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Fixed point theorems for mappings with condition (B)

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Abstract

In this article, a new type of mappings that satisfies condition (B) is introduced. We study Pazy's type fixed point theorems, demiclosed principles, and ergodic theorem for mappings with condition (B). Next, we consider the weak convergence theorems for equilibrium problems and the fixed points of mappings with condition (B).

Keywords: fixed point, equilibrium problem, Banach limit, generalized hybrid mapping, projection

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a mapping, and let $F(T)$ denote the set of fixed points of T. A mapping $T : C \to H$ is said to be nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in C$. A mapping $T : C \to H$ is said to be quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and $||Tx - Ty|| \leq ||x - y||$ for all $x \in C$ and $y \in F(T)$.

In 2008, Kohsaka and Takahashi [1] introduced nonspreading mapping, and obtained a fixed point theorem for a single nonspreading mapping, and a common fixed point theorem for a commutative family of nonspreading mappings in Banach spaces. A mapping $T : C \to C$ is called nonspreading [1] if

$$2\,\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. Indeed, $T : C \to C$ is a nonspreading mapping if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$ [2].

Recently, Takahashi and Yao [3] introduced two nonlinear mappings in Hilbert spaces. A mapping $T : C \to C$ is called a TY-1 mapping [3] if

$$2\,\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2$$

for all $x, y \in C$. A mapping $T : C \to C$ is called a TY-2 [3] mapping if

$$3\,\|Tx - Ty\|^2 \leq 2\,\|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$.

In 2010, Takahashi [4] introduced the hybrid mappings. A mapping $T : C \to C$ is hybrid [4] if

$$\| Tx - Ty\|^2 \leq \| x - y\|^2 + \langle x - Tx, y - Ty\rangle$$

for each $x, y \in C$. Indeed, $T : C \to C$ is a hybrid mapping if and only if

$$3 \| Tx - Ty\|^2 \leq \| x - y\|^2 + \| Tx - y\|^2 + \| Ty - x\|^2$$

for all $x, y \in C$ [4].

In 2010, Aoyoma et al. [5] introduced λ-hybrid mappings in a Hilbert space. Note that the class of λ-hybrid mappings contain the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings. Let λ be a real number. A mapping $T : C \to C$ is called λ-hybrid [5] if

$$\| Tx - Ty\|^2 \leq \| x - y\|^2 + 2\lambda\langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$.

In 2010, Kocourek et al. [6] introduced (α, β)-generalized hybrid mappings, and studied fixed point theorems and weak convergence theorems for such nonlinear mappings in Hilbert spaces. Let $\alpha, \beta \in \mathbb{R}$. A mapping $T : C \to H$ is (α, β)-generalized hybrid [6] if

$$\alpha \| Tx - Ty\|^2 + (1 - \alpha) \| Ty - x\|^2 \leq \beta \| Tx - y\|^2 + (1 - \beta) \| x - y\|^2$$

for all $x, y \in C$.

In 2011, Aoyama and Kohsaka [7] introduced α-nonexpansive mapping on Banach spaces. Let C be a nonempty closed convex subset of a Banach space E, and let α be a real number such that $\alpha < 1$. A mapping $T : C \to E$ is said to be α-nonexpansive if

$$\| Tx - Ty\|^2 \leq \alpha \| Tx - y\|^2 + \alpha \| Ty - x\|^2 + (1 - 2\alpha) \| x - y\|^2$$

for all $x, y \in C$.

Furthermore, we observed that Suzuki [8] introduced a new class of nonlinear mappings which satisfy condition (C) in Banach spaces. Let C be a nonempty subset of a Banach space E. Then, $T : C \to E$ is said to satisfy condition (C) if for all $x, y \in C$,

$$\frac{1}{2} \| x - Tx \| \leq \| x - y \| \Rightarrow \| Tx - Ty \| \leq \| x - y \| .$$

In fact, every nonexpansive mapping satisfies condition (C), but the converse may be false [8, Example 1]. Besides, if $T : C \to E$ satisfies condition (C) and $F(T) \neq \emptyset$, then T is a quasi-nonexpansive mapping. However, the converse may be false [8, Example 2].

Motivated by the above studies, we introduced Takahashi's $(\frac{1}{2}, \frac{1}{2})$-generalized hybrid mappings with Suzuki's sense on Hilbert spaces.

Definition 1.1. Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to H$ be a mapping. Then, we say T satisfies condition (B) if for all $x, y \in C$,

$$\frac{1}{2} \| x - Tx \| \leq \| x - y \| \Rightarrow \| Tx - Ty\|^2 + \| x - Ty\|^2 \leq \| Tx - y\|^2 + \| x - y\|^2.$$

Remark 1.1.

(i) In fact, if T is the identity mapping, then T satisfies condition (B).
(ii) Every $(\frac{1}{2}, \frac{1}{2})$-generalized hybrid mapping satisfies condition (B). But the converse may be false.
(iii) If $T : C \to C$ satisfies condition (B) and $F(T) \neq \emptyset$, then T is a quasi-nonexpansive mapping, and this implies that $F(T)$ is a closed convex subset of C [9].

Remark 1.2. Let $H = \mathbb{R}$, let C be nonempty closed convex subset of H, and let $T : C \to H$ be a function. In fact, we have

$$\begin{aligned}
&\frac{1}{2} \parallel x - Tx \parallel \le \parallel x - y \parallel \\
&\Leftrightarrow (Tx)^2 + x^2 - 2xTx \le 4x^2 + 4y^2 - 8xy \\
&\Leftrightarrow (Tx)^2 - 2xTx \le 3x^2 + 4y^2 - 8xy \\
&\Leftrightarrow Tx(Tx - 2x) \le (3x - 2y)(x - 2y),
\end{aligned}$$

and

$$\begin{aligned}
&|Tx - Ty|^2 + |x - Ty|^2 \le |Tx - y|^2 + |x - y|^2 \\
&\Leftrightarrow (Tx)^2 + (Ty)^2 - 2TxTy + x^2 + (Ty)^2 - 2xTy \le (Tx)^2 + y^2 - 2yTx + x^2 + y^2 - 2xy \\
&\Leftrightarrow 2(Ty)^2 - 2Ty(Tx + x) \le 2y^2 - 2y(Tx + x) \\
&\Leftrightarrow 2(Ty)^2 - 2y^2 \le 2(Ty - y)(Tx + x) \\
&\Leftrightarrow (Ty - y)(Ty + y) \le (Ty - y)(Tx + x) \\
&\Leftrightarrow (Ty - y)[(Ty + y) - (Tx + x)] \le 0.
\end{aligned}$$

