent important tool for finding the approximate solution of various types of variational and quasi-variational inequalities, the origin of which can be traced back to Lions and Stampacchia [31]. The projection type methods were developed in 1970's and 1980's. The main idea in this technique is to establish the equivalence between the variational inequalities and the fixed point problems using the concept of projection. This alternate formulation enables us to suggest some iterative methods for computing the approximate solution. Shi [50,51] and Robinson [48] considered the problem of solving a system of equations which are called the Wiener-Hopf equations or normal maps. Shi [50] and Robinson [48] proved that the variational inequalities and the Wiener-Hopf equations are equivalent by using the projection technique. It turned out that this alternative equivalent formulation is more general and flexible. It has shown in [48-53] that the Wiener-Hopf equations provide us a simple, elegant and convenient device for developing some efficient numerical methods for solving variational inequalities and complementarity problems.

It should be pointed that almost all the results regarding the existence and iterative schemes for solving variational inequalities and related optimizations problems are being considered in the convexity setting. Consequently, all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general, when the sets are nonconvex. It is known that the uniformly prox-regular sets are nonconvex and include the convex sets as special cases, for more details, see for example [23,28,29,46]. In recent years, Bounkhel et al. [23], Noor [36,41] and Pang et al. [45] have considered variational inequalities in the context of uniformly prox-regular sets.

On the other hand, related to the variational inequalities, we have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. It is natural to consider a unified approach to these two different problems. Motivated and inspired by the research going in this direction, Noor and Huang [43] considered the problem of finding the common element of the set of the solutions of variational inequalities and the set of the fixed points of the nonexpansive mappings. Noor [38] suggested and analyzed some three-step iterative algorithms for finding the common elements of the set of the solutions of the Noor variational inequalities and the set of the fixed points of nonexpansive mappings. He also discussed the convergence analysis of the suggested iterative algorithms under some conditions.

Recently, Qin and Noor [47] established the equivalence between general variational inequalities and general Wiener-Hopf equations. They proposed and analyzed a new iterative method for solving variational inequalities and related optimization problems. They also considered the problem of finding a comment element of fixed points of nonexpansive mappings and the set of solution of the general variational inequalities.

It is well known that every nonexpansive mapping is a Lipschitzian mapping. Lipschitzian mappings have been generalized by various authors. Sahu [53] introduced and investigated nearly uniformly Lipschitzian mappings as generalization of Lipschitzian mappings.

Motivated and inspired by the above works, at the present paper, some new classes of the extended general nonlinear regularized nonconvex variational inequalities and the extended general nonconvex Wiener-Hopf equations are introduced and studied, and by the projection technique, the equivalence between the extended general nonlinear regularized nonconvex variational inequalities and the fixed point problems as well as the extended general nonconvex Wiener-Hopf equations is proved. Then by using this equivalent formulation, the existence and uniqueness of solution of the problem of extended general nonlinear regularized nonconvex variational inequalities are discussed. Applying the equivalent alternative formulation and a nearly uniformly Lipschitzian mapping S, some new p-step projection iterative algorithms with mixed errors for finding an element of the set of fixed points of nearly uniformly Lipschitzian mapping S which is a unique solution of the problem of extended general nonlinear regularized nonconvex variational inequalities are defined. The convergence analysis of the suggested iterative schemes under some suitable conditions is discussed. Some remarks about established statements by Noor [38], Noor et al. [44] and Qin and Noor [47] are presented. Also, this fact that their statements are special cases of our results is shown. The results obtained in this paper may be viewed as an refinement and improvement of the previously known results.

2 Preliminaries and basic results

Throughout this article, we will let $\mathcal{H}$ be a real Hilbert space which is equipped with an inner product $\langle .,. \rangle$ and corresponding norm $||cdot||$ and K be a nonempty convex subset of $\mathcal{H}$. We denote by $d_K(\cdot)$ or $d(., K)$ the usual distance function to the subset K, i.e., $d_K(u) = \inf_{v \in K} ||u - v||$. Let us recall the following well-known definitions and some auxiliary results of nonlinear convex analysis and nonsmooth analysis [27-29,46].

Definition 2.1. Let $u \in \mathcal{H}$ is a point not lying in K. A point $v \in K$ is called *a closest point* or a *projection* of u onto K if $d_K(u) = ||u - v||$. The set of all such closest points is denoted by $P_K(u)$, i.e.,

$$P_K(u) := \{v \in K : d_K(u) = ||u - v||\}.$$

Definition 2.2. The proximal normal cone of K at a point $u \in \mathcal{H}$ with $u \notin K$ is given by

$$N_K^P(u) := \{\xi \in \mathcal{H} : u \in P_K(u + \alpha\xi) \text{ for some } \alpha > 0\}.$$

Clarke et al. [28], in Proposition 1.1.5, give a characterization of $N_K^P(u)$ as follows:

Lemma 2.3. *Let K be a nonempty closed subset in $\mathcal{H}$*. Then $\xi \in N_K^P(u)$ *if and only if there exists a constant $\alpha = \alpha(\xi, u) > 0$ such that $\langle \xi, v - u \rangle \leq \alpha||v - u||^2$ for all $v \in K$.*

The above inequality is called the *proximal normal inequality*. The special case in which K is closed and convex is an important one. In Proposition 1.1.10 of [28], the authors give the following characterization of the proximal normal cone the closed and convex subset $K \subset \mathcal{H}$:

Lemma 2.4. *Let K be a nonempty closed and convex subset in* $\mathcal{H}$. *Then* $\xi \in N_K^P(u)$ *if and only if* $\langle \xi, v - u \rangle \leq 0$ *for all* $v \in K$.

Definition 2.5. Let X is a real Banach space and $f: X \to \mathbb{R}$ be Lipschitzian with constant τ near a given point $x \in X$, that is, for some $\varepsilon > 0$, we have $|f(y) - f(z)| \leq \tau ||y - z||$ for all $y, z \in B(x; \varepsilon)$, where $B(x; \varepsilon)$ denotes the open ball of radius $r > 0$ and centered at x. The *generalized directional derivative* of f at x in the direction v, denoted by $f^\circ(x; v)$, is defined as follows:

$$f^\circ(x; v) = \limsup_{y \to x, t \downarrow 0} \frac{f(y + tv) - f(y)}{t},$$

where y is a vector in X and t is a positive scalar.

The generalized directional derivative defined earlier can be used to develop a notion of tangency that does not require K to be smooth or convex.

Definition 2.6. The *tangent cone* $T_K(x)$ to K at a point x in K is defined as follows:

$$T_K(x) := \{v \in \mathcal{H} : d_K^\circ(x; v) = 0\}.$$

Having defined a tangent cone, the likely candidate for the normal cone is the one obtained from $T_K(x)$ by polarity. Accordingly, we define the *normal cone* of K at x by polarity with $T_K(x)$ as follows:

$$N_K(x) := \{\xi : \langle \xi, v \rangle \leq 0, \quad \forall v \in T_K(x)\}.$$

Definition 2.7. The *Clarke normal cone*, denoted by $N_K^C(x)$, is given by $N_K^C(x) = \overline{co}[N_K^P(x)]$, where $\overline{co}[S]$ means the closure of the convex hull of S. It is clear that one always has $N_K^P(x) \subseteq N_K^C(x)$. The converse is not true in general. Note that $N_K^C(x)$ is always closed and convex cone, whereas $N_K^P(x)$ is always convex, but may not be closed (see [27,28,46]).

In 1995, Clarke et al. [29] introduced and studied a new class of nonconvex sets called proximally smooth sets; subsequently, Poliquin et al. in [46] investigated the aforementioned sets, under the name of uniformly prox-regular sets. These have been successfully used in many nonconvex applications in areas such as optimizations, economic models, dynamical systems, differential inclusions, etc. For such as applications see [20-22,24]. This class seems particularly well suited to overcome the difficulties which arise due to the nonconvexity assumptions on K. We take the following characterization proved in [29] as a definition of this class. We point out that the original definition was given in terms of the differentiability of the distance function (see [29]).

Definition 2.8. For any $r \in (0, +\infty]$, a subset K_r of $\mathcal{H}$ is called *normalized uniformly prox-regular* (or *uniformly r-prox-regular* [29]) if every nonzero proximal normal to K_r can be realized by an r-ball.

This means that for all $\bar{x} \in K_r$ and $0 \neq \xi \in N_{K_r}^P(\bar{x})$ with $||\xi|| = 1$,

$$\langle \xi, x - \bar{x} \rangle \leq \frac{1}{2r} ||x - \bar{x}||^2, \quad \forall x \in K_r.$$

Obviously, the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of $\mathcal{H}$, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets, see [25,29].

Lemma 2.9. [29]*A closed set $K \subseteq \mathcal{H}$ is convex if and only if it is proximally smooth of radius r for every r >0.*

If $r = +\infty$, then in view of Definition 2.8 and Lemma 2.9, the uniform *r*-prox-regularity of K_r is equivalent to the convexity of K_r, which makes this class of great importance. For the case of that $r = +\infty$, we set $K_r = K$.

The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel. The proof of this results can be found in [29,46].

Proposition 2.10. *Let r >0 and K_r be a nonempty closed and uniformly r-prox-regular subset of $\mathcal{H}$. Set $U(r) = \{u \in \mathcal{H} : 0 < d_{K_r}(u) < r\}$. Then the following statements hold:*

(a) *For all $x \in U(r)$, one has $P_{K_r}(x) \neq \emptyset$;*

(b) *For all $r' \in (0, r)$, P_{K_r} is Lipschitzian continuous with constant $\frac{r}{r-r'}$ on $U(r') = \{u \in \mathcal{H} : 0 < d_{K_r}(u) < r'\}$;*

(c) *The proximal normal cone is closed as a set-valued mapping.*

As a direct consequent of part (c) of Proposition 2.10, we have $N^C_{K_r}(x) = N^P_{K_r}(x)$. Therefore, we will define $N_{K_r}(x) := N^C_{K_r}(x) = N^P_{K_r}(x)$ for such a class of sets.

In order to make clear the concept of *r*-prox-regular sets, we state the following concrete example: The union of two disjoint intervals $[a, b]$ and $[c, d]$ is *r*-prox-regular with $r = \frac{c-b}{2}$. The finite union of disjoint intervals is also *r*-prox-regular and *r* depends on the distances between the intervals.

Definition 2.11. Let $T, g : \mathcal{H} \to \mathcal{H}$ be two single-valued operators. Then the operator T is said to be:

(a) *monotone* if

$$\langle T(x) - T(y), x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H};$$

(b) *r-strongly monotone* if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq r||x - y||^2, \quad \forall x, y \in \mathcal{H};$$

(c) *κ-strongly monotone* with respect to *g* if there exists a constant $\kappa > 0$ such that

$$\langle T(x) - T(y), g(x) - g(y)\rangle \geq \kappa||x - y||^2, \quad \forall x, y \in \mathcal{H};$$

(d) (ξ, ς)-*relaxed co-coercive* if there exist constants $\xi, \varsigma > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq -\xi \|T(x) - T(y)\|^2 + \varsigma \|x - y\|^2, \quad \forall x, y \in \mathcal{H};$$

(e) γ-*Lipschitzian continuous* if there exists a constant $\gamma > 0$ such that

$$\|T(x) - T(y)\| \leq \gamma \|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

In the next definitions, several generalizations of the nonexpansive mappings which have been introduced by various authors in recent years are stated.

Definition 2.12. A nonlinear mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be:

(a) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in \mathcal{H};$$

(b) *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in \mathcal{H};$$

(c) *generalized Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L(\|x - y\| + 1), \quad \forall x, y \in \mathcal{H};$$

(d) *generalized* (L, M)-*Lipschitzian* [53] if there exist two constants $L, M > 0$ such that

$$\|Tx - Ty\| \leq L(\|x - y\| + M), \quad \forall x, y \in \mathcal{H};$$

(e) *asymptotically nonexpansive* [54] if there exists a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that, for each $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in \mathcal{H};$$

(f) *pointwise asymptotically nonexpansive* [55] if, for each integer $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \leq \alpha_n(x) \|x - y\|, \quad \forall x, y \in \mathcal{H},$$

where $\alpha_n \to 1$ pointwise on X;

(g) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that, for each $n \in \mathbb{N}$,

$$||T^n x - T^n y|| \leq L||x - y||, \quad \forall x, y \in \mathcal{H}.$$

Definition 2.13. [53] A nonlinear mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be:

(a) *nearly Lipschitzian* with respect to the sequence $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$||T^n x - T^n y|| \leq k_n(||x - y|| + a_n), \quad \forall x, y \in \mathcal{H}, \tag{2.1}$$

where $\{a_n\}$ is a fix sequence in $[0, \infty)$ with $a_n \to 0$ as $n \to \infty$.
The infimum of constants k_n in (2.1) is called *nearly Lipschitz constant*, which is denoted by $\eta(T^n)$. Notice that

$$\eta(T^n) = \sup\left\{\frac{||T^n x - T^n y||}{||x - y|| + a_n} : x, y \in \mathcal{H}, x \neq y\right\}.$$

A nearly Lipschitzian mapping T with the sequence $\{(a_n, \eta(T^n))\}$ is said to be:
(b) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$, that is,

$$||T^n x - T^n y|| \leq ||x - y|| + a_n, \quad \forall x, y \in \mathcal{H};$$

(c) *nearly asymptotically nonexpansive* if $\eta(T^n) \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) = 1$, in other words, $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 1$;
(d) *nearly uniformly L-Lipschitzian* if $\eta(T^n) \leq L$ for all $n \in \mathbb{N}$, in other words, $k_n = L$ for all $n \in \mathbb{N}$.

Remark 2.14. It should be pointed that

(1) Every nonexpansive mapping is a asymptotically nonexpansive mapping and every asymptotically non-expansive mapping is a pointwise asymptotically nonexpansive mapping. Also, the class of Lipschitzian mappings properly includes the class of pointwise asymptotically nonexpansive mappings.
(2) It is obvious that every Lipschitzian mapping is a generalized Lipschitzian mapping. Furthermore, every mapping with a bounded range is a generalized Lipschitzian mapping. It is easy to see that the class of generalized (L, M)-Lipschitzian mappings is more general that the class of generalized Lipschitzian mappings.
(3) Clearly, the class of nearly uniformly L-Lipschitzian mappings properly includes the class of generalized (L, M)-Lipschitzian mappings and that of uniformly L-Lipschitzian mappings. Note that every nearly asymptotically nonexpansive mapping is nearly uniformly L-Lipschitzian.

Now, we present some new examples to investigate relations between these mappings.

Example 2.15. Let $\mathcal{H} = \mathbb{R}$ and define a mapping $T : \mathcal{H} \to \mathcal{H}$ as follow:

$$T(x) = \begin{cases} \frac{1}{\gamma} & x \in [0, \gamma], \\ 0 & x \in (-\infty, 0) \cup (\gamma, \infty), \end{cases}$$

where $\gamma > 1$ is a constant real number. Evidently, the mapping T is discontinuous at the points $x = 0, \gamma$. Since every Lipschitzian mapping is continuous, it follows that T is not Lipschitzian. For each $n \in \mathbb{N}$, take $a_n = \frac{1}{\gamma^n}$. Then,

$$|Tx - Ty| \le |x - y| + \frac{1}{\gamma} = |x - y| + a_1, \quad \forall x, y \in \mathbb{R}.$$

Since $T^n z = \frac{1}{\gamma}$ for all $z \in \mathbb{R}$ and $n \geq 2$, it follows that for all $x, y \in \mathbb{R}$ and $n \geq 2$,

$$|T^n x - T^n y| \le |x - y| + \frac{1}{\gamma^n} = |x - y| + a_n.$$

Hence T is a nearly nonexpansive mapping with respect to the sequence $\{a_n\} = \{\frac{1}{\gamma^n}\}$.

