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Weak and strong convergence theorems of implicit iteration process on Banach spaces

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Abstract

In this article, we first consider weak convergence theorems of implicit iterative processes for two nonexpansive mappings and a mapping which satisfies condition (*C*). Next, we consider strong convergence theorem of an implicit-shrinking iterative process for two nonexpansive mappings and a relative nonexpansive mapping on Banach spaces. Note that the conditions of strong convergence theorem are different from the strong convergence theorems for the implicit iterative processes in the literatures. Finally, we discuss a strong convergence theorem concerning two nonexpansive mappings and the resolvent of a maximal monotone operator in a Banach space.

1 Introduction

Let E be a Banach space, and let C be a nonempty closed convex subset of E. A mapping $T: C \to E$ *is* nonexpansive if $||Tx - Ty|| \le ||x - y||$ for every $x, y \in C$. Let $F(T) := \{x \in C: x = Tx\}$ denote the set of fixed points of T. Besides, a mapping $T: C \to E$ is quasinonexpansive if $F(T) \neq \emptyset$ and $||Tx - y|| \le ||x - y||$ for all $x \in C$ and $y \in F(T)$.

In 2008, Suzuki [1] introduced the following generalized nonexpansive mapping on Banach spaces. A mapping $T: C \to E$ is said to satisfy condition (C) if for all $x, y \in C$,

$$\frac{1}{2}||x - Tx|| \le ||x - y|| \Rightarrow ||Tx - Ty|| \le ||x - y||.$$

In fact, every nonexpansive mapping satisfies condition (C), but the converse may be false [1, Example 1]. Besides, if $T: C \to E$ satisfies condition (C) and $F(T) \neq \emptyset$, then T is a quasinonexpansive mapping. However, the converse may be false [1, Example 2].

Construction of approximating fixed points of nonlinear mappings is an important subject in the theory of nonlinear mappings and its applications in a number of applied areas.

Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T: C \to C$ be a mapping. In 1953, Mann [2] gave an iteration process:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \ge 0, \tag{1.1}$$

where x_0 is taken in C arbitrarily, and $\{\alpha_n\}$ is a sequence in [0,1].

In 2001, Soltuz [3] introduced the following Mann-type implicit process for a nonexpansive mapping $T: C \to C$:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N}, \tag{1.2}$$

where x_0 is taken in C arbitrarily, and $\{t_n\}$ is a sequence in [0,1].

In 2001, Xu and Ori [4] have introduced an implicit iteration process for a finite family of nonexpansive mappings. *Let* $T_1, T_2, \ldots, T_N$ be N self-mappings of C and suppose that $F := \cap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of T_i, $i = 1, 2, \ldots, N$. Let I: = $\{1, 2, \ldots, N\}$. Xu and Ori [4] gave an implicit iteration process for a finite family of nonexpansive mappings:

$$x_n = t_n x_{n-1} + (1 - t_n) T_n x_n, \quad n \in \mathbb{N}, \tag{1.3}$$

where x_0 is taken in C arbitrarily, $\{t_n\}$ is a sequence in [0,1], and $T_k = T_{k \bmod N}$. (Here the mod N function takes values in I.) And they proved the weak convergence of process (1.3) to a common fixed point in the setting of a Hilbert space.

In 2010, Khan et al. [5] presented an implicit iterative process for two nonexpansive mappings in Banach spaces. Let E be a Banach space, and let C be a nonempty closed convex subset of E, and let T, S: $C \to C$ be two nonexpansive mappings. Khan et al. [5] considered the following implicit iterative process:

$$x_n = \alpha_n x_{n-1} + \beta_n S x_n + \gamma_n T x_n, \quad n \in \mathbb{N}, \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in [0,1] with $\alpha_n + \beta_n + \gamma_n = 1$.

Motivated by the above works in [5], we want to consider the following implicit iterative process. Let E be a Banach space, C be a nonempty closed convex subset of E, and let $T_1, T_2 : C \to C$ be two nonexpansive mappings, and let S: $C \to C$ be a mapping which satisfy condition (C). We first consider the weak convergence theorems for the following implicit iterative process:

$$\begin{cases} x_0 \in C \text{ chosen arbitrary}, \\ x_n = a_n x_{n-1} + b_n S x_{n-1} + c_n T_1 x_n + d_n T_2 x_n, \end{cases} \tag{1.5}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ are sequences in [0,1] with $a_n + b_n + c_n + d_n = 1$.