Example 1.1. Let $H = C = \mathbb{R}$, and let $T : C \to H$ be defined by $Tx := -x$ for each $x \in C$. Hence, we have the following conditions:

(1) T is $(\frac{1}{2}, \frac{1}{2})$-generalized hybrid mapping, and T satisfies condition (B).
(2) T is not a nonspreading mapping. Indeed, if $x = 1$ and $y = -1$, then

$$2 \parallel Tx - Ty \parallel^2 = 8 > 0 = \parallel Tx - y \parallel^2 + \parallel Ty - x \parallel^2.$$

(3) T is not a TY-1 mapping. Indeed, if $x = 1$ and $y = -1$, then

$$2 \parallel Tx - Ty \parallel^2 = 8 > 4 = 4 + 0 = \parallel x - y \parallel^2 + \parallel Tx - y \parallel^2.$$

(4) T is not a TY-2 mapping. Indeed, if $x = 1$ and $y = -1$, then

$$3 \parallel Tx - Ty \parallel^2 = 12 > 0 = 2 \parallel Tx - y \parallel^2 + \parallel Ty - x \parallel^2.$$

(5) T is not a hybrid mapping. Indeed, if $x = 1$ and $y = -1$, then

$$3 \parallel Tx - Ty \parallel^2 = 12 > 4 = \parallel x - y \parallel^2 + \parallel Tx - y \parallel^2 + \parallel Ty - x \parallel^2.$$

(6) Now, we want to show that if $\alpha \neq 0$, then T is not a α-nonexpansive mapping. For $\alpha > 0$, let $x = 1$ and $y = -1$,

$$\parallel Tx - Ty \parallel^2 = 4 > 4 - 8\alpha = \alpha \parallel Tx - y \parallel^2 + \alpha \parallel Ty - x \parallel^2 + (1 - 2\alpha) \parallel x - y \parallel^2.$$

For $\alpha < 0$, let $x = y = 1$,

$$\| Tx - Ty\|^2 = 0 > 8\alpha = \alpha \| Tx - y\|^2 + \alpha \| Ty - x\|^2 + (1 - 2\alpha) \| x - y\|^2.$$

(7) Similar to (6), if $\alpha + \beta \neq 1$, then T is not a (α, β)-generalized hybrid mapping.

Example 1.2. Let $H = \mathbb{R}$, $C = [-1, 1]$, and let $T : C \to C$ be defined by

$$T(x) := \begin{cases} x & \text{if} \quad x \in [-1, 0], \\ -x & \text{if} \quad x \in (0, 1], \end{cases}$$

for each $x \in C$. First, we consider the following conditions:

(a) For $x \in [-1, 0]$ and $\frac{1}{2} \| x - Tx \| \leq \| x - y \|$, we know that

$(a)_1$ if $y \in [-1, 0]$, then $Ty = y$ and $(Ty - y)[(Ty + y) - (Tx + x)] = 0$;

$(a)_2$ if $y \in [0,1]$, then $Ty = -y$ and $(Ty - y)[(Ty + y) - (Tx + x)] = 4xy \leq 0$.

(b) For $x \in (0, 1]$ and $\frac{1}{2} \| x - Tx \| \leq \| x - y \|$, we know that

$(b)_1$ if $y \geq x$, then $x \leq y - x$, $Tx = -x$, and $Ty = -y$. So, $(Ty - y)[(Ty + y) - (Tx + x)] = 0$;

$(b)_2$ if $y < x$, then $x \leq x - y$ and this implies that $y \leq 0$. So, $(Ty - y)[(Ty + y) - (Tx + x)] = 0$.

By these conditions and Remark 1.2, we know that T satisfies condition (B). In fact, T is $(\frac{1}{2}, \frac{1}{2})$-generalized hybrid mapping. Furthermore, we know that the following conditions:

(1) T is a nonspreading mapping. Indeed, we know that the following conditions hold.

$(1)_1$ If $x > 0$ and $y > 0$, then

$$2 \| Tx - Ty\|^2 = 2 \| x - y\|^2 \leq 2 \| x + y\|^2 = \| Tx - y\|^2 + \| Ty - x\|^2;$$

$(1)_2$ If $x \leq 0$ and $y \leq 0$, then

$$2 \| Tx - Ty\|^2 = 2 \| x - y\|^2 = \| Tx - y\|^2 + \| Ty - x\|^2;$$

$(1)_3$ If $x > 0$ and $y \leq 0$, then $||Tx - Ty||^2 = ||Tx - y||^2 = || x+y||^2$, and $||Ty - x||^2 = ||x - y||^2$. Hence,

$$\| Tx - y\|^2 + \| Ty - x\|^2 - 2 \| Tx - Ty\|^2 = -4xy \geq 0.$$

(2) Similar to the above, we know that T is a TY-1 mapping, a TY-2 mapping, a hybrid mapping, (α, β)-generalized hybrid mapping, and T is a α-nonexpansive mapping.

On the other hand, the following iteration process is known as Mann's type iteration process [10] which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N},$$

where the initial guess x_0 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in [0,1].

In 1974, Ishikawa [11] gave an iteration process which is defined recursively by

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ y_n := (1 - \beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} := (1 - \alpha_n)x_n + \alpha_n Ty_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1].

In 1995, Liu [12] introduced the following modification of the iteration method and he called Ishikawa iteration method with errors: for a normed space E, and $T : E \to E$ a given mapping, the Ishikawa iteration method with errors is the following sequence

$$\begin{cases} x_1 \in E \text{ chosen arbitrary,} \\ y_n := (1 - \beta_n)x_n + \beta_n Tx_n + u_n, \\ x_{n+1} := (1 - \alpha_n)x_n + \alpha_n Ty_n + v_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1], and $\{u_n\}$ and $\{v_n\}$ are sequences in E with $\sum_{n=1}^{\infty} \| u_n \| < \infty$ and $\sum_{n=1}^{\infty} \| v_n \| < \infty$.

In 1998, Xu [13] introduced an Ishikawa iteration method with errors which appears to be more satisfactory than the one introduced by Liu [12]. For a nonempty convex subset C of E and $T : C \to C$ a given mapping, the Ishikawa iteration method with errors is generated by

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ y_n := a_n x_n + b_n Tx_n + c_n u_n, \\ x_{n+1} := a'_n x_n + b'_n Ty_n + c'_n v_n, \end{cases}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ are sequences in [0,1] with $a_n + b_n + c_n + = 1$ and $a'_n + b'_n + c'_n = 1$, and $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C.