The following example shows that the nearly uniformly L-Lipschitzian mappings are not necessarily continuous.

Example 2.16. Let $\mathcal{H} = [0, b]$, where $b \in (0, 1]$ is an arbitrary constant real number, and the self-mapping T of $\mathcal{H}$ be defined as below:

$$T(x) = \begin{cases} \gamma x & x \in [0, b), \\ 0 & x = b, \end{cases}$$

where $\gamma \in (0, 1)$ is also an arbitrary constant real number. It is plain that the mapping T is discontinuous in the point b. Hence T is not a Lipschitzian mapping. For each $n \in \mathbb{N}$, take $a_n = \gamma^{n-1}$. Then, for all $n \in \mathbb{N}$ and $x, y \in [0, b)$, we have

$$\begin{aligned} |T^n x - T^n y| &= |\gamma^n x - \gamma^n y| = \gamma^n |x - y| \le \gamma^n |x - y| + \gamma^n \\ &\le \gamma |x - y| + \gamma^n = \gamma(|x - y| + a_n). \end{aligned}$$

If $x \in [0, b)$ and $y = b$, then, for each $n \in \mathbb{N}$, we have $T^n x = \gamma^n x$ and $T^n y = 0$. Since $0 < |x - y| \le b \le 1$, it follows that, for all $n \in$ N,

$$\begin{aligned} |T^n x - T^n y| &= |\gamma^n x - 0| = \gamma^n x \le \gamma^n b \le \gamma^n < \gamma^n |x - y| + \gamma^n \\ &\le \gamma |x - y| + \gamma^n = \gamma(|x - y| + a_n). \end{aligned}$$

Hence T is a nearly uniformly γ-Lipschitzian mapping with respect to the sequence $\{a_n\} = \{\gamma^{n-1}\}$.

Obviously, every nearly nonexpansive mapping is a nearly uniformly Lipschitzian mapping. In the following example, we show that the class of nearly uniformly Lipschitzian mappings properly includes the class of nearly nonexpansive mappings.

Example 2.17. Let $\mathcal{H} = \mathbb{R}$ and the self-mapping T of $\mathcal{H}$ be defined as follow:

$$T(x) = \begin{cases} \frac{1}{2}\ x \in [0,1) \cup \{2\}, \\ 2\ x = 1, \\ 0\ x \in (-\infty, 0) \cup (1,2) \cup (2, +\infty). \end{cases}$$

Evidently, the mapping T is discontinuous in the points $x = 0, 1, 2$. Hence T is not a Lipschitzian mapping. Take for each $n \in$ N, $a_n = \frac{1}{2^n}$. Then T is not a nearly nonexpansive mapping with respect to the sequence $\{\frac{1}{2^n}\}$, because taking $x = 1$ and $y = \frac{1}{2}$, we have $Tx = 2$, $Ty = \frac{1}{2}$ and

$$|Tx - Ty| \ > \ |x - y| + \frac{1}{2} = \ |x - y| + \ a_1.$$

However,

$$|Tx - Ty| \ \leq 4\left(|x - y| + \ \frac{1}{2}\right) = 4(|x - y| + \ a_1), \quad \forall x, y \in \mathbb{R}$$

and for all $n \geq 2$,

$$|T^n x - T^n y| \ \leq 4\left(|x - y| + \ \frac{1}{2^n}\right) = 4(|x - y| + \ a_n), \quad \forall x, y \in \mathbb{R},$$

since $T^n z = \frac{1}{2}$ for all $z \in \mathbb{R}$ and $n \geq 2$. Hence, for each $L \geq 4$, T is a nearly uniformly L-Lipschitzian mapping with respect to the sequence $\{\frac{1}{2^n}\}$.

It is clear that every uniformly L-Lipschitzian mapping is a nearly uniformly L-Lipschitzian mapping. In the next example, we show that the class nearly uniformly L-Lipschitzian mappings properly includes the class of uniformly L-Lipschitzian mappings.

Example 2.18. Let $\mathcal{H} = \mathbb{R}$ and let the self-mapping T of $\mathcal{H}$ be defined the same as in Example 2.17. Then T is not a uniformly 4-Lipschitzian mapping. In fact, if $x = 1$ and $y \in (1, \frac{3}{2})$, then we have $|Tx - Ty| > 4|x - y|$ because $0 < \ |x - y| \ < \frac{1}{2}$. But, in view of Example 2.17, T is a nearly uniformly 4-Lipschitzian mapping.

The following example shows that the class of generalized Lipschitzian mappings properly includes the class of Lipschitzian mappings and that of mappings with bounded range.

Example 2.19. [26] Let $\mathcal{H} = \mathbb{R}$ and a mapping $T : \mathcal{H} \to \mathcal{H}$ be defined by

$$T(x) = \begin{cases} x - 1 & x \in (-\infty, -1), \\ x - \sqrt{1 - (x+1)^2} & x \in [-1, 0), \\ x + \sqrt{1 - (x-1)^2} & x \in [0, 1], \\ x + 1 & x \in (1, \infty). \end{cases}$$

Then T is a generalized Lipschitzian mapping which is not Lipschitzian and whose range is not bounded.

3 Extended general regularized nonconvex variational inequality

In this section, we introduce a new problem of extended general nonlinear regularized nonconvex variational inequality and some special cases of the problem in Hilbert spaces and investigate their relations.

Let $T, f, g : \mathcal{H} \to \mathcal{H}$ be three nonlinear single-valued operators such that $K_r \subseteq f(\mathcal{H})$. We consider the problem of finding $u \in \mathcal{H}$ such that $g(u) \in K_r$ and

$$\langle \rho T(u) + g(u) - f(u), f(v) - g(u) \rangle + \frac{1}{2r} \|f(v) - g(u)\|^2 \geq 0, \quad \forall v \in \mathcal{H} : f(v) \in K_r, \tag{3.1}$$

where $\rho > 0$ is a constant. The problem (3.1) is called the *extended general nonlinear regularized nonconvex variational inequality* involving three different nonlinear operators (EGNRNVID).

Proposition 3.1. *If K_r is a uniformly prox-regular set, then the problem* (3.1) *is equivalent to that of finding $u \in \mathcal{H}$ such that $g(u) \in K_r$ and*

$$0 \in \rho T(u) + g(u) - f(u) + N_{K_r}^P(g(u)), \tag{3.2}$$

where $N_{K_r}^P(s)$ denotes the P-normal cone of K_r at s in the sense of nonconvex analysis.

Proof. Let $u \in \mathcal{H}$ with $g(u) \in K_r$ be a solution of the problem (3.1). If $\rho T(u) + g(u) - f(u) = 0$, because the vector zero always belongs to any normal cone, we have $0 \in \rho T(u) + g(u) - f(u) + N_{K_r}^P(g(u))$. If $\rho T(u) + g(u) - f(u) \neq 0$, then for all $v \in \mathcal{H}$ with $f(v) \in K_r$, one has

$$\langle -(\rho T(u) + g(u) - f(u)), f(v) - g(u) \rangle \leq \frac{1}{2r} \|f(v) - g(u)\|^2.$$

Now, by using Lemma 2.3 conclude that $-(\rho T(u) + g(u) - f(u)) \in N_{K_r}^P(g(u))$ and so

$$0 \in \rho T(u) + g(u) - f(u) + N_{K_r}^P(g(u)).$$

Conversely, if $u \in \mathcal{H}$ with $g(u) \in K_r$ is a solution of the problem (3.2), then Definition 2.8 guarantees that $u \in \mathcal{H}$ with $g(u) \in K_r$ is a solution of the problem (3.1). This completes the proof.

The problem (3.2) is called the *extended general nonconvex variational inclusion* associated with EGNRNVID problem.

Some special cases of the problem (3.1) are as follows:

(1) If $g \equiv I$ (: the identity operator), then the problem (3.1) collapses to the following problem: Find $u \in K_r$ such that

$$\langle \rho T(u) + u - f(u), f(v) - u \rangle + \frac{1}{2r} \|f(v) - u\|^2 \geq 0, \quad \forall v \in \mathcal{H} : f(v) \in K_r, \tag{3.3}$$

which is a new problem of general nonlinear regularized nonconvex variational inequality involving two nonlinear operators (GNRNVID).

(2) If $f = g$, then the problem (3.1) reduces to the following problem: Find $u \in \mathcal{H}$ such that $g(u) \in K_r$ and

$$\langle \rho T(u), g(v) - g(u) \rangle + \frac{1}{2r}||g(v) - g(u)||^2 \ \geq 0, \quad \forall v \in \mathcal{H} : g(v) \in K_r, \tag{3.4}$$

which is also a new problem of general nonlinear regularized nonconvex variational inequality involving two nonlinear operators (GNRNVID).

(3) If $g \equiv I$, then the problem (3.4) collapses to the following problem: Find $u \in K_r$ such that

$$\langle \rho T(u), v - u \rangle + \frac{1}{2r}||v - u||^2 \ \geq 0, \quad \forall v \in K_r, \tag{3.5}$$

which is a new problem of nonlinear regularized nonconvex variational inequality (NRNVI).

(4) If $r = \infty$, i.e., $K_r = K$, the convex set in $\mathcal{H}$, then the problem (3.1) changes into that of finding $u \in \mathcal{H}$ such that $g(u) \in K$ and

$$\langle \rho T(u) + g(u) - f(u), f(v) - g(u) \rangle \geq 0, \quad \forall v \in \mathcal{H} : f(v) \in K. \tag{3.6}$$

The inequality of type (3.6) is introduced and studied by Noor [33,39].

(5) If $r = \infty$, then the problem (3.3) is equivalent to the problem: Find $u \lfloor K$ such that

$$\langle \rho T(u) + u - f(u), f(v) - u \rangle \geq 0, \quad \forall v \in \mathcal{H} : f(v) \in K. \tag{3.7}$$

The problem (3.7) is introduced and studied by Noor [34].

(6) If $r = \infty$, then the problem (3.4) reduces to the following problem: Find $u \in \mathcal{H}$ such that $g(u) \in K$ and

$$\langle T(u), g(v) - g(u) \rangle \geq 0, \quad \forall v \in \mathcal{H} : g(v) \in K, \tag{3.8}$$

which is known as the *general nonlinear variational inequality* introduced and studied by Noor [37] in 1988.

(7) If $r = \infty$, then the problem (3.5) changes into the problem: Find $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K. \tag{3.9}$$

The inequality of type (3.9) is called *variational inequality*, which was introduced and studied by Stampacchia [1] in 1964.

Now, we prove the existence and uniqueness theorem for solution of the problem of extended general nonlinear regularized nonconvex variational inequality (3.1). For this end, we need to the following lemma in which by using the projection operator technique, we verify the equivalence between the problem (3.1) and the fixed point problem.

Lemma 3.2. *Let T, f, g and* $\rho > 0$ *be the same as in the problem* (3.1). *Then* $u \in \mathcal{H}$ *with* $g(u) \in K_r$ *is a solution of the problem* (3.1) *if and only if*

$$g(u) = P_{K_r}(f(u) - \rho T(u)), \tag{3.10}$$

where P_{K_r} is the projection of $\mathcal{H}$ onto K_r.

Proof. Let $u \in \mathcal{H}$ with $g(u) \in K_r$ be a solution of the problem (3.1). Then, by using Proposition 3.1, we have

$$\begin{aligned} & 0 \in \rho T(u) + g(u) - f(u) + N^P_{K_r}(g(u)) \\ \Leftrightarrow \quad & f(u) - \rho T(u) \in g(u) + N^P_{K_r}(g(u)) \\ \Leftrightarrow \quad & f(u) - \rho T(u) \in (I + N^P_{K_r})(g(u)) \\ \Leftrightarrow \quad & g(u) = P_{K_r}(f(u) - \rho T(u)), \end{aligned}$$

where I is identity operator and we have used the well-known fact that $P_{K_r} = (I + N^P_{K_r})^{-1}$.

Theorem 3.3. *Let T, f, g and ρ be the same as in the problem* (3.1) *such that*

(a) *T is κ-strongly monotone with respect to f and σ-Lipschitz continuous;*
(b) *g is τ-strongly monotone and ι-Lipschitz continuous;*
(c) *f is ϖ-Lipschitz continuous.*

If the constant $\rho > 0$ satisfies the following condition:

$$\begin{cases} |\rho - \frac{\kappa}{\sigma^2}| < \frac{\sqrt{r^2\kappa^2 - \sigma^2(r^2\varpi^2 - (r-r')^2(1-\mu)^2)}}{r\sigma^2}, \\ r\kappa > \sigma\sqrt{r^2\varpi^2 - (r-r')^2(1-\mu)^2}, \\ r\varpi > (r - r')(1-\mu), \quad \mu = \sqrt{1 - (2\tau - \iota^2)} < 1, \\ 2\tau < 1 + \iota^2, \end{cases} \tag{3.11}$$

where $r' \in (0, r)$, then the problem (3.1) *admits a unique solution.*

Proof. Define the mapping $\phi : \mathcal{H} \to \mathcal{H}$ by

$$\phi(x) = x - g(x) + P_{K_r}(f(x) - \rho T(x)), \quad \forall x \in \mathcal{H} : g(x) \in K_r. \tag{3.12}$$

Now, we establish that φ is a contraction mapping. Let $x, \hat{x} \in \mathcal{H}$ with $g(x)$, $g(\hat{x}) \in K_r$ be given. It follows from Proposition 2.10 that

$$\begin{aligned} \|\phi(x) - \phi(\hat{x})\| \leq & \|x - \hat{x} - (g(x) - g(\hat{x}))\| + \|P_{K_r}(f(x) - \rho T(x)) - P_{K_r}(f(\hat{x}) - \rho T(\hat{x}))\| \\ \leq & \|x - \hat{x} - (g(x) - g(\hat{x}))\| + \frac{r}{r - r'}\|f(x) - f(\hat{x}) - \rho(T(x) - T(\hat{x}))\|. \end{aligned} \tag{3.13}$$

By using τ-strongly monotonicity and ι-Lipschitzian continuity of g, we have

$$\begin{aligned} \|x - \hat{x} - (g(x) - g(\hat{x}))\|^2 = & \|x - \hat{x}\|^2 - 2\langle g(x) - g(\hat{x}), x - \hat{x}\rangle + \|g(x) - g(\hat{x})\|^2 \\ \leq & (1 - 2\tau + \iota^2)\|x - \hat{x}\|^2. \end{aligned} \tag{3.14}$$

Since T is κ-strongly monotone with respect to f and σ-Lipschitzian continuous, and f is ϖ-Lipschitzian continuous, we gain

$$\begin{aligned} & \|f(x) - f(\hat{x}) - \rho(T(x) - T(\hat{x}))\|^2 \\ = & \|f(x) - f(\hat{x})\|^2 - 2\rho\langle T(x) - T(\hat{x}), f(x) - f(\hat{x})\rangle + \rho^2\|T(x) - T(\hat{x})\|^2 \\ \leq & (\varpi^2 - 2\rho\kappa + \rho^2\sigma^2)\|x - \hat{x}\|^2. \end{aligned} \tag{3.15}$$

Substituting (3.14) and (3.15) for (3.13), we obtain

$$||\phi(x) - \phi(\hat{x})|| \le \gamma ||x - \hat{x}|| \tag{3.16}$$

where

$$\gamma = \sqrt{1 - 2\tau + \iota^2} + \frac{r}{r - r'}\sqrt{\varpi^2 - 2\rho\kappa + \rho^2\sigma^2}. \tag{3.17}$$

In view of the condition (3.11), we note that $0 \le \gamma < 1$ and so from (3.16) conclude that the mapping φ is contraction. According to Banach fixed point theorem, φ has a unique fixed point in $\mathcal{H}$, that is, there exists a unique point $u \in \mathcal{H}$ with $g(u) \in K_r$ such that $\varphi(u) = u$. It follows from (3.12) that $g(u) = P_{K_r}(f(u) - \rho T(u))$. Now, Lemma 3.2 guarantees that $u \in \mathcal{H}$ with $g(u) \in K_r$ is a solution of the problem (3.1). This completes the proof.