Next, we also consider weak convergence theorems for another implicit iterative process:

$$\begin{cases} x_0 \in C \text{ chosen arbitrary}, \\ y_n = a_n x_{n-1} + b_n T_1 y_n + c_n T_2 y_n, \\ x_n = d_n y_n + (1 - d_n) S y_n, \end{cases} \tag{1.6}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ are sequences in [0,1] with $a_n + b_n + c_n = 1$.

In fact, for the above implicit iterative processes, most researchers always considered weak convergence theorems, and few researchers considered strong convergence theorem under suitable conditions. For example, one can see [5-7]. However, some conditions are not natural. For this reason, we consider the following shrinking-implicit iterative processes and study the strong convergence theorem. Let $\{x_n\}$ be defined by

$$\begin{cases} x_0 \in C \text{ chosen arbitrary and } C_0 = D_0 = C, \\ y_n = a_n x_{n-1} + b_n T_1 y_n + c_n T_2 y_n, \\ z_n = J^{-1}(d_n J y_n + (1 - d_n) J S y_n), \\ C_n = \{z \in C_{n-1} : \phi(z, z_n) \le \phi(z, y_n)\}, \\ D_n = \{z \in D_{n-1} : \|y_n - z\| \le \|x_{n-1} - z\|\}, \\ x_n = \Pi_{C_n \cap D_n} x_0, \end{cases} \tag{1.7}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ are sequences in (0, 1) with $a_n + b_n + c_n = 1$.

In this article, we first consider weak convergence theorems of implicit iterative processes for two nonexpansive mappings and a mapping which satisfy condition (C). And we generalize Khan et al.'s result [5] as special case. Next, we consider strong convergence theorem of an implicit-shrinking iterative process for two non-expansive mappings and a relative nonexpansive mapping on Banach spaces. Note that the conditions of strong convergence theorem are different from the strong convergence theorems for the implicit iterative processes in the literatures. Finally, we discuss a strong convergence theorem concerning two nonexpansive mappings and the resolvent of a maximal monotone operator in a Banach space.

2 Preliminaries

Throughout this article, let $\mathbb{N}$ and $\mathbb{R}$ be the sets of all positive integers and real numbers, respectively. Let E be a Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the weak convergence of $\{x_n\}$ to x and the strong convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

A Banach space E is said to satisfy Opial's condition if $\{x_n\}$ is a sequence in E with $x_n \rightharpoonup x$, then

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||, \quad \forall y \in E, y \neq x.$$

Let E be a Banach space. Then, the duality mapping $J : E \multimap E^*$ is defined by

$$Jx : \left\{x^* \in E^* : \left\langle x, x^*\right\rangle = ||x||^2 = ||x^*||^2\right\}, \quad \forall x \in E.$$

Let $S(E)$ be the unit sphere centered at the origin of E. Then, the space E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{||x + ty|| - ||x||}{t}$$

exists for all $x, y \in S(E)$. It is also said to be uniformly smooth if the limit exists uniformly in $x, y \in S(E)$. A Banach space E is said to be strictly convex if $\left\|\frac{x+y}{2}\right\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\left\|\frac{x+y}{2}\right\| < 1 - \delta$ whenever $x, y \in S(E)$ and $||x - y|| \geq \varepsilon$. Furthermore, we know that [8]

(i) if E in smooth, then J is single-valued;
(ii) if E is reflexive, then J is onto;
(iii) if E is strictly convex, then J is one-to-one;
(iv) if E is strictly convex, then J is strictly monotone;
(v) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

A Banach space E is said to have Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, then $x_n \to x$. It is known that if E uniformly convex, then E has the Kadec-Klee property [8].

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Throughout this article, define the function φ: $C \times C \to \mathbb{R}$ by

$$\phi(x,y) := ||x||^2 - 2\langle x, Jy\rangle + ||y||^2, \quad \forall x,y \in E.$$

Observe that, in a Hilbert space H, $\varphi(x, y) = ||x - y||^2$ for all $x, y \in H$. Furthermore, for each $x, y, z, w \in E$, we know that:

(1) $(||x|| - ||y||)^2 \le \varphi(x, y) \le (||x|| + ||y||)^2$;
(2) $\varphi(x, y) \ge 0$;
(3) $\varphi(x, y) = \varphi(x, z) + \varphi(z, y) + 2\langle x - z, Jz - Jy\rangle$;
(4) $2\langle x - y, Jz - Jw\rangle = \varphi(x, w) + \varphi(y, z) - \varphi(x, z) - \varphi(y, w)$;
(5) if E is additionally assumed to be strictly convex, then

$$\phi(x,y) = 0 \text{ if and only if } x = y;$$

(6) $\varphi(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz)) \le \lambda\varphi(x, y) + (1 - \lambda)\varphi(x, z)$.