Motivated by the above studies, we consider an Ishikawa iteration method with errors for mapping with condition (B).

We also consider the following iteration for mappings with condition (B). Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : C \times C \to \mathbb{R}$ be a function. Let $T : C \to H$ be a mapping. Let $\{a_n\}$, $\{b_n\}$, and $\{\theta_n\}$ be sequences in [0,1] with $a_n + b_n + \theta_n = 1$. Let $\{\omega_n\}$ be a bounded sequence in C. Let $\{r_n\}$ be a sequence of positive real numbers. Let $\{x_n\}$ be defined by $u_1 \in H$

$$\begin{cases} x_n \in C \text{ such that } G(x_n, y) + \dfrac{1}{r_n}\langle y - x_n, x_n - u_n \rangle \geq 0 \forall y \in C; \\ u_{n+1} := a_n x_n + b_n Tx_n + \theta_n \omega_n. \end{cases}$$

Furthermore, we observed that Phuengrattana [14] studied approximating fixed points of for a nonlinear mapping T with condition (C) by the Ishikawa iteration method on uniform convex Banach space with Opial property. Here, we also consider the Ishikawa iteration method for a mapping T with condition (C) and improve some conditions of Phuengrattana's result.

In this article, a new type of mappings that satisfies condition (B) is introduced. We study Pazy's type fixed point theorems, demiclosed principles, and ergodic theorem for mappings with condition (B). Next, we consider the weak convergence theorems for equilibrium problems and the fixed points of mappings with condition (B).

2 Preliminaries

Throughout this article, let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|| \cdot ||$, respectively. We denote the strongly convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [15], for each $x, y \in H$ and $\lambda \in [0,1]$, we have

$$\| \lambda x + (1 - \lambda)y\|^2 = \lambda \| x\|^2 + (1 - \lambda) \| y\|^2 - \lambda(1 - \lambda) \| x - y\|^2.$$

Hence, we also have

$$2\langle x - y, u - v\rangle = \| x - v\|^2 + \| y - u\|^2 - \| x - u\|^2 - \| y - v\|^2$$

for all $x, y, u, v \in H$. Furthermore, we know that

$$\| \alpha x+\beta y+\gamma z\|^2 = \alpha \| x\|^2+\beta \| y\|^2+\gamma \| z\|^2-\alpha\beta \| x-y\|^2-\alpha\gamma \| x-z\|^2-\beta\gamma \| y-z\|^2$$

for each $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$ [16].

Let ℓ^∞ be the Banach space of bounded sequences with the supremum norm. Let μ be an element of $(\ell^\infty)^*$(the dual space of ℓ^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in \ell^\infty$. Sometimes, we denote by $\mu_n x_n$ the value $\mu(f)$. A linear functional μ on ℓ^∞ is called a mean if $\mu(e) = ||\mu|| = 1$, where $e = (1, 1, 1, \ldots)$. For $x = (x_1, x_2, x_3, \ldots)$, A Banach limit on ℓ^∞ is an invariant mean, that is, $\mu_n x_n = \mu_n x_{n+1}$ for any $n \in \mathbb{N}$. If μ is a Banach limit on ℓ^∞, then for $f = (x_1, x_2, x_3, \ldots) \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n x_n \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in \ell^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n x_n = a$. For details, we can refer [17].

Lemma 2.1. [17]*Let C be a nonempty closed convex subset of a Hilbert space H,* $\{x_n\}$ *be a bounded sequence in H, and* μ *be a Banach limit. Let* $g : C \to \mathbb{R}$ *be defined by* $g(z) := \mu_n||x_n - z||^2$ *for all* $z \in C$*. Then there exists a unique* $z_0 \in C$ *such that* $g(z_0) = \min_{z\in C} g(z)$.

Lemma 2.2. [17]*Let C be a nonempty closed convex subset of a Hilbert space H. Let* P_C *be the metric projection from H onto C. Then for each* $x \in H$*, we have* $\langle x - P_C x, P_C x - y\rangle \geq 0$ *for all* $y \in C$.

Lemma 2.3. [17]*Let D be a nonempty closed convex subset of a real Hilbert space H. Let* P_D *be the metric projection from H onto D, and let* $\{x_n\}_{n\in \mathbb{N}}$ *be a sequence in H. If* $x_n \rightharpoonup x_0$ *and* $P_D x_n \to y_0$*, then* $P_D x_0 = y_0$.

Lemma 2.4. [18]*Let D be a nonempty closed convex subset of a real Hilbert space H. Let* P_D *be the metric projection from H onto D. Let* $\{x_n\}_{n\in \mathbb{N}}$ *be a sequence in H with* $||x_{n+1} - u||^2 \leq (1 + \lambda_n) ||x_n - u||^2 + \delta_n$ *for all* $u \in D$ *and* $n \in \mathbb{N}$*, where* $\{\lambda_n\}$ *and* $\{\delta_n\}$ *are* *sequences of nonnegative real numbers such that* $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $\sum_{n=1}^{\infty} \delta_n < \infty$*. Then* $\{P_D x_n\}$ *converges strongly to an element of D.*

Lemma 2.5. [19]*Let* $\{s_n\}$ *and* $\{t_n\}$ *be two nonnegative sequences satisfying* $s_{n+1} \leq s_n + t_n$ *for each* $n \in \mathbb{N}$*. If* $\sum_{n=1}^{\infty} t_n < \infty$*, then* $\lim_{n\to\infty} s_n$ *exists.*

The equilibrium problem is to find $z \in C$ such that

$$G(z, y) \geq 0 \quad \text{foreach} \quad y \in C. \tag{2.1}$$

The solution set of equilibrium problem (2.1) is denoted by (EP). For solving the equilibrium problem, let us assume that the bifunction $G : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $G(x, x) = 0$ for each $x \in C$;

(A2) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} G(tz + (1 - t)x, y) \leq G(x, y)$;

(A4) for each $x \in C$, the scalar function $y \to G(x, y)$ is convex and lower semicontinuous.