As in the proof of Theorem 3.3, one can prove the existence and uniqueness theorem for solution of the problems (3.3)-(3.5) and we omit their proofs.

Theorem 3.4. *Assume that T, f and ρ are the same as in the problem* (3.3) *such that*

(a) *T is κ-strongly monotone with respect to f and σ-Lipschitz continuous;*
(b) *f is ϖ-Lipschitz continuous.*

If the constant ρ >0 satisfies the following condition:

$$\begin{cases} |\rho - \frac{\kappa}{\sigma^2}| < \frac{\sqrt{r^2\kappa^2 - \sigma^2(r^2\varpi^2 - (r-r')^2)}}{r\sigma^2}, \\ r\kappa > \sigma\sqrt{r^2\varpi^2 - (r - r')^2}, \\ r\varpi > (r - r'), \end{cases} \tag{3.18}$$

where r' ∈ (0, *r*), *then the problem* (3.3) *admits a unique solution.*

Theorem 3.5. *Let T, g and ρ be the same as in the problem* (3.4) *such that*

(a) *T is κ-strongly monotone with respect to f and σ-Lipschitz continuous;*
(b) *g is τ-strongly monotone and ι-Lipschitz continuous.*

If the constant ρ >0 satisfies the following condition:

$$\begin{cases} |\rho - \frac{\kappa}{\sigma^2}| < \frac{\sqrt{r^2\kappa^2 - \sigma^2(r^2\iota^2 - (r-r')^2(1-\mu)^2)}}{r\sigma^2}, \\ r\kappa > \sigma\sqrt{r^2\iota^2 - (r - r')^2(1-\mu)^2}, \\ r\iota > (r - r')(1 - \mu), \quad \mu = \sqrt{1 - (2\tau - \iota^2)} < 1, \\ 2\tau < 1 + \iota^2, \end{cases} \tag{3.19}$$

where r' ∈ (0, *r*), *then the problem* (3.4) *admits a unique solution.*

Theorem 3.6. *Suppose that T and ρ are the same as in the problem* (3.5) *such that T is κ-strongly monotone and σ-Lipschitz continuous. If the constant ρ >0 satisfies the following condition:*

$$|\rho - \frac{\kappa}{\sigma^2}| < \frac{\sqrt{r^2\kappa^2 - \sigma^2 r'(r' - 2r)}}{r\sigma^2}, \tag{3.20}$$

where r' ∈ (0, r), then the problem (3.5) admits a unique solution.

4 Nearly uniformly Lipschitzian mappings and finite step projection iterative algorithms

In this section, applying a nearly uniformly Lipschitzian mapping S and by using the fixed point formulation (3.10), we suggest and analyze some new p-step projection iterative algorithms with mixed errors for finding an element of set of the fixed points of nearly uniformly Lipschitzian mapping S which is unique solution of the problem of extended general nonlinear regularized nonconvex variational inequality (3.1).

Let $S : K_r \to K_r$ be a nearly uniformly Lipschitzian mapping. We denote the set of all the fixed points of S by Fix(S) and the set of all the solutions of the problem (3.1) by EGNRNVID(K_r, T, f, g). We now characterize the problem. If $u \in$ Fix(S) ∩ EGNRNVID(K_r, T, f, g), then it follows from Lemma 3.2 that, for each $n \geq 0$,

$$u = S^n u = u - g(u) + P_{K_r}(f(u) - \rho T(u)) = S^n\{u - g(u) + P_{K_r}(f(u) - \rho T(u))\}. \quad (4.1)$$

The fixed point formulation (4.1) enables us to define the following p-step projection iterative algorithms with mixed errors for finding a common element of two different sets of solutions of the fixed points of the nearly uniformly Lipschitzian mappings and the extended general nonlinear regularized nonconvex variational inequalities (3.1).

Algorithm 4.1. Let T, f, g and ρ be the same as in the problem (3.1). For arbitrary chosen initial point $x_0 \in K_r$, compute the iterative sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative process

$$\begin{aligned} x_{n+1} &= (1 - \alpha_{n,1} - \beta_{n,1})x_n + \alpha_{n,1}(S^n\Psi(y_{n,1}) + e_{n,1}) + \beta_{n,1}l_{n,1} + r_{n,1}, \\ y_{n,i} &= (1 - \alpha_{n,i+1} - \beta_{n,i+1})x_n + \alpha_{n,i+1}(S^n\Psi(y_{n,i+1}) + e_{n,i+1}) + \beta_{n,i+1}l_{n,i+1} + r_{n,i+1}, \\ &\cdots \\ y_{n,p-1} &= (1 - \alpha_{n,p} - \beta_{n,p})x_n + \alpha_{n,p}(S^n\Psi(x_n) + e_{n,p}) + \beta_{n,p}l_{n,p} + r_{n,p}, \\ & i = 1, 2, \ldots, p-2, \end{aligned} \quad (4.2)$$

.

where

$$\begin{cases} \Psi(y_{n,i}) = y_{n,i} - g(y_{n,i}) + P_{K_r}(f(y_{n,i}) - \rho T(y_{n,i})), \\ \Psi(x_n) = x_n - g(x_n) + P_{K_r}(f(x_n) - \rho T(x_n)), \\ i = 1, 2, \ldots, p-2, \end{cases}$$

$S : K_r \to K_r$ is a nearly uniformly Lipschitzian mapping, $\{\alpha_{n,i}\}_{n=0}^{\infty}$, $\{\beta_{n,i}\}_{n=0}^{\infty} (i = 1, 2, \ldots, p)$ are $2p$ sequences in interval [0,1] such that $\sum_{n=0}^{\infty} \beta_{n,i} < \infty$, $\sum_{n=0}^{\infty} \beta_{n,i} < \infty$, and $\{e_{n,i}\}_{n=0}^{\infty}$, $\{l_{n,i}\}_{n=0}^{\infty}$, $\{r_{n,i}\}_{n=0}^{\infty} (i = 1, 2, \ldots, p)$ are $3p$ sequences in $\mathcal{H}$ to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions: $\{l_{n,i}\}_{n=0}^{\infty} (i = 1, 2, \ldots, p)$ are p bounded sequences in $\mathcal{H}$ and $\sum_{n=0}^{\infty} \beta_{n,i} < \infty$, $\{r_{n,i}\}_{n=0}^{\infty}$ are $2p$ sequences in $\mathcal{H}$ such that

$$\begin{cases} e_{n,i} = e'_{n,i} + e''_{n,i}, \quad n \geq 0, \quad i = 1, 2, \ldots, p, \\ \lim\limits_{n\to\infty} \|e'_{n,i}\| = 0, \quad i = 1, 2, \ldots, p, \\ \sum\limits_{n=0}^{\infty} \|e''_{n,i}\| < \infty, \quad \sum\limits_{n=0}^{\infty} \|r_{n,i}\| < \infty, \quad i = 1, 2, \ldots, p. \end{cases} \tag{4.3}$$

Algorithm 4.2. Assume that T, f and ρ are the same as in the problem (3.3). For arbitrary chosen initial point $x_0 \in K_r$, compute the iterative sequence $\{x_n\}_{n=0}^{\infty}$ by the iterative process

$$\begin{cases} x_{n+1} = (1 - \alpha_{n,1} - \beta_{n,1})x_n + \alpha_{n,1}(S^n P_{K_r}(f(y_{n,1}) - \rho T(y_{n,1})) + e_{n,1}) + \beta_{n,1} l_{n,1} + r_{n,1}, \\ y_{n,i} = (1 - \alpha_{n,i+1} - \beta_{n,i+1})x_n + \alpha_{n,i+1}(S^n P_{K_r}(f(y_{n,i+1}) - \rho T(y_{n,i+1})) + e_{n,i+1}) + \beta_{n,i+1} l_{n,i+1} + r_{n,i+1}, \\ \cdots \\ y_{n,p-1} = (1 - \alpha_{n,p} - \beta_{n,p})x_n + \alpha_{n,p}(S^n P_{K_r}(f(x_n) - \rho T(x_n)) + e_{n,p}) + \beta_{n,p} l_{n,p} + r_{n,p}, \\ i = 1, 2, \ldots, p - 2, \end{cases}$$

where S, $\{\alpha_{n,i}\}_{n=0}^{\infty}$, $\{\beta_{n,i}\}_{n=0}^{\infty}$, $\{e_{n,i}\}_{n=0}^{\infty}$, $\{l_{n,i}\}_{n=0}^{\infty}$, $\{r_{n,i}\}_{n=0}^{\infty}(i = 1, 2, \ldots, p)$ are the same as in Algorithm 4.1.

Algorithm 4.3. Let T, g and ρ be the same as in the problem (3.4). For arbitrary chosen initial point $x_0 \in K_r$, compute the iterative sequence $\{x_n\}_{n=0}^{\infty}$ as follows:

$$\begin{cases} x_{n+1} = (1 - \alpha_{n,1} - \beta_{n,1})x_n + \alpha_{n,1}(S^n \Phi(y_{n,1}) + e_{n,1}) + \beta_{n,1} l_{n,1} + r_{n,1}, \\ y_{n,i} = (1 - \alpha_{n,i+1} - \beta_{n,i+1})x_n + \alpha_{n,i+1}(S^n \Phi(y_{n,i+1}) + e_{n,i+1}) + \beta_{n,i+1} l_{n,i+1} + r_{n,i+1}, \\ \cdots \\ y_{n,p-1} = (1 - \alpha_{n,p} - \beta_{n,p})x_n + \alpha_{n,p}(S^n \Phi(x_n) + e_{n,p}) + \beta_{n,p} l_{n,p} + r_{n,p}, \\ i = 1, 2, \ldots, p - 2, \end{cases}$$

where

$$\begin{cases} \Phi(y_{n,i}) = y_{n,i} - g(y_{n,i}) + P_{K_r}(g(y_{n,i}) - \rho T(y_{n,i})), \\ \Phi(x_n) = x_n - g(x_n) + P_{K_r}(g(x_n) - \rho T(x_n)), \\ i = 1, 2, \ldots, p - 2, \end{cases}$$

and S, $\{\alpha_{n,i}\}_{n=0}^{\infty}$, $\{\beta_{n,i}\}_{n=0}^{\infty}$, $\{e_{n,i}\}_{n=0}^{\infty}$, $\{l_{n,i}\}_{n=0}^{\infty}$, $\{r_{n,i}\}_{n=0}^{\infty}(i = 1, 2, \ldots, p)$ are the same as in Algorithm 4.1.

Algorithm 4.4. Let T and ρ be the same as in the problem (3.5). For arbitrary chosen initial point $x_0 \in K_r$, compute the iterative sequence $\{x_n\}_{n=0}^{\infty}$ by using

$$\begin{cases} x_{n+1} = (1 - \alpha_{n,1} - \beta_{n,1})x_n + \alpha_{n,1}(S^n P_{K_r}(y_{n,1} - \rho T(y_{n,1})) + e_{n,1}) + \beta_{n,1} l_{n,1} + r_{n,1}, \\ y_{n,i} = (1 - \alpha_{n,i+1} - \beta_{n,i+1})x_n + \alpha_{n,i+1}(S^n P_{K_r}(y_{n,i+1} - \rho T(y_{n,i+1})) + e_{n,i+1}) + \beta_{n,i+1} l_{n,i+1} + r_{n,i+1}, \\ \cdots \\ y_{n,p-1} = (1 - \alpha_{n,p} - \beta_{n,p})x_n + \alpha_{n,p}(S^n P_{K_r}(x_n - \rho T(x_n)) + e_{n,p}) + \beta_{n,p} l_{n,p} + r_{n,p}, \\ i = 1, 2, \ldots, p - 2, \end{cases}$$

where S, $\{\alpha_{n,i}\}_{n=0}^{\infty}$, $\{\beta_{n,i}\}_{n=0}^{\infty}$, $\{e_{n,i}\}_{n=0}^{\infty}$, $\{l_{n,i}\}_{n=0}^{\infty}$, $\{r_{n,i}\}_{n=0}^{\infty}(i = 1, 2, \ldots, p)$ are the same as in Algorithm 4.1.

Remark 4.5. It should be pointed out that

(1) If $e_{n,i} = r_{n,i} = 0$, for all $n \geq 0$ and $i = 1, 2,..., p$, then Algorithms 4.1-4.4 change into the perturbed iterative process with mean errors.

(2) When $e_{n,i} = l_{n,i} = r_{n,i} = 0$, for all $n \geq 0$ and $i = 1, 2,..., p$, then Algorithms 4.1-4.4 reduce to the perturbed iterative process without error.

Remark 4.6. Algorithms 2.1-2.6 in [38] and Algorithm 2.1 in [44] are special cases of Algorithms 4.1-4.4. In brief, for a suitable and appropriate choice of the operators T, f, g, the constant ρ, and the sequences $\{\alpha_{n,i}\}_{n=0}^{\infty}$, $\{\beta_{n,i}\}_{n=0}^{\infty}$, $\{e_{n,i}\}_{n=0}^{\infty}$, $\{l_{n,i}\}_{n=0}^{\infty}$, $\{r_{n,i}\}_{n=0}^{\infty}(i = 1, 2, \ldots, p)$, one can obtain a number of new and previously known iterative schemes for solving the problems (3.1) and (3.3)-(3.5) and related problems. This clearly shows that Algorithms 4.1-4.4 are quite general and unifying.

Now, we discuss the convergence analysis of the suggested iterative Algorithms 4.1-4.4 under some suitable conditions. For this end, we need to the following lemma:

Lemma 4.7. *Let* -a_n}, -b_n} *and* -c_n} *be three nonnegative real sequences satisfying the following condition: there exists a natural number* n_0 *such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0,$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $lim_{n\to\infty}\, b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $\lim_{n\to 0} a_n = 0$.

Proof. The proof directly follows from Lemma 2 in Liu [32].

Theorem 4.8. *Let* T, f, g *and* ρ *be the same as in Theorem 3.3 such that the conditions* (a)-(c) *and* (3.11) *in Theorem 3.3 hold. Assume that* $S : K_r \to K_r$ *is a nearly uniformly L-Lipschitzian mapping with the sequence* $\{b_n\}_{n=0}^{\infty}$*such that* Fix(S) ∩ EGNRNVID (K_r, T, f, g) ≠ ∅. *Further, let* $L\gamma < 1$, *where* γ *is the same as in* (3.17). *If there exists a constant* $\alpha > 0$ *such that* $\prod_{i=1}^{p} \alpha_{n,i} > \alpha$ *for each* $n \geq 0$, *then the iterative sequence* $\{x_n\}_{n=0}^{\infty}$ *generated by Algorithm 4.1 converges strongly to the only element of* Fix(S) ∩ EGNRNVID(K_r. T, f, g).