Lemma 2.6. [20]*Let C be a nonempty closed convex subset of a real Hilbert space H. Let* $G : C \times C \to \mathbb{R}$ *be a bifunction which satisfies conditions (A1)-(A4). Let* $r > 0$ *and* $\times \in H$*. Then there exists* $z \in C$ *such that*

$$G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

Furthermore, if

$$T_r(x) := \{z \in C : G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C\},$$

then we have:

(i) T_r *is single-valued;*

(ii) T_r *is firmly nonexpansive, that is,* $||T_r x - T_r y||^2 \leq \langle T_r x - T_r y, x - y\rangle$ *for each* $x, y \in H$*;*

(iii) (EP) is a closed convex subset of C;

(iv) $(EP) = F(T_r)$*.*

3 Fixed point theorems

Proposition 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let* $T : C \to C$ *be a mapping with condition (B). Then for each* $x, y \in C$*, we have:*

(i) $||Tx - T^2x||^2 + ||x - T^2x|| \leq ||x - Tx||^2$;

(ii) $||Tx - T^2x|| \leq ||x - Tx||$ *and* $||x - T^2x|| \leq ||x - Tx||$;

(iii) *either* $\frac{1}{2}\| x - Tx \| \leq \| x - y \|$ *or* $\frac{1}{2}\| Tx - T^2x \| \leq \| Tx - y \|$ *holds;*

(iv) *either*

$$\| Tx - Ty \|^2 + \| x - Ty \|^2 \leq \| Tx - y \|^2 + \| x - y \|^2$$

$$\| T^2x - Ty \|^2 + \| Tx - Ty \|^2 \leq \| T^2x - y\|^2 + \| Tx - y \|^2$$

holds;

(v) $\lim_{n\to\infty} \| T^n x - T^{n+2}x \| = 0$.

Proof Since $\frac{1}{2} \| x - Tx \| \leq \| x - Tx \|$, it is easy to see (i) and (ii) are satisfied. (iii) Suppose that

$$\frac{1}{2} \| x - Tx \| > \| x - y \| \quad \text{and} \quad \frac{1}{2} \| Tx - T^2x \| > \| Tx - y \|$$

holds. So,

$$\begin{aligned} \| x - Tx \| &\leq \| x - y \| + \| y - Tx \| \\ &< \frac{1}{2} \| x - Tx \| + \frac{1}{2} \| Tx - T^2x \| \\ &\leq \frac{1}{2} \| x - Tx \| + \frac{1}{2} \| x - Tx \| = \| x - Tx \| . \end{aligned}$$

This is a contradiction. Therefore, we obtain the desired result. Next, it is easy to get (iv) by (iii).

(v): By (i), we know that

$$\| T^{n+1}x - T^{n+2}x \|^2 + \| T^n x - T^{n+2}x \|^2 \leq \| T^n x - T^{n+1}x \|^2 .$$

Then $\{\|T^n x - T^{n+1}x\|\}$ is a decreasing sequence, and $\lim_{n\to\infty} \| T^n x - T^{n+1}x \|$ exists. Furthermore, we have:

$$\lim_{n\to\infty} \| T^n x - T^{n+2}x\|^2 \leq \lim_{n\to\infty} \| T^n x - T^{n+1}x\|^2 - \lim_{n\to\infty} \| T^{n+1}x - T^{n+2}x\|^2 = 0.$$

So, $\lim_{n\to\infty} \| T^n x - T^{n+2}x \| = 0$.

Proposition 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a mapping with condition (B). Then for each $x, y \in C$,*

$$\langle Tx - Ty, y - Ty\rangle \leq \langle x - y, Ty - y\rangle + \| T^2x - x \| \cdot \| Ty - y \| .$$

Proof By Proposition 3.1(iv), for each $x, y \in C$, either

$$\| Tx - Ty\|^2 + \| x - Ty\|^2 \leq \| Tx - y\|^2 + \| x - y\|^2$$

or

$$\| T^2x - Ty\|^2 + \| Tx - Ty\|^2 \leq \| T^2x - y\|^2 + \| Tx - y\|^2$$

holds. In the first case, we have

$$\begin{aligned} & \| Tx - Ty\|^2 + \| x - Ty\|^2 \leq \| Tx - y\|^2 + \| x - y\|^2 \\ \Rightarrow & \| Tx - Ty\|^2 + \| x - y\|^2 + 2\langle x - y, y - Ty\rangle + \| Ty - y\|^2 \leq \| Tx - Ty\|^2 + \\ & 2\langle Tx - Ty, Ty - y\rangle + \| Ty - y\|^2 + \| x - y\|^2 \\ \Rightarrow & \langle x - y, y - Ty\rangle \leq \langle Tx - Ty, Ty - y\rangle \\ \Rightarrow & \langle Tx - Ty, y - Ty\rangle \leq \langle x - y, Ty - y\rangle. \end{aligned}$$

In the second case, we have

$$\begin{aligned}
&\| T^2x - Ty\|^2 + \| Tx - Ty\|^2 \le \| Tx - y\|^2 + \| T^2x - y\|^2 \\
\Rightarrow &\| T^2x - y\|^2 + 2\langle T^2x - y, y - Ty\rangle + \| y - Ty\|^2 + \| Tx - Ty\|^2 \le \| Tx - Ty\|^2 + \\
&2\langle Tx - Ty, Ty - y\rangle + \| y - Ty\|^2 + \| T^2x - y\|^2 \\
\Rightarrow &\langle Tx - Ty, y - Ty\rangle \le \langle T^2x - y, Ty - y\rangle \\
&\le \langle x - y, Ty - y\rangle + \| T^2x - x \| \cdot \| Ty - y \| .
\end{aligned}$$

Therefore, the proof is completed.

Remark 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a mapping with condition (B). Then for each $x, y \in C$, we have:

(a) $||Tx - Ty||^2 + ||x - Ty||^2 \le ||x - y||^2 + ||Tx - y||^2 + ||T^2x - x|| \cdot ||Ty - y||$.

(b) $\langle Tx - Ty, y - Ty\rangle \le \langle x - y, Ty - y\rangle + ||Tx - x|| \cdot ||Ty - y||$.

Proof By Proposition 3.2, it is easy to prove Remark 3.1.

The following theorem shows that demiclosed principle is true for mappings with condition (B).

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let* $T : C \to C$ *be a mapping with condition* (B). *Let* $\{x_n\}$ *be a sequence in C with* $x_n \rightharpoonup x$ *and* $\lim_{n\to\infty} \| x_n - Tx_n \| = 0$. *Then* $Tx = x$.

Proof By Remark 3.1, we get:

$$\langle Tx_n - Tx, x - Tx\rangle \le \langle x_n - x, Tx - x\rangle + \| x_n - Tx_n \| \cdot \| x - Tx \|$$

for each $n \in \mathbb{N}$. By assumptions, $\langle x - Tx , x - Tx \rangle \le 0$. So, $Tx = x$.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let* $T : C \to C$ *be a mapping with condition* (B). *Then* $\{T^n x\}$ *is a bounded sequence for some* $x \in C$ *if and only if* $F(T) \neq \emptyset$.

Proof For each $n \in \mathbb{N}$, let $x_n := T^n x$. Clearly, $\{x_n\}$ is a bounded sequence. By Lemma 2.1, there is a unique $z \in C$ such that $\mu_n \| x_n - z\|^2 = \min_{y\in C} \mu_n \| x_n - y\|^2$. By Proposition 3.2, for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\langle x_{n+1} - Tz, z - Tz\rangle \le \langle x_n - z, Tz - z\rangle + \| x_n - x_{n+2} \| \cdot \| z - Tz \| \\
\Rightarrow &\tfrac{1}{2} \| x_{n+1} - Tz\|^2 + \tfrac{1}{2} \| Tz - z\|^2 - \tfrac{1}{2} \| x_{n+1} - z\|^2 \\
&\le \tfrac{1}{2} \| x_n - z\|^2 + \tfrac{1}{2} \| z - Tz\|^2 - \tfrac{1}{2} \| x_n - Tz\|^2 + \| x_n - x_{n+2} \| \cdot \| z - Tz \| \\
\Rightarrow &\mu_n \| x_n - Tz\|^2 \le \mu_n \| x_n - z\|^2 + \mu_n \| x_n - x_{n+2} \| \cdot \| z - Tz \| .
\end{aligned}$$

By Proposition 3.1(v), $\mu_n||x_n - Tz||^2 \le \mu_n||x_n - z||^2$. This implies that $Tz = z$ and $F(T) \neq \emptyset$. Conversely, it is easy to see.

Corollary 3.1. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H, and let* $T : C \to C$ *be a mapping with condition* (B). *Then* $F(T) \neq \emptyset$.

The following theorem shows that Ballion's type Ergodic's theorem is also true for the mapping with condition (B).

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let* $T : C \to C$ *be a mapping with condition* (B). *Then the following conditions are equivalent:*

(i) *for each* $x \in C$, $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$ *converges weakly to an element of* C;

(ii) $F(T) \neq \emptyset$.

In fact, if $F(T) \neq \emptyset$, *then for each* $x \in C$, *we know that* $S_n x \rightharpoonup v$, *where* $v = \lim_{n\to\infty} P_{F(T)} T^n x$ *and* $P_{F(T)}$ *is the metric projection from H onto* $F(T)$.

Proof (i)⇒ (ii): Take any $x \in C$ and let x be fixed. Then there exists $v \in C$ such that $S_n x \rightharpoonup v$. By Proposition 3.2, for each $k \in \mathbb{N}$, we have:

$$\begin{aligned}
&\langle TT^k x - Tv, v - Tv\rangle \le \langle T^k x - v, Tv - v\rangle + \| T^2 T^k x - T^k x \| \cdot \| Tv - v \| \\
&\Rightarrow \langle T^{k+1} x - Tv, v - Tv\rangle \le \langle T^k x - v, Tv - v\rangle + \| T^{k+2} x - T^k x \| \cdot \| Tv - v \| \\
&\Rightarrow \sum_{k=0}^{n-2} \langle T^{k+1} x - Tv, v - Tv\rangle \\
&\quad \le \sum_{k=0}^{n-2} \langle T^k x - Tv, Tv - v\rangle + \sum_{k=0}^{n-2} \| T^{k+2} x - T^k x \| \cdot \| Tv - v \| \\
&\Rightarrow \langle nS_n x - x - (n-1)Tv, v - Tv\rangle \le \langle (n-1)S_{n-1} x - (n-1)Tv, Tv - v\rangle + \\
&\quad \sum_{k=0}^{n-2} \| T^{k+2} x - T^k x \| \cdot \| Tv - v \| \\
&\Rightarrow \langle \tfrac{n}{n-1} S_n x - \tfrac{x}{n-1} - Tv, v - Tv\rangle \\
&\quad \le \langle S_{n-1} x - Tv, Tv - v\rangle + \frac{1}{n-1}\sum_{k=0}^{n-2} \| T^{k+2} x - T^k x \| \cdot \| Tv - v \|.
\end{aligned}$$

By Proposition 3.1(v), $\lim_{n\to\infty} \| T^{k+2} x - T^k x \| = 0$. This implies that

$$\lim_{n\to\infty} \frac{1}{n-1}\sum_{k=0}^{n-2} \| T^{k+2} x - T^k x \| = 0.$$

Since $S_n x \rightharpoonup v$, we have:

$$\langle v - Tv, v - Tv\rangle \le \langle v - Tv, Tv - v\rangle.$$

So, $Tv = v$.

(ii)⇒ (i): Take any $x \in C$ and $u \in F(T)$, and let x and u be fixed. Since T satisfies condition (B), $||T^n x - u|| \le ||T^{n-1} x - u||$ for each $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} \| T^n x - u \|$ exists and this implies that $\{T^n x\}$ is a bounded sequence. By Lemma 2.4, there exists $z \in F(T)$ such that $\lim_{n\to\infty} P_{F(T)} T^n x = z$. Clearly, $z \in F(T)$. Besides, we have:

$$\| S_n x - u \| \le \frac{1}{n}\sum_{k=0}^{n-1} \| T^k x - u \| \le \| x - u \|.$$

So, $\{S_n x\}$ is a bounded sequence. Then there exist a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ and $v \in C$ such that $S_{n_i} x \rightharpoonup v$. By the above proof, we have:

$$\langle \tfrac{n}{n-1} S_n x - \tfrac{x}{n-1} - Tv, v - Tv\rangle \le \langle S_{n-1}x - Tv, Tv - v\rangle + \frac{1}{n-1}\sum_{k=0}^{n-2} \| T^{k+2}x - T^k x \| \cdot \| Tv - v \|.$$

This implies that

$$\begin{aligned} &\langle \tfrac{n_i}{n_i-1} S_{n_i} x - \tfrac{x}{n_i-1} - Tv, v - Tv\rangle \\ &\le \langle S_{n_i-1}x - \tfrac{x}{n_i-1} - Tv, Tv - v\rangle + \frac{1}{n_i-1}\sum_{k=0}^{n_i-2} \| T^{k+2}x - T^k x \| \cdot \| Tv - v \| . \end{aligned}$$

Since $S_{n_i}x \rightharpoonup v$, $\{T^n x\}$ is a bounded sequence, and $\lim_{n\to\infty} \frac{1}{n-1}\sum_{k=0}^{n-2} \| T^{k+2}x - T^k x \| = 0$, it is easy to see that $Tv = v$. So, $v \in F(T)$.