Proof. According to Theorem 3.3, the problem (3.1) has a unique solution $x^* \in \mathcal{H}$ with $g(x^*) \in K_r$. Hence, in view of Lemma 3.2, $g(x^*) = P_{K_r}(f(x^*) - \rho T(x^*))$. Since EGNRNVID($K_r$, T, f, g) is a singleton set, it follows from Fix(S) ∩ EGNRNVID(K_r, T, f, g) ≠ ∅ that $x^* \in$ Fix(S). Accordingly, for each $n \geq 0$ and $i \in \{1, 2,..., p\}$, we can write

$$x^* = (1 - \alpha_{n,i} - \beta_{n,i})x^* + \alpha_{n,i}S^n\{x^* - g(x^*) + P_{K_r}(f(x^*) - \rho T(x^*))\} + \beta_{n,i}x^*, \tag{4.4}$$

where the sequences $\{\alpha_{n,i}\}_{n=0}^{\infty}$ and $\{\beta_{n,i}\}_{n=0}^{\infty}(i = 1, 2, \ldots, p)$ are the same as in Algorithm 4.1. Let $\Gamma = \sup_{n\geq 0}\{||l_{n,i} - x^*||: i = 1, 2,. .., p\}$. It follows from (4.2), (4.4), Proposition 2.10 and the assumptions that

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq (1 - \alpha_{n,1} - \beta_{n,1})||x_n - x^*|| + \alpha_{n,1}||S^n\{y_{n,1} - g(y_{n,1}) + P_{K_r}(f(y_{n,1}) - \rho T(y_{n,1}))\} \\ &\quad - S^n\{x^* - g(x^*) + P_{K_r}(f(x^*) - \rho T(x^*))\}|| + \beta_{n,1}||l_{n,1} - x^*|| + \alpha_{n,1}||e_{n,1}|| + ||r_{n,1}|| \\ &\leq (1 - \alpha_{n,1} - \beta_{n,1})||x_n - x^*|| + \alpha_{n,1}L(||y_{n,1} - x^* - (g(y_{n,1}) - g(x^*))|| \\ &\quad + \frac{r}{r - r'}||f(y_{n,1}) - f(x^*) - \rho(T(y_{n,1}) - T(x^*))|| + b_n) \\ &\quad + \alpha_{n,1}||e'_{n,1}|| + ||e''_{n,1}|| + ||r_{n,1}|| + \beta_{n,1}\Gamma. \end{aligned} \tag{4.5}$$

Since T is κ-strongly monotone with respect to f and σ-Lipschitz continuous, g is τ-strongly monotone and ι-Lipschitz continuous, in similar way to the proofs (3.14) and (3.15), we can prove that

$$||y_{n,1} - x^* - (g(y_{n,1}) - g(x^*))|| \leq \sqrt{1 - 2\tau + \iota^2}||y_{n,1} - x^*|| \tag{4.6}$$

and

$$||f(y_{n,1}) - f(x^*) - \rho(T(y_{n,1}) - T(x^*))|| \leq \sqrt{\varpi^2 - 2\rho\kappa + \rho^2\sigma^2}||y_{n,1} - x^*||. \tag{4.7}$$

Substituting (4.6) and (4.7) for (4.5), we obtain

$$\begin{aligned}||x_{n+1}-x^*|| \ &\le (1-\alpha_{n,1}-\beta_{n,1})||x_n-x^*|| + \ \alpha_{n,1}L\gamma||y_{n,1}-x^*|| \\ &+\alpha_{n,1}Lb_n+\alpha_{n,1}||e'_{n,1}||+||e''_{n,1}||+||r_{n,1}||+\beta_{n,1}\Gamma.\end{aligned} \tag{4.8}$$

Like in the proofs of (4.5)-(4.8), we can establish that, for each $i \in \{1, 2,..., p - 2\}$,

$$\begin{aligned}||y_{n,i}-x^*|| &\le (1-\alpha_{n,i+1}-\beta_{n,i+1})||x_n-x^*|| + \ \alpha_{n,i+1}L\gamma||y_{n,i+1}-x^*|| \\ &+\alpha_{n,i+1}Lb_n+\alpha_{n,i+1}||e'_{n,i+1}||+||e''_{n,i+1}||+||r_{n,i+1}||+\beta_{n,i+1}\Gamma\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}||y_{n,p-1}-x^*|| &\le (1-\alpha_{n,p}-\beta_{n,p})||x_n-x^*|| + \ \alpha_{n,p}L\gamma||x_n-x^*|| \\ &+\alpha_{n,p}Lb_n+\alpha_{n,p}||e'_{n,p}||+||e''_{n,p}||+||r_{n,p}||+\beta_{n,p}\Gamma.\end{aligned} \tag{4.10}$$

By using (4.9) and (4.10), we get

$$\begin{aligned}||y_{n,p-2}-x^*|| \ &\le (1-\alpha_{n,p-1}-\beta_{n,p-1})||x_n-x^*|| + \ \alpha_{n,p-1}L\gamma||y_{n,p-1}-x^*|| \\ &\quad +\alpha_{n,p-1}Lb_n+\alpha_{n,p-1}||e'_{n,p-1}||+||e''_{n,p-1}||+||r_{n,p-1}||+\beta_{n,p-1}\Gamma \\ &\le (1-\alpha_{n,p-1}-\beta_{n,p-1})||x_n-x^*|| + \ \alpha_{n,p-1}L\gamma[(1-\alpha_{n,p}-\beta_{n,p})||x_n-x^*|| \\ &\quad +\alpha_{n,p}L\gamma||x_n-x^*||+\alpha_{n,p}Lb_n+\alpha_{n,p}||e'_{n,p}||+||e''_{n,p}||+||r_{n,p}||+ \ \beta_{n,p}\Gamma] \\ &\quad +\alpha_{n,p-1}Lb_n+\alpha_{n,p-1}||e'_{n,p-1}||+||e''_{n,p-1}||+||r_{n,p-1}||+\beta_{n,p-1}\Gamma \\ &= (1-\alpha_{n,p-1}-\beta_{n,p-1}+\alpha_{n,p-1}(1-\alpha_{n,p}-\beta_{n,p})L\gamma+\alpha_{n,p-1}\alpha_{n,p}L^2\gamma^2)||x_n-x^*|| \\ &\quad +(\alpha_{n,p-1}L+\alpha_{n,p-1}\alpha_{n,p}L^2\gamma)b_n+\alpha_{n,p-1}||e'_{n,p-1}||+ \ \alpha_{n,p-1}\alpha_{n,p}L\gamma||e'_{n,p}|| \\ &\quad +||e''_{n,p-1}||+ \ \alpha_{n,p-1}L\gamma||e''_{n,p}||+||r_{n,p-1}||+ \ \alpha_{n,p-1}L\gamma||r_{n,p}|| \\ &\quad +(\beta_{n,p-1}+\alpha_{n,p-1}\beta_{n,p}L\gamma)\Gamma.\end{aligned} \tag{4.11}$$

As in the proof of (4.11), applying (4.9) and (4.11), we have

$$\begin{aligned}||y_{n,p-3}-x^*|| \ &\le \big(1-\alpha_{n,p-2}-\beta_{n,p-2}+\alpha_{n,p-2}(1-\alpha_{n,p-1}-\beta_{n,p-1})L\gamma \\ &\quad +\alpha_{n,p-2}\alpha_{n,p-1}(1-\alpha_{n,p}-\beta_{n,p})L^2\gamma^2+\alpha_{n,p-2}\alpha_{n,p-1}\alpha_{n,p}L^3\gamma^3\big)\,||x_n-x^*|| \\ &\quad +\alpha_{n,p-2}||e'_{n,p-2}||+ \ \alpha_{n,p-2}\alpha_{n,p-1}L\gamma||e'_{n,p-1}||+ \ \alpha_{n,p-2}\alpha_{n,p-1}\alpha_{n,p}L^2\gamma^2||e'_{n,p}|| \\ &\quad +(\alpha_{n,p-2}L+\alpha_{n,p-2}\alpha_{n,p-1}L^2\gamma+\alpha_{n,p-2}\alpha_{n,p-1}\alpha_{n,p}L^3\gamma^2)b_n \\ &\quad +||e''_{n,p-2}||+ \ \alpha_{n,p-2}L\gamma||e''_{n,p-1}||+ \ \alpha_{n,p-2}\alpha_{n,p-1}L^2\gamma^2||e''_{n,p}|| \\ &\quad +||r_{n,p-2}||+ \ \alpha_{n,p-2}L\gamma||r_{n,p-1}||+ \ \alpha_{n,p-2}\alpha_{n,p-1}L^2\gamma^2||r_{n,p}|| \\ &\quad +(\beta_{n,p-2}+\alpha_{n,p-2}\beta_{n,p-1}L\gamma+\alpha_{n,p-2}\alpha_{n,p-1}\beta_{n,p}L^2\gamma^2)\Gamma.\end{aligned} \tag{4.12}$$

Continuing this procedure in (4.10)-(4.12), we gain

$$\begin{aligned}||y_{n,1}-x^*|| &\le \Big(1-\alpha_{n,2}-\beta_{n,2}+\alpha_{n,2}(1-\alpha_{n,3}-\beta_{n,3})L\gamma+\alpha_{n,2}\alpha_{n,3}(1-\alpha_{n,4}-\beta_{n,4})L^2\gamma^2 \\ &\quad +\cdots+\prod_{i=2}^{p-1}\alpha_{n,i}(1-\alpha_{n,p}-\beta_{n,p})L^{p-2}\gamma^{p-2}+\prod_{i=2}^{p}\alpha_{n,i}L^{p-1}\gamma^{p-1}\Big)\,||x_n-x^*|| \\ &\quad +\left(\alpha_{n,2}L+\alpha_{n,2}\alpha_{n,3}L^2\gamma+\alpha_{n,2}\alpha_{n,3}\alpha_{n,4}L^3\gamma^2+\cdots+\prod_{i=2}^{p}\alpha_{n,i}L^{p-1}\gamma^{p-2}\right)b_n \\ &\quad +\alpha_{n,2}||e'_{n,2}||+ \ \alpha_{n,2}\alpha_{n,3}L\gamma||e'_{n,3}||+\cdots+\prod_{i=2}^{p}\alpha_{n,i}L^{p-2}\gamma^{p-2}||e'_{n,p}|| \\ &\quad +||e''_{n,2}||+ \ \alpha_{n,2}L\gamma||e''_{n,3}||+ \ \alpha_{n,2}\alpha_{n,3}L^2\gamma^2||e''_{n,4}||+\cdots+\prod_{i=2}^{p-1}\alpha_{n,i}L^{p-2}\gamma^{p-2}||e''_{n,p}|| \\ &\quad +||r_{n,2}||+ \ \alpha_{n,2}L\gamma||r_{n,3}||+ \ \alpha_{n,2}\alpha_{n,3}L^2\gamma^2||r_{n,4}||+\cdots+\prod_{i=2}^{p-1}\alpha_{n,i}L^{p-2}\gamma^{p-2}||r_{n,p}|| \\ &\quad +\left(\beta_{n,2}+\alpha_{n,2}\beta_{n,3}L\gamma+\alpha_{n,2}\alpha_{n,3}\beta_{n,4}L^2\gamma^2+\cdots+\prod_{i=2}^{p-1}\alpha_{n,i}\beta_{n,p}L^{p-2}\gamma^{p-2}\right)\Gamma.\end{aligned} \tag{4.13}$$

It follows from (4.8) and (4.13) that

$$\begin{aligned}
\|x_{n+1}-x^*\| &\le (1-\alpha_{n,1}-\beta_{n,1})\|x_n-x^*\| + \alpha_{n,1}L\gamma\|y_{n,1}-x^*\| \\
&+ \alpha_{n,1}Lb_n + \alpha_{n,1}\|e'_{n,1}\| + \|e''_{n,1}\| + \|r_{n,1}\| + \beta_{n,1}\Gamma \\
&\le (1-\alpha_{n,1}-\beta_{n,1}+\alpha_{n,1}(1-\alpha_{n,2}-\beta_{n,2})L\gamma + \alpha_{n,1}\alpha_{n,2}(1-\alpha_{n,3}-\beta_{n,3})L^2\gamma^2 \\
&+ \cdots + \prod_{i=1}^{p-1}\alpha_{n,i}(1-\alpha_{n,p}-\beta_{n,p})L^{p-1}\gamma^{p-1} + \prod_{i=1}^{p}\alpha_{n,i}L^p\gamma^p\Bigg)\|x_n-x^*\| \\
&+ \left(\alpha_{n,1}L + \alpha_{n,1}\alpha_{n,2}L^2\gamma + \alpha_{n,1}\alpha_{n,2}\alpha_{n,3}L^3\gamma^2 + \cdots + \prod_{i=1}^{p}\alpha_{n,i}L^p\gamma^{p-1}\right)b_n \\
&+ \alpha_{n,1}\|e'_{n,1}\| + \alpha_{n,1}\alpha_{n,2}L\gamma\|e'_{n,2}\| + \cdots + \prod_{i=1}^{p}\alpha_{n,i}L^{p-1}\gamma^{p-1}\|e'_{n,p}\| \\
&+ \|e''_{n,1}\| + \alpha_{n,1}L\gamma\|e''_{n,2}\| + \alpha_{n,1}\alpha_{n,2}L^2\gamma^2\|e''_{n,3}\| + \cdots + \prod_{i=1}^{p-1}\alpha_{n,i}L^{p-1}\gamma^{p-1}\|e''_{n,p}\| \\
&+ \|r_{n,1}\| + \alpha_{n,1}L\gamma\|r_{n,2}\| + \alpha_{n,1}\alpha_{n,2}L^2\gamma^2\|r_{n,3}\| + \cdots \prod_{i=1}^{p-1}\alpha_{n,i}L^{p-1}\gamma^{p-1}\|r_{n,p}\| \\
&+ \left(\beta_{n,1} + \alpha_{n,1}\beta_{n,2}L\gamma + \alpha_{n,1}\alpha_{n,2}\beta_{n,3}L^2\gamma^2 + \cdots + \prod_{i=1}^{p-1}\alpha_{n,i}\beta_{n,p}L^{p-1}\gamma^{p-1}\right)\Gamma \\
&\le [1-(1-L\gamma)\prod_{i=1}^{p}\alpha_{n,i}L^{p-1}\gamma^{p-1}]\|x_n-x^*\| + \sum_{i=1}^{p}\prod_{j=1}^{i}\alpha_{n,j}L^i\gamma^{i-1}b_n \\
&+ \sum_{i=1}^{p}\prod_{j=1}^{i}\alpha_{n,j}L^{i-1}\gamma^{i-1}\|e'_{n,i}\| + \|e''_{n,1}\| + \sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}L^{i-1}\gamma^{i-1}\|e''_{n,i}\| + \|r_{n,1}\| \\
&+ \sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}L^{i-1}\gamma^{i-1}\|r_{n,i}\| + \left(\beta_{n,1} + \sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}\beta_{n,i}L^{i-1}\gamma^{i-1}\right)\Gamma \\
&= [1-(1-L\gamma)\prod_{i=1}^{p}\alpha_{n,i}L^{p-1}\gamma^{p-1}]\|x_n-x^*\| \\
&+ (1-L\gamma)\prod_{i=1}^{p}\alpha_{n,i}L^{p-1}\gamma^{p-1}\frac{\sum_{i=1}^{p}\prod_{j=1}^{i}\alpha_{n,j}L^i\gamma^{i-1}b_n + \sum_{i=1}^{p}\prod_{j=1}^{i}\alpha_{n,j}L^{i-1}\gamma^{i-1}\|e'_{n,i}\|}{\alpha(1-L\gamma)L^{p-1}\gamma^{p-1}} \\
&+ \sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}L^{i-1}\gamma^{i-1}\|e''_{n,i}\| + \sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}L^{i-1}\gamma^{i-1}\|r_{n,i}\| \\
&+ \|e''_{n,1}\| + \|r_{n,1}\| + \left(\beta_{n,1} + \sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}\beta_{n,i}L^{i-1}\gamma^{i-1}\right)\Gamma.
\end{aligned} \tag{4.14}$$

Since $L\gamma < 1$ and $\lim_{n\to\infty} b_n = 0$, in view of (4.3), it is clear that all the conditions of Lemma 4.7 are satisfied and so Lemma 4.7 and (4.14) guarantee that $x_n \to x^*$, as $n \to \infty$. Thus the sequence $\{x_n\}_{n=0}^{\infty}$ generated by Algorithm 4.1 converges strongly to the only element of $\mathrm{Fix}(S) \cap \mathrm{EGNRNVID}(K_r, T, f, g)$. This completes the proof.