By Lemma 2.2, for each $k \in \mathbb{N}$, $\langle T^k x - P_{F(T)}T^k x, P_{F(T)}T^k x - u\rangle \ge 0$. This implies that

$$\begin{aligned} &\langle T^k x - P_{F(T)}T^k x, u - z\rangle \\ \le\ &\langle T^k x - P_{F(T)}T^k x, P_{F(T)}T^k x - z\rangle \\ \le\ &\| T^k x - P_{F(T)}T^k x \| \cdot \| P_{F(T)}T^k x - z \| \\ \le\ &\| T^k x - z \| \cdot \| P_{F(T)}T^k x - z \| \\ \le\ &\| x - z \| \cdot \| P_{F(T)}T^k x - z \| . \end{aligned}$$

Adding these inequalities from $k = 0$ to $k = n - 1$ and dividing by n, we have

$$\left\langle S_n x - \frac{1}{n}\sum_{k=0}^{n-1} P_{F(T)}T^k x, u - z \right\rangle \le \frac{\| x - z \|}{n}\sum_{k=0}^{n-1} \| P_{F(T)}T^k x - z \| .$$

Since $S_{n_k}x \rightharpoonup v$ and $P_{F(T)}T^k x \to z$, we get $\langle v - z, u - z\rangle \le 0$. Since u is any point of $F(T)$, we know that $v = z = \lim_{n\to\infty} P_{F(T)}T^n x$.

Furthermore, if $\{S_{n_i}x\}$ is a subsequence of $\{S_n x\}$ and $S_{n_i}x \rightharpoonup q$, then $q = v$ by following the same argument as the above proof. Therefore, $S_n x \rightharpoonup v = \lim_{n\to\infty} PT^n x$, and the proof is completed.

4 Weak convergence theorems with errors

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T_1, T_2 : C \to C$ be two mappings with condition (B) and $\Omega := F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{\theta_n\}$, and $\{\lambda_n\}$ be sequences in [0,1] with*

$$a_n + c_n + \theta_n = b_n + d_n + \lambda_n = 1, n \in \mathbb{N}.$$

Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in C. Let $\{x_n\}$ and $\{y_n\}$ be defined by

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ y_n := a_n x_n + c_n T_1 x_n + \theta_n u_n, \\ x_{n+1} := b_n x_n + d_n T_2 y_n + \lambda_n v_n. \end{cases}$$

Assume that:

(i) $\liminf_{n\to\infty} a_n c_n > 0$ *and* $\liminf_{n\to\infty} b_n d_n > 0$;

(ii) $\sum_{n=1}^{\infty} \theta_n < \infty$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Then $x_n \rightharpoonup z$ *and* $y_n \rightharpoonup z$, *where* $z = \lim_{n\to\infty} P_\Omega x_n$.

Proof Take any $w \in \Omega$ and let w be fixed. Then for each $n \in \mathbb{N}$, we have:

$$\begin{aligned}
&\|y_n - w\|^2\\
=\ &\|a_n x_n + c_n T_1 x_n + \theta_n u_n - w\|^2\\
=\ &a_n\|x_n - w\|^2 + c_n\|T_1x_n - w\|^2 + \theta_n\|u_n - w\|^2\\
&-a_nc_n\|x_n - T_1x_n\|^2 - a_n\theta_n\|x_n - w\|^2 - c_n\theta_n\|T_1x_n - u_n\|^2\\
\le\ &a_n\|x_n - w\|^2 + c_n\|x_n - w\|^2 + \theta_n\|u_n - w\|^2\\
&-a_nc_n\|x_n - T_1x_n\|^2 - a_n\theta_n\|x_n - w\|^2 - c_n\theta_n\|T_1x_n - u_n\|^2\\
\le\ &\|x_n - w\|^2 + \theta_n\|u_n - w\|^2,
\end{aligned}$$

and

$$\begin{aligned}
&\|x_{n+1} - w\|^2\\
=\ &\|b_nx_n + d_nT_2y_n + \lambda_nv_n - w\|^2\\
=\ &b_n\|x_n - w\|^2 + d_n\|T_2y_n - w\|^2 + \lambda_n\|v_n - w\|^2\\
&-b_nd_n\|x_n - T_2y_n\|^2 - b_n\lambda_n\|x_n - v_n\|^2 - d_n\lambda_n\|T_2y_n - v_n\|^2\\
\le\ &b_n\|x_n - w\|^2 + d_n\|y_n - w\|^2 + \lambda_n\|v_n - w\|^2\\
&-b_nd_n\|x_n - T_2y_n\|^2 - b_n\lambda_n\|x_n - v_n\|^2 - d_n\lambda_n\|T_2y_n - v_n\|^2\\
\le\ &b_n\|x_n - w\|^2 + d_n(\|x_n - w\|^2 + \theta_n\|u_n - w\|^2) + \lambda_n\|v_n - w\|^2\\
&-b_nd_n\|x_n - T_2y_n\|^2 - b_n\lambda_n\|x_n - v_n\|^2 - d_n\lambda_n\|T_2y_n - v_n\|^2\\
\le\ &\|x_n - w\|^2 + d_n\theta_n\|u_n - w\|^2 + \lambda_n\|v_n - w\|^2\\
&-b_nd_n\|x_n - T_2y_n\|^2 - b_n\lambda_n\|x_n - v_n\|^2 - d_n\lambda_n\|T_2y_n - v_n\|^2\\
\le\ &\|x_n - w\|^2 + d_n\theta_n\|u_n - w\|^2 + \lambda_n\|v_n - w\|^2.
\end{aligned}$$

By Lemma 2.5, $\lim_{n\to\infty}\|x_n - w\|$ exists. So, $\{x_n\}$ is a bounded sequence. Now, we set $\lim_{n\to\infty}\|x_n - w\| = t$. Besides,

$$\begin{aligned}
&b_nd_n\|x_n - T_2y_n\|^2 + b_n\lambda_n\|x_n - v_n\|^2 + d_n\lambda_n\|T_2y_n - v_n\|^2\\
\le\ &\|x_n - w\|^2 + d_n\theta_n\|u_n - w\|^2 + \lambda_n\|v_n - w\|^2 - \|x_{n+1} - w\|^2.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} b_nd_n\|x_n - T_2y_n\|^2 = 0.$$