As in the proof of Theorem 3.5, one can prove the convergence of iterative sequences generated by Algorithms 4.2-4.4 and we omit their proofs.

Theorem 4.9. *Assume that T, f and ρ are the same as in Theorem 3.4 such that the conditions* (a), (b) *and* (3.18) *in Theorem 3.4 hold. Let $S : K_r \to K_r$ be a nearly uniformly L-Lipschitzian mapping with the sequence $\{b_n\}_{n=0}^{\infty}$ such that* $\mathrm{Fix}(S) \cap \mathrm{GNRNVID}\ (K_r, T, f) \neq \emptyset$, *where* GNRNVID (K_r, T, f) *is the set of the solutions of the problem* (3.3). *Further, let $L\theta < 1$, where $\theta = \frac{r}{r-r'}\sqrt{\varpi^2 - 2\rho\kappa + \rho^2\sigma^2}$. If there exists a constant $\alpha > 0$ such that $\prod_{i=1}^{p}\alpha_{n,i} > \alpha$ for each $n \geq 0$, then the iterative sequence $\{x_n\}_{n=0}^{\infty}$ generated by Algorithm 4.2 converges strongly to the only element of* $\mathrm{Fix}(S) \cap \mathrm{GNRNVID}\ (K_r, T, f)$.

Theorem 4.10. *Suppose that T, g and ρ are the same as in Theorem 3.5 such that the conditions* (a), (b) *and* (3.19) *in Theorem 3.5 hold. Let* $S : K_r \to K_r$ *be a nearly uniformly L-Lipschitzian mapping with the sequence* $\{b_n\}_{n=0}^{\infty}$ *such that* Fix(*S*) ∩ GNRNVID (K_r, *T*, *f*) ≠ ∅, *where* GNRNVID (K_r, *T*, *g*) *is the set of the solutions of the problem* (3.4). *Further, let* $L\tilde{\theta} < 1$, *where*

$$\tilde{\theta} = \sqrt{1 - 2\tau + \iota^2} + \frac{r}{r - r'}\sqrt{\iota^2 - 2\rho\kappa + \rho^2\sigma^2}.$$

If there exists a constant α >0 *such that* $\prod_{i=1}^{p} \alpha_{n,i} > \alpha$ *for each n* ≥ 0, *then the iterative sequence* $\{x_n\}_{n=0}^{\infty}$ *generated by Algorithm 4.3 converges strongly to the only element of* Fix(*S*) ∩ GNRNVID (K_r, *T*, *g*).

Theorem 4.11. *Let T and ρ be the same as in Theorem 3.6 such that the condition* (3.20) *in Theorem 3.6 holds. Assume that* $S : K_r \to K_r$ *is a nearly uniformly L-Lipschitzian mapping with the sequence* $\{b_n\}_{n=0}^{\infty}$ *such that* Fix(*S*) ∩ NRNVI (K_r, *T*) ≠ ∅, *where* NRNVI (K_r, *T*) *is the set of the solutions of the problem* (3.5). *Moreover, let Lη* <1, *where* $\eta = \frac{r}{r-r'}\sqrt{1 - 2\rho\kappa + \rho^2\sigma^2}$. *Then the iterative sequence* $\{x_n\}_{n=0}^{\infty}$ *generated by Algorithm 4.4 converges strongly to the only element of* Fix(*S*) ∩ NRNVI (K_r, *T*).

5 Extended general nonconvex Wiener-Hopf equations

In this section, we introduce a new class of extended general nonconvex Wiener-Hopf equations and some new special cases from it, and by using the projection method, we establish that the aforesaid class is equivalent with the class of extended general nonlinear regularized nonconvex variational inequalities (3.1).

Let *T*, *f*, *g* and *ρ* be the same as in the problem (3.1) and suppose that the inverse of the operator *g* exists. Associated with the problem (3.1), the problem of finding $z \in \mathcal{H}$ such that

$$Tg^{-1}P_{K_r}z + \rho^{-1}Q_{K_r}z = 0, \tag{5.1}$$

Where $Q_{K_r} = I - fg^{-1}P_{K_r}$ with *I* the identity operator and P_{K_r} the projection operator is considered.

The problem (5.1) is called the *extended general nonconvex Wiener-Hopf equation* (EGNWHE) associated with the problem of extended general nonlinear regularized nonconvex variational inequality (3.1). Next, we denote by EGNWHE(K_r, *T*, *f*, *g*) the set of the solutions of the extended general nonconvex Wiener-Hopf equation (5.1).

Now, we state some special cases of the problem (5.1) as follows:

(1) If *g* ≡ *I*, then the problem (5.1) is equivalent to the following problem: Find $z \in \mathcal{H}$ such that

$$TP_{K_r}z + \rho^{-1}Q_{K_r}z = 0, \tag{5.2}$$

where $Q_{K_r} = I - fP_{K_r}$ and is called the *general nonconvex Wiener-Hopf equation* (GNWHE) associated with the problem of general nonlinear regularized nonconvex variational inequality (3.3). We denote by GNWHE(K_r, *T*, *f*) the set of the solutions of the general nonconvex Wiener-Hopf equation (5.2).

(2) If $f = g$, then the problem (5.1) changes into the following problem: Find $z \in \mathcal{H}$ such that

$$Tg^{-1}P_{K_r}z + \rho^{-1}Q_{K_r}z = 0, \tag{5.3}$$

where $Q_{K_r} = I - P_{K_r}$ and is called also the *general nonconvex Wiener-Hopf equation* (GNWHE) associated with the problem of general nonlinear regularized nonconvex variational inequality (3.4). We denote by GNWHE (K_r, T, g) the set of the solutions of the general nonconvex Wiener-Hopf equation (5.2).

(3) If $f = g \equiv I$, then the problem (5.1) collapses to the following problem: Find $z \in \mathcal{H}$ such that

$$TP_{K_r}z + \rho^{-1}Q_{K_r}z = 0, \tag{5.4}$$

where Q_{K_r} is the same as in Eq. (5.3). The equation of the type (5.4) is called the *nonconvex Wiener-Hopf equation* (NWHE) associated with the problem of nonlinear regularized nonconvex variational inequality (3.4).

We denote by NWHE (K_r, T) the set of the solutions of the nonconvex Wiener-Hopf equation (5.2).

(4) If $r = \infty$, that is $K_r = K$, then the problem (5.1) reduces to the following problem: Find $z \in \mathcal{H}$ such that

$$Tg^{-1}P_Kz + \rho^{-1}Q_Kz = 0, \tag{5.5}$$

Where $Q_K = I - f_g^{-1} P_K$. The equations of the type (5.5) were introduced and studied by Noor [39].

(5) If $r = \infty$, then the problem (5.2) is equivalent to the following problem: Find $z \in \mathcal{H}$ such that

$$TP_Kz + \rho^{-1}Q_Kz = 0, \tag{5.6}$$

where $Q_K = I - P_K$. The problem (5.6) is introduced and studied by Noor [35].

(6) If $r = \infty$, then the problem (5.3) changes into the following problem: Find $z \in \mathcal{H}$ such that

$$Tg^{-1}P_Kz + \rho^{-1}Q_Kz = 0, \tag{5.7}$$

where $Q_K = I - P_K$. The equations of the type (5.7) were introduced and studied by Noor [40].

(7) If $r = \infty$, then the problem (5.4) reduces to the following problem: Find $z \in \mathcal{H}$ such that

$$TP_Kz + \rho^{-1}Q_Kz = 0, \tag{5.8}$$

where Q_K is the same as in (5.7). The equation (5.8) is the original Wiener-Hopf equation mainly due to Shi [50].

Remark 5.1. It has been shown that the Wiener-Hopf equations have played an important and significant role in developing several numerical techniques for solving variational inequalities and related optimizations problems (see, for example, [30,31,35,39,50] and references therein).

The following lemma shows that the extended general nonlinear regularized nonconvex variational inequality (3.1) and the extended general nonconvex Wiener-Hopf equation (5.1) are equivalent.

Lemma 5.2. *Let T, f, g and ρ be the same as in the problem* (3.1) *and suppose that the inverse of the operator g exists. Then $u \in \mathcal{H}$ with $g(u) \in K_r$ is a solution of the problem* (3.1) *if and only if the extended general nonconvex Wiener-Hopf equation* (5.1) *has a solution $z \in \mathcal{H}$ satisfying the following:*

$$g(u) = P_{K_r}z, \quad z = f(u) - \rho T(u).$$

Proof. Let $u \in \mathcal{H}$ with $g(u) \in K_r$ be a solution of the problem (3.1). Then, from Lemma 3.2, it follows that

$$g(u) = P_{K_r}(f(u) - \rho T(u)). \tag{5.9}$$

Taking $z = f(u) - \rho T(u)$ in (5.9), we have $g(u) = P_{K_r}z$, which leads to

$$u = g^{-1}P_{K_r}z. \tag{5.10}$$

Applying (5.10) and this fact that $z = f(u) - \rho T(u)$, we have

$$z = fg^{-1}P_{K_r}z - \rho Tg^{-1}P_{K_r}z.$$

Evidently, the above equality is equivalent to the following:

$$Tg^{-1}P_{K_r}z + \rho^{-1}Q_{K_r}z = 0, \tag{5.11}$$

where Q_{K_r} is the same as in (5.1). Now, (5.11) guarantees that $z \in \mathcal{H}$ is a solution of the extended general nonconvex Wiener-Hopf equation (5.1).

Conversely, if $z \in \mathcal{H}$ is a solution of the problem (5.1) satisfying the following:

$$g(u) = P_{K_r}z, \quad z = f(u) - \rho T(u),$$

then it follows from Lemma 3.2 that $u \in \mathcal{H}$ with $g(u) \in K_r$ is a solution of the problem (3.1). This completes the proof.

In similar way to the proof of Lemma 5.2, one can prove the following statements.

Lemma 5.3. *Let T, f and ρ be the same as in the problem* (3.3). *Then $u \in K_r$ is a solution of the problem* (3.3) *if and only if the general nonconvex Wiener-Hopf equation* (5.2) *has a solution $z \in \mathcal{H}$ satisfying*

$$u = P_{K_r}z, \quad z = f(u) - \rho T(u).$$

Lemma 5.4. *Suppose that T, g and ρ are the same as in the problem* (3.4) *and let the inverse of the operator g exists. Then $u \in \mathcal{H}$ with $g(u) \in K_r$ is a solution of the problem* (3.4) *if and only if the general nonconvex Wiener-Hopf equation* (5.3) *has a solution $z \in \mathcal{H}$ satisfying the following:*

$$g(u) = P_{K_r}z, \quad z = g(u) - \rho T(u).$$

Lemma 5.5. *Assume that* T *and* ρ *are the same as in the problem* (3.5). *Then* $u \in K_r$ *is a solution of the problem* (3.5) *if and only if the nonconvex Wiener-Hopf equation* (5.4) *has a solution* $z \in \mathcal{H}$ *satisfying the following:*

$$u = P_{K_r}z, \quad z = u - \rho T(u).$$

6 Some new perturbed *p*-step projection iterative methods

In this section, by using the problems (5.1)-(5.4) and four Lemmas 5.2-5.5, we get some fixed point formulations for constructing a number of the new perturbed p-step projection iterative algorithms with mixed errors for solving the problems (3.1) and (3.3)-(3.5).

(I) By using (5.1) and Lemma 5.2, we have

$$\begin{aligned} Tg^{-1}P_{K_r}z + \rho^{-1}Q_{K_r}z = 0 &\Leftrightarrow \rho Tg^{-1}P_{K_r}z + Q_{K_r}z = 0 \\ &\Leftrightarrow \rho Tg^{-1}P_{K_r}z + z - fg^{-1}P_{K_r}z = 0 \\ &\Leftrightarrow z = fg^{-1}P_{K_r}z - \rho Tg^{-1}P_{K_r}z \\ &\Leftrightarrow z = f(u) - \rho T(u). \end{aligned}$$

This fixed point formulation enables us to define the following p-step projection iterative algorithm with mixed errors for solving the problem (3.1).

Algorithm 6.1. Let T, f, g and ρ be the same as in the problem (3.1) such that g be an onto operator. For arbitrary chosen initial point $z_0 \in \mathcal{H}$, compute the iterative sequence $\{z_n\}_{n=0}^{\infty}$ by

$$\begin{cases} g(u_n) = S^n P_{K_r} z_n, \\ z_{n+1} = (1 - \alpha_{n,1} - \beta_{n,1})z_n + \alpha_{n,1}(f(v_{n,1}) - \rho T(v_{n,1}) + e_{n,1}) + \beta_{n,1} l_{n,1} + r_{n,1}, \\ v_{n,i} = (1 - \alpha_{n,i+1} - \beta_{n,i+1})z_n + \alpha_{n,i+1}(f(v_{n,i+1}) - \rho T(v_{n,i+1}) + e_{n,i+1}) + \beta_{n,i+1} l_{n,i+1} + r_{n,i+1}, \\ \cdots \\ v_{n,p-1} = (1 - \alpha_{n,p} - \beta_{n,p})z_n + \alpha_{n,p}(f(u_n) - \rho T(u_n) + e_{n,p}) + \beta_{n,p} l_{n,p} + r_{n,p}, \\ i = 1, 2, \ldots, p-2, \end{cases} \tag{6.1}$$

where S, $\{\alpha_{n,i}\}_{n=0}^{\infty}$, $\{\beta_{n,i}\}_{n=0}^{\infty}$, $\{e_{n,i}\}_{n=0}^{\infty}$, $\{l_{n,i}\}_{n=0}^{\infty}$, $\{r_{n,i}\}_{n=0}^{\infty} (i = 1, 2, \ldots, p)$, are the same as in Algorithm 4.1.

(II) From (5.1) and Lemma 5.2, it follows that

$$\begin{aligned} Tg^{-1}P_{K_r}z + \rho^{-1}Q_{K_r}z = 0 &\Leftrightarrow Q_{K_r}z = Q_{K_r}z - Tg^{-1}P_{K_r}z - \rho^{-1}Q_{K_r}z \\ &\Leftrightarrow Q_{K_r}z = -Tg^{-1}P_{K_r}z + (1 - \rho^{-1})Q_{K_r}z \\ &\Leftrightarrow z = fg^{-1}P_{K_r}z - Tg^{-1}P_{K_r}z + (1 - \rho^{-1})Q_{K_r}z \\ &\Leftrightarrow z = f(u) - T(u) + (1 - \rho^{-1})Q_{K_r}z. \end{aligned}$$

By using this fixed point formulation, we can construct the following p-step projection iterative algorithm with mixed errors for solving the problem (3.1).

Algorithm 6.2. Assume that T, f, g and ρ are the same as in Algorithm 6.1. For arbitrary chosen initial point $z_0 \in \mathcal{H}$, compute the iterative sequence $\{z_n\}_{n=0}^{\infty}$ as follows:

$$\begin{cases} g(u_n) = S^n P_{K_r} z_n, \\ z_{n+1} = (1 - \alpha_{n,1} - \beta_{n,1})z_n + \alpha_{n,1}(\Psi(v_{n,1}, z_n) + e_{n,1}) + \beta_{n,1} l_{n,1} + r_{n,1}, \\ v_{n,i} = (1 - \alpha_{n,i+1} - \beta_{n,i+1})z_n + \alpha_{n,i+1}(\Psi(v_{n,i+1}, z_n) + e_{n,i+1}) + \beta_{n,i+1} l_{n,i+1} + r_{n,i+1}, \\ \cdots \\ v_{n,p-1} = (1 - \alpha_{n,p} - \beta_{n,p})z_n + \alpha_{n,p}(\Psi(u_n, z_n) + e_{n,p}) + \beta_{n,p} l_{n,p} + r_{n,p}, \\ i = 1, 2, \ldots, p-2, \end{cases}$$

where

$$\begin{cases} \Psi(v_{n,i}, z_n) = f(v_{n,i}) - T(v_{n,i}) + (1-\rho^{-1})Q_{K_r}z_n, \\ \Psi(u_n, z_n) = f(u_n) - T(u_n) + (1-\rho^{-1})Q_{K_r}z_n, \\ i = 1, 2, \ldots, p-2, \end{cases}$$

and S, $\{\alpha_{n,i}\}_{n=0}^{\infty}$, $\{\beta_{n,i}\}_{n=0}^{\infty}$, $\{e_{n,i}\}_{n=0}^{\infty}$, $\{l_{n,i}\}_{n=0}^{\infty}$, $\{r_{n,i}\}_{n=0}^{\infty}(i = 1, 2, \ldots, p)$ are the same as in Algorithm 4.1.

(III) Like in the proof (I), by using (5.2) and Lemma 5.3, we have

$$TP_{K_r}z + \rho^{-1}Q_{K_r}z = 0 \quad \Leftrightarrow \quad z = f(u) - \rho T(u).$$

This fixed point formulation allows us to construct the following p-step projection iterative algorithm with mixed errors for solving the problem (3.3).

Algorithm 6.3. Let T, f and ρ be the same as in the problem (3.3). For arbitrary chosen initial point $z_0 \in \mathcal{H}$, compute the iterative sequence $\{z_n\}_{n=0}^{\infty}$ by

$$\begin{cases} u_n = S^n P_{K_r} z_n, \\ z_{n+1} = (1-\alpha_{n,1}-\beta_{n,1})z_n + \alpha_{n,1}(f(v_{n,1}) - \rho T(v_{n,1}) + e_{n,1}) + \beta_{n,1}l_{n,1} + r_{n,1}, \\ v_{n,i} = (1-\alpha_{n,i+1}-\beta_{n,i+1})z_n + \alpha_{n,i+1}(f(v_{n,i+1}) - \rho T(v_{n,i+1}) + e_{n,i+1}) + \beta_{n,i+1}l_{n,i+1} + r_{n,i+1}, \\ \cdots \\ v_{n,p-1} = (1-\alpha_{n,p}-\beta_{n,p})z_n + \alpha_{n,p}(f(u_n) - \rho T(u_n) + e_{n,p}) + \beta_{n,p}l_{n,p} + r_{n,p}, \\ i = 1, 2, \ldots, p-2, \end{cases}$$

where S, $\{\alpha_{n,i}\}_{n=0}^{\infty}$, $\{\beta_{n,i}\}_{n=0}^{\infty}$, $\{e_{n,i}\}_{n=0}^{\infty}$, $\{l_{n,i}\}_{n=0}^{\infty}$, $\{r_{n,i}\}_{n=0}^{\infty}(i = 1, 2, \ldots, p)$ are the same as in Algorithm 4.1.

(IV) In similar way, from (5.3) and Lemma 5.4, it follows that

$$Tg^{-1}P_{K_r}z + \rho^{-1}Q_{K_r}z = 0 \quad \Leftrightarrow \quad z = g(u) - \rho T(u).$$

By using the above fixed point formulation, we can define the following p-step projection iterative algorithm with mixed errors for solving the problem (3.4).

Algorithm 6.4. Let T, g and ρ be the same as in the problem (3.4) such that g be an onto operator. For arbitrary chosen initial point $z_0 \in \mathcal{H}$, compute the iterative sequence $\{z_n\}_{n=0}^{\infty}$ by

$$\begin{cases} g(u_n) = S^n P_{K_r} z_n, \\ z_{n+1} = (1-\alpha_{n,1}-\beta_{n,1})z_n + \alpha_{n,1}(g(v_{n,1}) - \rho T(v_{n,1}) + e_{n,1}) + \beta_{n,1}l_{n,1} + r_{n,1}, \\ v_{n,i} = (1-\alpha_{n,i+1}-\beta_{n,i+1})z_n + \alpha_{n,i+1}(g(v_{n,i+1}) - \rho T(v_{n,i+1}) + e_{n,i+1}) + \beta_{n,i+1}l_{n,i+1} + r_{n,i+1}, \\ \cdots \\ v_{n,p-1} = (1-\alpha_{n,p}-\beta_{n,p})z_n + \alpha_{n,p}(g(u_n) - \rho T(u_n) + e_{n,p}) + \beta_{n,p}l_{n,p} + r_{n,p}, \\ i = 1, 2, \ldots, p-2, \end{cases}$$

where S, $\{\alpha_{n,i}\}_{n=0}^{\infty}$, $\{\beta_{n,i}\}_{n=0}^{\infty}$, $\{e_{n,i}\}_{n=0}^{\infty}$, $\{l_{n,i}\}_{n=0}^{\infty}$, $\{l_{n,i}\}_{n=0}^{\infty}$ are the same as in Algorithm 4.1.

(V) Similarly, by using (5.4) and Lemma 5.5, we have

$$TP_{K_r}z + \rho^{-1}Q_{K_r}z = 0 \quad \Leftrightarrow \quad z = f(u) - \rho T(u).$$

This fixed point formulation enables us to construct the following p-step projection iterative algorithm with mixed errors for solving the problem (3.5).

Algorithm 6.5. Let T and ρ be the same as in the problem (3.5). For arbitrary chosen initial point $z_0 \in \mathcal{H}$, compute the iterative sequence $\{z_n\}_{n=0}^{\infty}$ by the iterative scheme

$$\begin{cases} u_n = S^n P_{K_r} z_n, \\ z_{n+1} = (1-\alpha_{n,1}-\beta_{n,1})z_n + \alpha_{n,1}(v_{n,1} - \rho T(v_{n,1}) + e_{n,1}) + \beta_{n,1} l_{n,1} + r_{n,1}, \\ v_{n,i} = (1-\alpha_{n,i+1}-\beta_{n,i+1})z_n + \alpha_{n,i+1}(v_{n,i+1} - \rho T(v_{n,i+1}) + e_{n,i+1}) + \beta_{n,i+1} l_{n,i+1} + r_{n,i+1}, \\ \cdots \\ v_{n,p-1} = (1-\alpha_{n,p}-\beta_{n,p})z_n + \alpha_{n,p}(u_n - \rho T(u_n) + e_{n,p}) + \beta_{n,p} l_{n,p} + r_{n,p}, \\ i = 1, 2, \ldots, p-2, \end{cases}$$

where S, $\{\alpha_{n,i}\}_{n=0}^{\infty}$, $\{\beta_{n,i}\}_{n=0}^{\infty}$, $\{e_{n,i}\}_{n=0}^{\infty}$, $\{l_{n,i}\}_{n=0}^{\infty}$, $\{r_{n,i}\}_{n=0}^{\infty}(i = 1, 2, \ldots, p)$ are the same as in Algorithm 4.1.

Remark 6.6. In similar to Remark 4.5, for a suitable and appropriate choice of the sequences $\{e_{n,i}\}_{n=0}^{\infty}$, $\{l_{n,i}\}_{n=0}^{\infty}$ and $\{r_{n,i}\}_{n=0}^{\infty}(i = 1, 2, \ldots, p)$, Algorithms 6.1-6.5 reduce to algorithms with mean errors and without errors.

Remark 6.7. Algorithm 3.1 in [42] is a special case of Algorithms 6.1 and 6.4. Algorithm 3.2 in [42] is a special case of Algorithm 6.2. Also, Algorithms 3.1-3.3 in [44] and Algorithms 2.1-2.3 in [47] are special cases of Algorithms 6.1, 6.2 and 6.4.

Now, we discuss the convergence analysis of iterative sequences generated by perturbed projection iterative Algorithms 6.1-6.5.

Theorem 6.8. *Let T, f, g and ρ be the same as in the problem* (3.1) and suppose that all the conditions of Theorem 3.3 *hold. Assume that $S : K_r \to K_r$ is a nearly uniformly L-Lipschitzian mapping with the sequence $\{b_n\}_{n=0}^{\infty}$such that, for any $u \in$* EGNRNVID (K_r, T, f, g), $g(u) \in$ Fix(S). *Further, assume that $L\gamma < 1$, where γ is the same as in* (3.17). *If there exists a constant $\alpha > 0$ such that $\prod_{i=1}^{p} \alpha_{n,i} > \alpha$ for each $n \geq 0$, then the iterative sequence $\{z_n\}_{n=0}^{\infty}$generated by Algorithm 6.1 converges strongly to the only element of* EGNWHE(K_r, T, f, g).

Proof. Theorem 3.3 guarantees the existence a unique solution $u^* \in \mathcal{H}$ with $g(u^*) \in K_r$ for the problem (3.1). Hence, in view of Lemma 5.2, there exists a unique point $z \in \mathcal{H}$ satisfying the following:

$$g(u^*) = P_{K_r} z, \quad z = f(u^*) - \rho T(u^*). \tag{6.2}$$

Since $g(u^*) \in$ Fix(S), it follows from (6.2) that, for each $n \geq 0$,

$$g(u^*) = S^n P_{K_r} z, \quad z = f(u^*) - \rho T(u^*). \tag{6.3}$$

Let $\Gamma = \sup_{n\geq 0}\{||l_{n,i} - z||, ||z - u^*||: i = 1, 2,..., p\}$. By using (6.1), (6.2) and the assumptions, we have

$$\begin{aligned}
||z_{n+1}-z|| &\le (1-\alpha_{n,1}-\beta_{n,1})||z_n-z|| + \alpha_{n,1}||f(v_{n,1})-f(u^*)-\rho(T(v_{n,1})-T(u^*))|| \\
&\quad +\beta_{n,1}||l_{n,1}-z|| + \alpha_{n,1}||e_{n,1}|| + ||r_{n,1}|| \\
&\le (1-\alpha_{n,1}-\beta_{n,1})||z_n-z|| + \alpha_{n,1}\sqrt{\varpi^2-2\rho\kappa+\rho^2\sigma^2}||v_{n,1}-u^*|| \\
&\quad +\beta_{n,1}||l_{n,1}-z|| + \alpha_{n,1}(||e'_{n,1}||+||e''_{n,1}||) + ||r_{n,1}|| \\
&\le (1-\alpha_{n,1}-\beta_{n,1})||z_n-z|| + \alpha_{n,1}\sqrt{\varpi^2-2\rho\kappa+\rho^2\sigma^2}||v_{n,1}-z|| \\
&\quad + \alpha_{n,1}\sqrt{\varpi^2-2\rho\kappa+\rho^2\sigma^2}||z-u^*|| + \alpha_{n,1}||e'_{n,1}|| + ||e''_{n,1}|| + ||r_{n,1}|| + \beta_{n,1}\Gamma \\
&\le (1-\alpha_{n,1}-\beta_{n,1})||z_n-z|| + \alpha_{n,1}\sqrt{\varpi^2-2\rho\kappa+\rho^2\sigma^2}||v_{n,1}-z|| \\
&\quad + (\alpha_{n,1}\sqrt{\varpi^2-2\rho\kappa+\rho^2\sigma^2}+\beta_{n,1})\Gamma + \alpha_{n,1}||e'_{n,1}|| + ||e''_{n,1}|| + ||r_{n,1}||.
\end{aligned} \tag{6.4}$$

In similar way to the proof (6.4), for each $i \in \{1, 2,..., p - 2\}$, we can get

$$\begin{aligned}
||v_{n,i}-z|| &\le (1-\alpha_{n,i+1}-\beta_{n,i+1})||z_n-z|| \\
&\quad + \alpha_{n,i+1}\sqrt{\varpi^2-2\rho\kappa+\rho^2\sigma^2}||v_{n,i+1}-z|| \\
&\quad + (\alpha_{n,i+1}\sqrt{\varpi^2-2\rho\kappa+\rho^2\sigma^2}+\beta_{n,i+1})\Gamma \\
&\quad + \alpha_{n,i+1}||e'_{n,i+1}|| + ||e''_{n,i+1}|| + ||r_{n,i+1}||
\end{aligned} \tag{6.5}$$

and

$$\begin{aligned}
||v_{n,p-1}-z|| &\le (1-\alpha_{n,p}-\beta_{n,p})||z_n-z|| + \alpha_{n,p}\sqrt{\varpi^2-2\rho\kappa+\rho^2\sigma^2}||u_n-u^*|| \\
&\quad + \alpha_{n,p}||e'_{n,p}|| + ||e''_{n,p}|| + ||r_{n,p}|| + \beta_{n,p}\Gamma.
\end{aligned} \tag{6.6}$$

Now, we make an estimation for $||u_n - u^*||$. Applying (6.1), (6.3) and Proposition 2.10, we find that

$$\begin{aligned}
||u_n-u^*|| &\le ||u_n-u^*-(g(u_n)-g(u^*))|| + ||S^nP_{K_r}z_n - S^nP_{K_r}z|| \\
&\le \sqrt{1-2\tau+\iota^2}||u_n-u^*|| + L\left(\frac{r}{r-r'}||z_n-z|| + b_n\right),
\end{aligned}$$

which leads to

$$||u_n-u^*|| \le \frac{rL}{(r-r')(1-\sqrt{1-2\tau+\iota^2})}||z_n-z|| + \frac{Lb_n}{1-\sqrt{1-2\tau+\iota^2}}. \tag{6.7}$$

By using (6.6) and (6.7), we conclude that

$$\begin{aligned}
||v_{n,p-1}-z|| &\le (1-\alpha_{n,p}-\beta_{n,p})||z_n-z|| \\
&\quad + \alpha_{n,p}L\frac{r\sqrt{\varpi^2-2\rho\kappa+\rho^2\sigma^2}}{(r-r')(1-\sqrt{1-2\tau+\iota^2})}||z_n-z|| \\
&\quad + \alpha_{n,p}L\frac{\sqrt{\varpi^2-2\rho\kappa+\rho^2\sigma^2}}{1-\sqrt{1-2\tau+\iota^2}}b_n \\
&\quad + \alpha_{n,p}||e'_{n,p}|| + ||e''_{n,p}|| + ||r_{n,p}|| + \beta_{n,p}\Gamma \\
&\le (1-\alpha_{n,p}-\beta_{n,p})||z_n-z|| + \alpha_{n,p}L\vartheta||z_n-z|| \\
&\quad + \alpha_{n,p}\frac{(r-r')L\vartheta}{r}b_n + \alpha_{n,p}||e'_{n,p}|| + ||e''_{n,p}|| + ||r_{n,p}|| + \beta_{n,p}\Gamma,
\end{aligned} \tag{6.8}$$

where $\vartheta = \frac{r\sqrt{\varpi^2-2\rho\kappa+\rho^2\sigma^2}}{(r-r')(1-\sqrt{1-2\tau+\iota^2})}$ In view of the condition (3.11), we have $\vartheta < 1$. From $r' \in (0, r)$ and (6.8), we have

$$\begin{aligned}||v_{n,p-1} - z|| &\le (1 - \alpha_{n,p} - \beta_{n,p})||z_n - z|| + \alpha_{n,p}L\vartheta||z_n - z|| \\ &+ \alpha_{n,p}L\vartheta b_n + \alpha_{n,p}||e'_{n,p}|| + ||e''_{n,p}|| + ||r_{n,p}|| + \beta_{n,p}\Gamma.\end{aligned} \tag{6.9}$$