By assumption, $\lim_{n\to\infty}\|x_n - T_2y_n\| = 0$. Furthermore, we have:

$$\|x_{n+1} - w\|^2 \le b_n\|x_n - w\|^2 + d_n\|y_n - w\|^2 + \lambda_n\|v_n - w\|^2.$$

This implies that

$$\begin{aligned}
&b_nd_n(\|x_n - w\|^2 + \theta_n\|u_n - w\|^2 - \|y_n - w\|^2)\\
&\le d_n(\|x_n - w\|^2 + \theta_n\|u_n - w\|^2 - \|y_n - w\|^2)\\
&\le (1 - b_n)\|x_n - w\|^2 + d_n\theta_n\|u_n - w\|^2 - d_n\|y_n - w\|^2\\
&\le \|x_n - w\|^2 - \|x_{n+1} - w\|^2 + \lambda_n\|v_n - w\|^2 + d_n\theta_n\|u_n - w\|^2.
\end{aligned}$$

Hence, $\lim_{n\to\infty} b_n d_n (\| x_n - w\|^2 + \theta_n \| u_n - w\|^2 - \| y_n - w\|^2) = 0$. By assumption,

$$\lim_{n\to\infty} (\| x_n - w\|^2 + \theta_n \| u_n - w\|^2 - \| y_n - w\|^2) = 0.$$

Since $\lim_{n\to\infty} \theta_n \| u_n - w\|^2 = 0$,

$$\lim_{n\to\infty} (\| x_n - w\|^2 - \| y_n - w\|^2) = 0.$$

Hence, $\lim_{n\to\infty} \| y_n - w \| = \lim_{n\to\infty} \| x_n - w \| = t$. Similar to the above proof, we also get

$$\lim_{n\to\infty} \| x_n - T_1 x_n \| = 0.$$

Besides,

$$\begin{aligned} \|y_n - x_n\| &= \| a_n x_n + c_n T_1 x_n + \theta_n u_n - x_n \| \\ &\le c_n \| T_1 x_n - x_n \| + \theta_n \| x_n - u_n \| \\ &\le \| T_1 x_n - x_n \| + \theta_n \| x_n - u_n \| . \end{aligned}$$

This implies that $\lim_{n\to\infty} \| y_n - x_n \| = 0$ and $\lim_{n\to\infty} \| y_n - T_2 y_n \| = 0$. Since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z$. By Theorem 3.1, $z = T_1 z$.

If x_{n_j} is a subsequence of $\{x_n\}$ and $x_{n_j} \rightharpoonup q$, then $T_1 q = q$. Suppose that $q \neq z$. Then we have:

$$\begin{aligned} \liminf_{k\to\infty} \| x_{n_k} - z \| &< \liminf_{k\to\infty} \| x_{n_k} - q \| = \lim_{n\to\infty} \| x_n - q \| = \lim_{j\to\infty} \| x_{n_j} - q \| \\ &< \liminf_{j\to\infty} \| x_{n_j} - z \| = \lim_{n\to\infty} \| x_n - z \| = \liminf_{k\to\infty} \| x_{n_k} - z \| . \end{aligned}$$

And this leads to a contradiction. Then every weakly convergent subsequence of x_n has the same limit. So, $x_n \rightharpoonup z \in F(T_1)$. Since $x_n \rightharpoonup z$ and $\lim_{n\to\infty} \| x_n - y_n \| = 0$, $y_n \rightharpoonup z$. By Theorem 3.1, $z \in F(T_2)$. Hence, $z \in \Omega$.

Next, by Lemma 2.4, $P_\Omega x_n$ converges. Then there exists $v \in \Omega$ such that $\lim_{n\to\infty} P_\Omega x_n = v$. By Lemma 2.3, $P_\Omega z = v$. Since $z \in \Omega$, $z = v = \lim_{n\to\infty} P_\Omega x_n$, and the proof is completed.

In Theorem 4.1, if $\theta_n = \lambda_n = 0$ for each $n \in \mathbb{N}$, then we have the following result.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T_1, T_2 : C \to C$ be two mappings with condition* (*B*) *and* $\Omega := F(T_1) \cap F(T_2) \neq \emptyset$. *Let $\{a_n\}$ and (b_n) be two sequences in* [0,1]. *Let $\{x_n\}$ be defined by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ y_n := a_n x_n + (1 - a_n) T_1 x_n, \\ x_{n+1} := b_n x_n + (1 - b_n) T_2 y_n. \end{cases}$$

Assume that $\liminf_{n\to\infty} a_n(1 - a_n) > 0$ *and* $\liminf_{n\to\infty} b_n(1 - b_n) > 0$. *Then $x_n \rightharpoonup z$ and $y_n \rightharpoonup z$, where* $z = \lim_{n\to\infty} P_\Omega x_n$.

Furthermore, we also have the following corollaries from Theorem 4.2.

Corollary 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a mapping with condition* (*B*) *and* $F(T) \neq \emptyset$. *Let $\{a_n\}$ and $\{b_n\}$ be* *two sequences in* [0,1]. *Let* $\{x_n\}$ *be defined by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ y_n := a_n x_n + (1 - a_n) T x_n, \\ x_{n+1} := b_n x_n + (1 - b_n) T y_n. \end{cases}$$

Assume that $\liminf_{n\to\infty} a_n(1 - a_n) > 0$ *and* $\liminf_{n\to\infty} b_n(1 - b_n) > 0$. *Then* $x_n \rightharpoonup z$ *and* $y_n \rightharpoonup z$, *where* $z = \lim_{n\to\infty} P_{F(T)} x_n$.

Corollary 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let* $T : C \to C$ *be a mapping with condition* (*B*) *and* $F(T) \neq \emptyset$. *Let* $\{b_n\}$ *be a sequence in* [0,1]. *Let* $\{x_n\}$ *be defined by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrary}, \\ x_{n+1} := b_n x_n + (1 - b_n) T x_n. \end{cases}$$

Assume that $\liminf_{n\to\infty} b_n(1 - b_n) > 0$. *Then* $x_n \rightharpoonup z$, *where* $z = \lim_{n\to\infty} P_{F(T)} x_n$.

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let* $G : C \times C \to \mathbb{R}$ *be a function satisfying (A1)-(A4). Let* $T : C \to C$ *be a mapping with condition* (*B*) *and* $\Omega := F(T) \cap (EP) \neq \emptyset$. *Let* $\{a_n\}$, $\{b_n\}$, *and* $\{\theta_n\}$ *be sequences in* [0,1] *with* $a_n + b_n + \theta_n = 1$. *Let* $\{\omega_n\}$ *be a bounded sequence in C. Let* $\{r_n\} \subseteq [a, \infty)$ *for some* $a > 0$. *Let* $\{x_n\}$ *be defined by* $u_1 \in H$

$$\begin{cases} x_n \in C \text{ such that } G(x_n, y) + \frac{1}{r_n}\langle y - x_n, x_n - u_n \rangle \geq 0 \quad \forall y \in C; \\ u_{n+1} := a_n x_n + b_n T x_n + \theta_n \omega_n. \end{cases}$$

Assume that: $\liminf_{n\to\infty} a_n b_n > 0$, *and* $\sum_{n=1}^{\infty} \theta_n < \infty$. *Then* $x_n \rightharpoonup z$, *where* $z = \lim_{n\to\infty} P_{(EP)\cap F(T)} x_n$.

Proof Take any $w \in \Omega$ and let w be fixed. Putting $x_n = T_{r_n} u_n$ for each $n \in \mathbb{N}$. Then we have:

$$\begin{aligned} & \| x_{n+1} - w \|^2 \\ = \; & \| T_{r_{n+1}} u_{n+1} - w \|^2 \\ \leq \; & \| u_{n+1} - w \|^2 \\ \leq \; & \| a_n x_n + b_n T x_n + \theta_n \omega_n - w \|^2 \\ \leq \; & a_n \| x_n - w \|^2 + b_n \| T x_n - w \|^2 + \theta_n \| \omega_n - w \|^2 - a_n b_n \| x_n - T x_n \|^2 \\ \leq \; & a_n \| x_n - w \|^2 + b_n \| x_n - w \|^2 + \theta_n \| \omega_n - w \|^2 - a_n b_n \| x_n - T x_n \|^2 \\ \leq \; & \| x_n - w \|^2 + \theta_n \| \omega_n - w \|^2 - a_n b_n \| x_n - T x_n \|^2. \end{aligned}$$

By Lemma 2.5, $\lim_{n\to\infty} \| x_n - w \|$ exists. So, $\{x_n\}$ is bounded. Furthermore, we have:

(a) $\lim_{n\to\infty} a_n b_n \| x_n - T x_n \|^2 = 0$;

(b) $\lim_{n\to\infty} \| x_n - T x_n \| = 0$;

(c) $\| u_{n+1} - x_n \| = \| b_n T x_n - b_n x_n + \theta_n w_n - \theta_n x_n \| \leq \| T x_n - x_n \| + \theta_n \| w_n - x_n \|$;

(d) $\lim_{n\to\infty} \| u_{n+1} - x_n \| = 0$;

(e) $\lim_{n\to\infty} \| u_n - w \| = \lim_{n\to\infty} \| x_n - w \|$.

Following the same argument as the proof of Theorem 4.2, there exists $z \in C$ such that $x_n \rightharpoonup z$ and $Tz = z$. Besides, we also have

$$\begin{aligned}\|x_{n+1} - w\|^2 &= \| T_{r_{n+1}} u_{n+1} - w\|^2 \\ &= \| T_{r_{n+1}} u_{n+1} - T_{r_{n+1}} w\|^2 \\ &\le \langle T_{r_{n+1}} u_{n+1} - T_{r_{n+1}} w, \ u_{n+1} - w\rangle \\ &\le \langle x_{n+1} - w, u_{n+1} - w\rangle \\ &= \frac{1}{2} \| x_{n+1} - w\|^2 + \frac{1}{2} \| u_{n+1} - w\|^2 - \frac{1}{2} \| x_{n+1} - u_{n+1}\|^2.\end{aligned}$$

This implies that

$$\| x_{n+1} - u_{n+1}\|^2 \le \| u_{n+1} - w\|^2 - \| x_{n+1} - w\|^2.$$

By (e), $\lim_{n\to\infty} \| x_n - u_n \| = 0$. Next, we want to show that $z \in (EP)$. Since $x_n = T_{r_n} u_n$,

$$G(x_n, y) + \frac{1}{r_n}\langle y - x_n, x_n - u_n\rangle \ge 0 \quad \forall y \in C.$$

By (A2),

$$\frac{1}{r_n}\langle y - x_n, x_n - u_n\rangle \ge G(y, x_n) \quad \forall y \in C.$$

By (A4), (i), and $\lim_{n\to\infty} \| x_n - u_n \| = 0$, we get

$$0 \ge \lim_{n\to\infty} G(y, x_n) \ge G(y, z) \quad \forall y \in C.$$

By (A2), $G(z, y) \ge 0$ for all $y \in C$. So, $z \in (EP) \cap F(T) = \Omega$. By Lemma 2.4, there exists $v \in (EP) \cap F(T)$ such that $\lim_{n\to\infty} P_{(EP)\cap F(T)} x_n = v$. By Lemma 2.3, $z = P_{(EP)\cap F(T)} z = v$, and the proof is completed.

In Theorem 4.3, if $\theta_n = 0$ for each $n \in \mathbb{N}$, then we have the following result.

Theorem 4.4. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : C \times C \to \mathbb{R}$ be a function satisfying (A1)-(A4). Let $T : C \to C$ be a mapping with condition (B) and $\Omega := F(T) \cap (EP) \neq \emptyset$. Let $\{a_n\}$ be a sequence in [0,1]. Let $\{x_n\}$ be defined by $u_1 \in H$

$$\begin{cases} x_n \in C \text{ such that } G(x_n, y) + \frac{1}{r_n}\langle y - x_n, x_n - u_n\rangle \ge 0 \quad \forall y \in C; \\ u_{n+1} := a_n x_n + (1 - a_n) T x_n. \end{cases}$$

Assume that: $\{r_n\} \subseteq [a, \infty)$ for some $a > 0$ and $\liminf_{n\to\infty} a_n(1 - a_n) > 0$. Then $x_n \rightharpoonup z$, where $z = \lim_{n\to\infty} P_{(EP)\cap F(T)} x_n$.

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Authors' contributions

LJL is responsible for problem resign, coordinator, discussion, revise the important part, and submit. CSC is responsible for the important results of this article, discuss, and draft. ZTY is responsible for giving the examples of this types of problems, discussion. All authors read and approved the final manuscript.

Competing interests

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