Since $\sqrt{1 - 2\tau + \iota^2} + \frac{r}{r-r'}\sqrt{\varpi^2 - 2\rho\kappa + \rho^2\sigma^2} < 1$ and $r' \in (0, r)$, deduce that

$$\lambda = \sqrt{\varpi^2 - 2\rho\kappa + \rho^2\sigma^2} < 1. \tag{6.10}$$

By using (6.10), the inequality (6.5), for each $i = 1, 2, ..., p - 2$, can be written as follows:

$$\begin{aligned}||v_{n,i} - z|| &\le (1 - \alpha_{n,i+1} - \beta_{n,i+1})||z_n - z|| + \alpha_{n,i+1}\lambda||v_{n,i+1} - z|| \\ &+ \alpha_{n,i+1}||e'_{n,i+1}|| + ||e''_{n,i+1}|| + ||r_{n,i+1}|| + (\alpha_{n,i+1}\lambda + \beta_{n,i+1})\Gamma.\end{aligned} \tag{6.11}$$

Thus it follows from (6.9) and (6.11) that

$$\begin{aligned}||v_{n,p-2} - z|| &\le (1 - \alpha_{n,p-1} - \beta_{n,p-1})||z_n - z|| + \alpha_{n,p-1}\lambda\{(1 - \alpha_{n,p} - \beta_{n,p})||z_n - z|| \\ &+ \alpha_{n,p}L\vartheta||z_n - z|| + \alpha_{n,p}L\vartheta b_n + \alpha_{n,p}||e'_{n,p}|| + ||e''_{n,p}|| + ||r_{n,p}|| + \beta_{n,p}\Gamma\} \\ &+ \alpha_{n,p-1}||e'_{n,p-1}|| + ||e''_{n,p-1}|| + ||r_{n,p-1}|| + (\alpha_{n,p-1}\lambda + \beta_{n,p-1})\Gamma \\ &= \left(1 - \alpha_{n,p-1} - \beta_{n,p-1} + \alpha_{n,p-1}(1 - \alpha_{n,p} - \beta_{n,p})\lambda + \alpha_{n,p-1}\alpha_{n,p}\lambda L\vartheta\right)||z_n - z|| \\ &+ \alpha_{n,p-1}\alpha_{n,p}\lambda L\vartheta b_n + \alpha_{n,p-1}\alpha_{n,p}\lambda||e'_{n,p}|| + \alpha_{n,p-1}||e'_{n,p-1}|| \\ &+ \alpha_{n,p-1}\lambda||e''_{n,p}|| + ||e''_{n,p-1}|| + \alpha_{n,p-1}\lambda||r_{n,p}|| + ||r_{n,p-1}|| \\ &+ (\alpha_{n,p-1}\beta_{n,p}\lambda + \alpha_{n,p-1}\lambda + \beta_{n,p-1})\Gamma.\end{aligned} \tag{6.12}$$

Similarly, by using (6.11) and (6.12), we obtain

$$\begin{aligned}||v_{n,p-3} - z|| &\le \left(1 - \alpha_{n,p-2} - \beta_{n,p-2} + \alpha_{n,p-2}(1 - \alpha_{n,p-1} - \beta_{n,p-1})\lambda\right. \\ &\left.+ \alpha_{n,p-2}\alpha_{n,p-1}(1 - \alpha_{n,p} - \beta_{n,p})\lambda^2 + \alpha_{n,p-2}\alpha_{n,p-1}\alpha_{n,p}\lambda^2 L\vartheta\right)||z_n - z|| \\ &+ \alpha_{n,p-2}\alpha_{n,p-1}\alpha_{n,p}\lambda^2||e'_{n,p}|| + \alpha_{n,p-2}\alpha_{n,p-1}\lambda||e'_{n,p-1}|| + \alpha_{n,p-2}||e'_{n,p-2}|| \\ &+ \alpha_{n,p-2}\alpha_{n,p-1}\alpha_{n,p}\lambda^2 L\vartheta b_n + \alpha_{n,p-2}\alpha_{n,p-1}\lambda^2||e''_{n,p}|| + \alpha_{n,p-2}\lambda||e''_{n,p-1}|| + ||e''_{n,p-2}|| \\ &+ \alpha_{n,p-2}\alpha_{n,p-1}\lambda^2||r_{n,p}|| + \alpha_{n,p-2}\lambda||r_{n,p-1}|| + ||r_{n,p-2}|| + (\alpha_{n,p-2}\lambda + \alpha_{n,p-2}\alpha_{n,p-1}\lambda^2)\Gamma \\ &+ \left(\beta_{n,p-2} + \alpha_{n,p-2}\beta_{n,p-1}\lambda + \alpha_{n,p-2}\alpha_{n,p-1}\beta_{n,p}\lambda^2\right)\Gamma.\end{aligned}$$

Continuing the same procedures, we get

$$\begin{aligned}||v_{n,1} - z|| &\le (1 - \alpha_{n,2} - \beta_{n,2} + \alpha_{n,2}(1 - \alpha_{n,3} - \beta_{n,3})\lambda + \alpha_{n,2}\alpha_{n,3}(1 - \alpha_{n,4} - \beta_{n,4})\lambda^2 \\ &+ \cdots + \prod_{i=2}^{p-1}\alpha_{n,i}(1 - \alpha_{n,p} - \beta_{n,p})\lambda^{p-2} + \prod_{i=2}^{p}\alpha_{n,i}\lambda^{p-2}L\vartheta)||z_n - z|| \\ &+ \alpha_{n,2}||e'_{n,2}|| + \alpha_{n,2}\alpha_{n,3}\lambda||e'_{n,3}|| + \alpha_{n,2}\alpha_{n,3}\alpha_{n,4}\lambda^2||e'_{n,4}|| + \cdots + \prod_{i=2}^{p}\alpha_{n,i}\lambda^{p-2}||e'_{n,p}|| \\ &+ \prod_{i=2}^{p}\alpha_{n,i}\lambda^{p-2}L\vartheta b_n + ||e''_{n,2}|| + \alpha_{n,2}\lambda||e''_{n,3}|| + \alpha_{n,2}\alpha_{n,3}\lambda^2||e''_{n,4}|| + \cdots + \prod_{i=2}^{p-1}\alpha_{n,i}\lambda^{p-2}||e''_{n,p}|| \\ &+ ||r_{n,2}|| + \alpha_{n,2}\lambda||r_{n,3}|| + \alpha_{n,2}\alpha_{n,3}\lambda^2||r_{n,4}|| + \cdots + \prod_{i=2}^{p-1}\alpha_{n,i}\lambda^{p-2}||r_{n,p}|| \\ &+ \left(\alpha_{n,2}\lambda + \alpha_{n,2}\alpha_{n,3}\lambda^2 + \alpha_{n,2}\alpha_{n,3}\alpha_{n,4}\lambda^3 + \cdots + \prod_{i=2}^{p-1}\alpha_{n,i}\lambda^{p-2}\right)\Gamma \\ &+ \left(\beta_{n,2} + \alpha_{n,2}\beta_{n,3}\lambda + \alpha_{n,2}\alpha_{n,3}\beta_{n,4}\lambda^2 + \cdots + \prod_{i=2}^{p-1}\alpha_{n,i}\beta_{n,p}\lambda^{p-2}\right)\Gamma.\end{aligned} \tag{6.13}$$

Thus, applying (6.4) and (6.13), one has

$$
\begin{aligned}
||z_{n+1}-z|| \le{}& \Big(1-\alpha_{n,1}-\beta_{n,1}+\alpha_{n,1}(1-\alpha_{n,2}-\beta_{n,2})\lambda+\alpha_{n,1}\alpha_{n,2}(1-\alpha_{n,3}-\beta_{n,3})\lambda^2 \\
&+\cdots+\prod_{i=1}^{p-1}\alpha_{n,i}(1-\alpha_{n,p}-\beta_{n,p})\lambda^{p-1}+\prod_{i=1}^{p}\alpha_{n,i}\lambda^{p-1}L\vartheta\Big)||z_n-z|| \\
&+\alpha_{n,1}||e'_{n,1}||+\alpha_{n,1}\alpha_{n,2}\lambda||e'_{n,2}||+\alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\lambda^2||e'_{n,3}||+\cdots+\prod_{i=1}^{p}\alpha_{n,i}\lambda^{p-1}||e'_{n,p}|| \\
&+\prod_{i=1}^{p}\alpha_{n,i}\lambda^{p-1}L\vartheta b_n+||e''_{n,1}||+\alpha_{n,1}\lambda||e''_{n,2}||+\alpha_{n,1}\alpha_{n,2}\lambda^2||e''_{n,3}||+\cdots+\prod_{i=1}^{p-1}\alpha_{n,i}\lambda^{p-1}||e''_{n,p}|| \\
&+||r_{n,1}||+\alpha_{n,1}\lambda||r_{n,2}||+\alpha_{n,1}\alpha_{n,2}\lambda^2||r_{n,3}||+\cdots+\prod_{i=1}^{p-1}\alpha_{n,i}\lambda^{p-1}||r_{n,p}|| \\
&+\left(\alpha_{n,1}\lambda+\alpha_{n,1}\alpha_{n,2}\lambda^2+\alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\lambda^3+\cdots+\prod_{i=1}^{p-1}\alpha_{n,i}\lambda^{p-1}\right)\Gamma \\
&+\left(\beta_{n,1}+\alpha_{n,1}\beta_{n,2}\lambda+\alpha_{n,1}\alpha_{n,2}\beta_{n,3}\lambda^2+\cdots+\prod_{i=1}^{p-1}\alpha_{n,i}\beta_{n,p}\lambda^{p-1}\right)\Gamma \\
\le{}& \left(1-(1-L\vartheta)\prod_{i=1}^{p}\alpha_{n,i}\lambda^{p-1}\right)||z_n-z||+\sum_{i=1}^{p}\prod_{j=1}^{i}\alpha_{n,j}\lambda^{i-1}||e'_{n,i}|| \\
&+\prod_{i=1}^{p}\alpha_{n,i}\lambda^{p-1}L\vartheta b_n+\sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}\lambda^{i-1}||e''_{n,i}||+\sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}\lambda^{i-1}||r_{n,i}|| \\
&+||e''_{n,1}||+||r_{n,1}||+\left(\sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}\lambda^{i-1}+\sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}\beta_{n,i}\lambda^{i-1}+\beta_{n,1}\right)\Gamma \\
\le{}& \left(1-(1-L\vartheta)\prod_{i=1}^{p}\alpha_{n,i}\lambda^{p-1}\right)||z_n-z||+(1-L\vartheta)\prod_{i=1}^{p}\alpha_{n,i}\lambda^{p-1}\frac{\frac{\sum_{i=1}^{p}\prod_{j=1}^{i}\alpha_{n,j}\lambda^{i-1}||e'_{n,i}||}{\alpha\lambda^{p-1}}+L\vartheta b_n}{1-L\vartheta} \\
&+\sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}\lambda^{i-1}||e''_{n,i}||+\sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}\lambda^{i-1}||r_{n,i}||+||e''_{n,1}||+||r_{n,1}|| \\
&+\left(\sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}\lambda^{i-1}+\sum_{i=2}^{p}\prod_{j=1}^{i-1}\alpha_{n,j}\beta_{n,i}\lambda^{i-1}+\beta_{n,1}\right)\Gamma.
\end{aligned}
\tag{6.14}
$$

If $L \ge 1$, then $L\gamma < 1$, where γ is the same as in (3.17), implies that

$$\sqrt{1-2\tau+\iota^2}+\frac{rL}{r-r'}\sqrt{\varpi^2-2\rho\kappa+\rho^2\sigma^2}<1,$$

whence deduce that $L\vartheta < 1$. For the case that $L < 1$, it is plain that $L\vartheta < 1$. In view of (4.3), we note that all the conditions of Lemma 4.7 hold and so, (6.14) and Lemma 4.7 guarantee that the sequence $\{z_n\}_{n=0}^{\infty}$ generated by Algorithm 6.1 converges strongly to the solution $z \in \mathcal{H}$ of the problem (5.1) and there is nothing to prove. This completes the proof.

As in the proof of Theorem 6.8, we can prove the convergence of iterative sequences generated by Algorithms 6.2-6.5 and we omit their proofs.

Theorem 6.9. *Suppose that T, f, g, ρ and S are the same as in Theorem 6.8 and let all the conditions of Theorem 6.8 hold. If there exists a constant* $\alpha > 0$ *such that* $\prod_{i=1}^{p}\alpha_{n,i} > \alpha$ *for each* $n \ge 0$, *then the iterative sequence* $\{z_n\}_{n=0}^{\infty}$ *generated by Algorithm 6.2 converges strongly to the only element of* EGNWHE(K_r, T, f, g).

Theorem 6.10. *Let T, f, ρ and S be the same as in* Theorem 4.9 *and let all the conditions of Theorem 4.9 hold. If there exists a constant* $\alpha > 0$ *such that* $\prod_{i=1}^{p}\alpha_{n,i} > \alpha$ *for each* $n \ge 0$, *then the iterative sequence* $\{z_n\}_{n=0}^{\infty}$ *generated by Algorithm 6.3 converges strongly to the only element of* GNWHE(K_r, T, f).

Theorem 6.11. *Let T, g and ρ be the same as in Theorem 3.5 and suppose that the conditions* (a), (b) *and* (3.19) *in Theorem 3.5 hold. Let $S : K_r \to K_r$ be a nearly uniformly L-Lipschitzian mapping with the sequence $\{b_n\}_{n=0}^{\infty}$ such that for any $u \in$* GNRNVID(K_r, T, g), *$g(u) \in$* Fix(S). *Further, let $L\tilde{\theta} < 1$, where $\tilde{\theta}$ is the same as in* Theorem 4.10. *If there exists a constant $\alpha > 0$ such that $\prod_{i=1}^{p} \alpha_{n,i} > \alpha$ for each $n \geq 0$, then the iterative sequence $\{z_n\}_{n=0}^{\infty}$ generated by Algorithm 6.4 converges strongly to the only element of* GNWHE(K_r, T, g).

Theorem 6.12. *Assume that T, ρ and S are the same as in Theorem 4.11 and let all the conditions of Theorem 4.11 hold. If there exists a constant $\alpha > 0$ such that $\prod_{i=1}^{p} \alpha_{n,i} > \alpha$ for each $n \geq 0$, then the iterative sequence $\{z_n\}_{n=0}^{\infty}$ generated by Algorithm 6.5 converges strongly to the only element of* NWHE(K_r, T).

7 Some remarks

In view of Definition 2.11, we note that the condition relaxed cocoercivity of the operator T is weaker than the condition strongly monotonicity of T. In other words, the class of relaxed cocoercive operators is more general than the class of strongly monotone operators. In the present section, we shall show that unlike claims of Noor [38], Noor et al. [44], Qin and Noor [47], they studied the convergence analysis of the proposed iterative algorithms under the condition of strongly monotonicity, not the mild condition relaxed cocoercivity.

Noor [38] proposed the following three-step iterative algorithm and its special forms for finding a common element of two different sets of solutions of the fixed points of the nonexpansive mappings and the general variational inequalities (3.9) and studied convergence analysis of the suggested iterative algorithm under some conditions.

Algorithm 7.1. (Algorithm 2.1 [38]) *For any $x_0 \in \mathcal{H}$, compute the approximate solution x_n by the iterative schemes*

$$\begin{cases} z_n = (1 - c_n)x_n + c_n S\{x_n - g(x_n) + P_K[g(x_n) - \rho T x_n]\}, \\ y_n = (1 - b_n)x_n + b_n S\{z_n - g(z_n) + P_K[g(z_n) - \rho T z_n]\}, \\ x_{n+1} = (1 - a_n)x_n + a_n S\{y_n - g(y_n) + P_K[g(y_n) - \rho T y_n]\}, \end{cases}$$

where $a_n, b_n, c_n \in [0, 1]$ for all $n \geq 0$ and S is the nonexpansive operator.

Theorem 7.2. (Theorem 3.1 [38]) *Let K be a closed convex subset of a real Hilbert space $\mathcal{H}$. Let T be a relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping of K into $\mathcal{H}$. Let g be a relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian mapping of K into $\mathcal{H}$ and S be a nonexpansive mapping of K into K such that $F(S) \cap \mathrm{GVI}(K, T, g) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by Algorithm 7.1, for any initial point $x_0 \in K$, with the following conditions:*

$$||\rho - \frac{r - \gamma\mu^2}{\mu^2}|| \leq \frac{\sqrt{(r - \gamma\mu^2)^2 - \mu^2 k(2 - k)}}{\mu^2}, \quad r > \gamma\mu^2 + \mu\sqrt{k(2 - k)}, \quad k < 1,$$

where

$$k = 2\sqrt{1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2},$$

a_n, b_n, $c_n \in [0, 1]$ and $\sum_{n=0}^{\infty} a_n = \infty$, then x_n obtained from Algorithm 7.1 converges strongly to $x^* \in F(S) \cap \mathrm{GVI}(K, T, g)$.

From $k = 2\sqrt{1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2}$, it follows that $2(r_1 - \gamma_1\mu_1^2) < 1 + \mu_1^2$. Accordingly, the condition $2(r_1 - \gamma_1\mu_1^2) < 1 + \mu_1^2$ should be added to the conditions of Theorem 7.2. On the other hand, the conditions $r > \gamma\mu^2 + \mu\sqrt{k(2-k)}$ and $k < 1$ imply that $r > \gamma\mu^2$. Since T is (γ, r)-relaxed cocoercive and μ-Lipschitz continuous, the condition $r > \gamma\mu^2$ guarantees that the operator T is $(r - \gamma\mu^2)$-strongly monotone. Therefore, one can rewrite Theorem 7.2 as follows.

Theorem 7.3. *Let K be a closed convex subset of a real Hilbert space $\mathcal{H}$ and let T be a ξ-strongly monotone and μ-Lipschitzian mapping of K into $\mathcal{H}$. Let g be a ξ_1-strongly monotone and μ_1-Lipschitzian mapping of K into $\mathcal{H}$ and S be a nonexpansive mapping of K into K such that $F(S) \cap \mathrm{GVI}(K, T, g) \neq \emptyset$. If the constant ρ satisfies the following condition:*

$$\begin{cases} |\rho - \frac{\xi^2}{\mu^2}| \le \frac{\sqrt{\xi^2 - \mu^2 k(2-k)}}{\mu^2}, \\ \xi > \mu\sqrt{k(2-k)}, \\ k = 2\sqrt{1 - 2\xi_1^2 + \mu_1^2} < 1, \\ 2\xi_1^2 < 1 + \mu_1^2, \end{cases}$$

and the sequence $\{a_n\}$ satisfies $k = 2\sqrt{1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2}$, then the iterative sequence $\{x_n\}$ generated by Algorithm 7.1 converges strongly to the only element of $F(S) \cap \mathrm{GVI}(K, T, g)$.

Remark 7.4. Theorem 3.2 in [38] is stated with the condition relaxed cocoercivity of the operators T and g. Similarly, by using the conditions of Theorem 3.2 [38], we note that the operators T and g are in fact strongly monotone. Hence, Theorem 3.2 [38] is proved with the condition strongly monotonicity of the operators T and g instead of the mild condition relaxed cocoercivity.

Noor et al. [44] presented the following iterative scheme and its special forms for finding the common element of the solution sets of the general variational inequalities (3.9) and the nonexpansive mappings.

Algorithm 7.5. (Algorithm 3.1 [44]) *For a given*

$$\begin{aligned} Tg^{-1}P_{K_r}z + \rho^{-1}Q_{K_r}z = 0 &\Leftrightarrow \rho Tg^{-1}P_{K_r}z + Q_{K_r}z = 0 \\ &\Leftrightarrow \rho Tg^{-1}P_{K_r}z + z - fg^{-1}P_{K_r}z = 0 \\ &\Leftrightarrow z = fg^{-1}P_{K_r}z - \rho Tg^{-1}P_{K_r}z \\ &\Leftrightarrow z = f(u) - \rho T(u). \end{aligned}$$

and some sequence $\{a_n\}$, $a_n \in$ [0, 1], compute the approximate solution z_{n+1} by the iterative schemes

$$\begin{cases} g(u_n) = SP_K z_n, \\ z_{n+1} = (1 - a_n)z_n + a_n\{g(u_n) - \rho Tu_n\}, \end{cases}$$

where S is a non-expansive operator.

They also studied convergence analysis of the suggested iterative algorithm under some conditions as follows:

Theorem 7.6. (Theorem 3.1 [44]) *Let K be a closed convex subset of a real Hilbert space $\mathcal{H}$. Let T be a relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping. Let g be a* *relaxed* (γ_1, r_1)*-cocoercive and* μ_1*-Lipschitzian mapping of K into* $\mathcal{H}$ *and S be a nonexpansive mapping of* $\mathcal{H}$ *into* $\mathcal{H}$ *such that* $F(S) \cap \text{GWHE}(\mathcal{H}, T, g, S) \neq \emptyset$. *Let* $\{z_n\}$ *be a sequence defined by Algorithm 7.5, for any initial point* $z_0 \in K$*, with the following conditions:*

$$||\rho - \frac{r - \gamma\mu^2}{\mu^2}|| \leq \frac{\sqrt{(r - \gamma\mu^2)^2 - \mu^2 k(2-k)}}{\mu^2},$$
$$r > \gamma\mu^2 + \mu\sqrt{k(2-k)}, \quad k < 1,$$

where

$$k = 2\sqrt{1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2},$$

$a_n \in [0,1]$ *and* $k = 2\sqrt{1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2},$, *then* z_n *obtained from* Algorithm 7.5 *converges strongly to* $z^* \in F(S) \cap \text{GWHE}(\mathcal{H}, T, g, S)$.

As in Theorem 7.2, since $k = 2\sqrt{1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2}$, it follows that $2(r_1 - \gamma_1\mu_1^2) < 1 + \mu_1^2$. Hence the condition $2(r_1 - \gamma_1\mu_1^2) < 1 + \mu_1^2$ should be added to Theorem 7.6. Moreover, by using the conditions $r > \gamma\mu^2 + \mu\sqrt{k(2-k)}$ and $k < 1$, (γ, r)-relaxed cocoercivity and μ-Lipschitz continuity of the operator T and (γ_1, r_1)-relaxed cocoercivity and μ_1-Lipschitz continuity of the operator g, we note that the operators T and g are $(r - \gamma\mu^2)$-strongly monotone and $(r_1 - \gamma_1\mu_1^2)$-strongly monotone, respectively. Therefore, Theorem 7.6 collapses to the following theorem.

Theorem 7.7. *Let K be a closed convex subset of a real Hilbert space* $\mathcal{H}$ *and let* $T : \mathcal{H} \to \mathcal{H}$ *be a* ξ*-strongly monotone and* μ*-Lipschitzian mapping. Let* $g : \mathcal{H} \to \mathcal{H}$ *be a* ξ_1*-strongly monotone and* μ_1*-Lipschitzian mapping and* $S : \mathcal{H} \to \mathcal{H}$ *be a nonexpansive mapping such that* $F(S) \cap \text{GVI}(K, T, g) \neq \emptyset$. *If the constant* ρ *satisfies the following condition:*

$$\begin{cases} |\rho - \frac{\xi^2}{\mu^2}| \leq \frac{\sqrt{\xi^2 - \mu^2 k(2-k)}}{\mu^2}, \\ \xi > \mu\sqrt{k(2-k)}, \\ k = 2\sqrt{1 - 2\xi_1^2 + \mu_1^2} < 1, \\ 2\xi_1^2 < 1 + \mu_1^2, \end{cases}$$

and the sequence $\{a_n\}$ *satisfies* $\sum_{n=0}^{\infty} a_n = \infty$, *then the iterative sequence* $\{z_n\}$ *generated by Algorithm 7.5 converges strongly to the only element of* $F(S) \cap \text{GWHE}(\mathcal{H}, T, g, S)$.

Therefore, Noor et al. [44] proved the strongly convergence of the iterative sequence $\{z_n\}$ generated by Algorithm 7.5, under the condition strongly monotonicity of the operators T and g, not under the mild condition relaxed cocoercivity.

Qin and Noor [47] proposed the following iterative algorithm and its special forms for solving the general variational inequalities (3.9).

Algorithm 7.8. (Algorithm 2.1 [47]) *For any* $z_0 \in K$*, compute the sequence* $\{z_n\}$ *by the iterative processes*

$$\begin{cases} g(u_n) = SP_K z_n, \\ z_{n+1} = (1-\alpha_n)z_n + \alpha_n[g(u_n) - \rho A u_n], \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* [0, 1] *and S is a nonexpansive mapping.*

They studied convergence analysis of the suggested iterative algorithm under some conditions as follows.

Theorem 7.9. (Theorem 3.1 [47]) *Let K be a closed convex subset of a real Hilbert space* $\mathcal{H}$. *Let* $g : \mathcal{H} \to \mathcal{H}$ *be a relaxed* (u_1, v_1)*-cocoercive and* μ_1*-Lipschitz continuous mapping,* $A : \mathcal{H} \to \mathcal{H}$ *be a relaxed* (u_2, v_2)*-cocoercive and* μ_2*-Lipschitz continuous mapping and S be a nonexpansive mapping from K into itself such that* $F(S) \neq \emptyset$. *Let* $\{z_n\}$, $\{u_n\}$ *and* $\{g(u_n)\}$ *be the sequences generated by Algorithm 7.8 and* $\{\alpha_n\}$ *be a sequence in* [0, 1]. *Assume that the following conditions are satisfied:*

(C1) $2\theta_1 + \theta_2 < 1$, *where* $\theta_1 = \sqrt{1 + \mu_1^2 - 2v_1 + 2u_1\mu_1^2}$ *and* $\theta_2 = \sqrt{1 + \rho^2\mu_2^2 - 2\rho v_2 + 2\rho u_2 \mu_2^2}$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the sequences $\{z_n\}$, $\{u_n\}$ *and* $\{g(u_n)\}$ *converge strongly to* $z^* \in \mathrm{WHE}(\mathcal{H}, A, S)$, $u^* \in \mathrm{VI}(K, A)$ *and* $g(u^*) \in F(S)$, *respectively.*

From the condition (C1), it follows that $2(v_1 - u_1\mu_1^2) < 1 + \mu_1^2$ and $2\rho(v_2 - u_2\mu_2^2) < 1 + \mu_2^2$. Therefore, these conditions should be added to Theorem 7.9. On the other hand, the condition (C1) implies that $v_i > u_i\mu_i^2$, for $i = 1, 2$. Because g is (u_1, v_1)-relaxed cocoercive and μ_1-Lipschitz continuous, the condition $v_1 > u_1\mu_1^2$ guarantees $(v_1 - u_1\mu_1^2)$-strongly monotonicity of the operator g. Similarly, from (u_2, v_2)-relaxed cocoercivity and μ_2-Lipschitz continuity of the operator A and the condition $v_2 > u_2\mu_2^2$, it follows that the operator A is $(v_2 - u_2\mu_2^2)$-strongly monotone. Hence Theorem 7.9 reduces to the following theorem:

Theorem 7.10. *Let K be a closed convex subset of a real Hilbert space* $\mathcal{H}$ *and let* $g : \mathcal{H} \to \mathcal{H}$ *be a* ξ_1*-strongly monotone and* μ_1*-Lipschitz continuous mapping,* $A : \mathcal{H} \to \mathcal{H}$ *be* ξ_2*-strongly monotone and* μ_2*-Lipschitz continuous mapping and let S be a nonexpansive mapping from K into itself such that* $F(S) \neq \emptyset$. *Let* $\{z_n\}$, $\{u_n\}$ *and* $\{g(u_n)\}$ *be sequences generated by Algorithm 7.8. If the following conditions hold:*

(C1) $2\theta_1 + \theta_2 < 1$, *where* $\theta_1 = \sqrt{1 + \mu_1^2 - 2\xi_1}$ *and* $\theta_2 = \sqrt{1 + \rho^2\mu_2^2 - 2\rho\xi_2}$;

(C2) $2\xi_1 < 1 + \mu_1^2$, $2\rho\xi_2 < 1 + \rho\mu_1^2$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

then the iterative sequences $\{z_n\}$, $\{u_n\}$ *and* $\{g(u_n)\}$ *generated by Algorithm 7.5 converge strongly to* $z^* \in \mathrm{WHE}(\mathcal{H}, A, S)$, $u^* \in \mathrm{VI}(K, A)$ *and* $g(u^*) \in F(S)$, *respectively.*

Remark 7.11. (1) Qin and Noor in Remark 3.2 [47] claimed that Theorem 7.9 is obtained under the mild condition relaxed cocoercivity of the operators g and A. But, in view of the above facts, their results are obtained under the condition strongly monotonicity of the operators g and A not under the mild condition relaxed cocoercivity.

(2) The operators A and g in Corollaries 3.3 and 3.4 [47] are relaxed cocoercive. But we note that the conditions of the aforesaid corollaries guarantee that the operators A and g in these corollaries are in fact strongly monotone. Accordingly, Corollaries 3.3 and 3.4 in [47] are stated with the condition strongly monotonicity of the operators A and g instead of the mild condition relaxed cocoercivity.

Remark 7.12. In view of the above facts, we note that Theorems 4.8 and 4.10 generalize and improve Theorem 3.1 in [38]. Theorems 4.8, 4.10 and 4.11 generalize and improve Theorem 3.2 in [38]. Theorems 6.8-6.11 improve and generalize Theorem 3.2 in [42], Theorem 3.1 in [44] and [47] and Corollaries 3.3 and 3.4 in [47].

8 Conclusion

In this paper, we have introduced and considered some new classes of extended general nonlinear regularized nonconvex variational inequalities and the extended general nonconvex Wiener-Hopf equations involving three different nonlinear operators. By the projection operator technique, we have established the equivalence between the extended general nonlinear regularized nonconvex variational inequalities and the fixed point problems as well as the extended general nonconvex Wiener-Hopf equations. Then by this equivalent formulation, we have discussed the existence and uniqueness theorem for solution of the problem of extended general nonlinear regularized nonconvex variational inequalities. This equivalence and a nearly uniformly Lipschitzian mapping S are used to suggest and analyze some new perturbed p-step projection iterative algorithms with mixed errors for finding an element of the set of the fixed points of the nearly uniformly Lipschitzian mapping S which is unique solution of the problem of extended general nonlinear regularized nonconvex variational inequalities. We have presented some remarks about established statements by Noor [38], Noor et al. [44], Qin and Noor [47] and also have shown that their statements are special cases of our results. Several special cases are also discussed. It is expected that the results proved in this paper may simulate further research regarding the numerical methods and their applications in various fields of pure and applied sciences.

Acknowledgements
The authors are thankful to the referees for their helpful corrections, comments and valuable suggestions in preparation of the paper. The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant Number: 2011-0021821).

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 1 July 2011 Accepted: 24 November 2011 Published: 24 November 2011

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doi:10.1186/1687-1812-2011-86
Cite this article as: Balooee *et al.*: **The Wiener-Hopf Equation Technique for Solving General Nonlinear Regularized Nonconvex Variational Inequalities.** *Fixed Point Theory and Applications* 2011 **2011**:86.