Lemma 2.1. [9] Let E be a uniformly convex Banach space and let $r > 0$. Then, there exists a strictly increasing, continuous, and convex function g: $[0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and

$$||ax + by + cz + dw||^2 \le a||x||^2 + b||y||^2 + c||z||^2 + d||w||^2 - abg(||x - y||)$$

for all $x, y, z, w \in B_r$ and $a, b, c, d \in [0,1]$ with $a + b + c + d = 1$, where $B_r := \{z \in E: ||z|| \le r\}$.

Lemma 2.2. [10] Let E be a uniformly convex Banach space and let $r > 0$. Then, there exists a strictly increasing, continuous, and convex function g: $[0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and

$$\phi(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)) \le \lambda\phi(x,y) + (1-\lambda)\phi(x,z) - \lambda(1-\lambda)g(||Jy - Jz||)$$

for all $x, y, z \in B_r$ and $\lambda \in [0,1]$, where $B_r := \{z \in E: ||z|| \le r\}$.

Lemma 2.3. [11] Let E be a uniformly convex Banach space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < b \le \alpha_n \le c < 1$ for all $n \in \mathbb{N}$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of E such that $\limsup_{n\to\infty} ||x_n|| \le a$, $\limsup_{n\to\infty} ||y_n|| \le a$, and $\lim_{n\to\infty} ||\alpha_n x_n + (1 - \alpha_n)y_n|| = a$ for some $a \ge 0$. Then, $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Lemma 2.4. [12] Let E be a smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \varphi(x_n, y_n) = 0$, then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Remark 2.1. [13] Let E be a uniformly convex and uniformly smooth Banach space. If $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E, then

$$\lim_{n\to\infty}\phi(x_n, y_n) = 0 \Leftrightarrow \lim_{n\to\infty}||x_n - y_n|| = 0 \Leftrightarrow \lim_{n\to\infty}||Jx_n - Jy_n|| = 0.$$

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E. For an arbitrary point x of E, the set

$$\left\{z \in C : \phi(z, x) = \min_{y \in C} \phi(y, x)\right\}$$

is always nonempty and a singleton [14]. Let us define the mapping Π_C from E onto C by $\Pi_C x = z$, that is,

$$\phi(\Pi_C x, x) = \min_{y \in C} \phi(y, x)$$

for every $x \in E$. Such Π_C is called the generalized projection from E onto C [14].

Lemma 2.5. [14,15] Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, and let $(x, z) \in E \times C$. Then:

(i) $z = \Pi_C x$ if and only if $\langle y - z, Jx - Jz\rangle \le 0$ for all $y \in C$;
(ii) $\varphi(z, \Pi_C x) + \varphi(\Pi_C x, x) \le \varphi(z, x)$.

Lemma 2.6. [16] Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and T: $C \to C$ is a nonexpansive mapping. Let $\{x_n\}$ be a sequence in C with $x_n \rightharpoonup x \in C$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. Then, $Tx = x$.

Lemma 2.7. [1] Let C be a nonempty subset of a Banach space E with the Opial property. Assume that T: $C \to E$ satisfies condition (C). Let $\{x_n\}$ be a sequence in C with $x_n \rightharpoonup x \in C$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. Then, $Tx = x$.

Lemma 2.8. [1] Let T be a mapping on a closed subset C of a Banach space E. Assume that T satisfies condition (C). Then, $F(T)$ is a closed set. Moreover, if E is strictly convex and C is convex, then $F(T)$ is also convex.

Lemma 2.9. [17] Let C be a nonempty closed convex subset of a strictly convex Banach space E, and T: $C \to C$ be a nonexpansive mapping. Then, $F(T)$ is a closed convex subset of C.

3 Weak convergence theorems

Lemma 3.1. Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E, and let T_1, $T_2 : C \to C$ be two nonexpansive mappings, and let S: $C \to C$ be a mapping with condition (C). Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences with $0 < a \le a_n, b_n, c_n, d_n \le b < 1$ and $a_n + b_n + c_n + d_n = 1$. Suppose that $\Omega := F(S) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Define a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C \text{ chosen arbitrary,} \\ x_n = a_n x_{n-1} + b_n S x_{n-1} + c_n T_1 x_n + d_n T_2 x_n. \end{cases}$$

Then, we have:

(i) $\lim_{n\to\infty} ||x_n - p||$ exists for each $p \in \Omega$.
(ii) $\lim_{n\to\infty} ||x_n - Sx_n|| = \lim_{n\to\infty} ||x_n - T_1 x_n|| = \lim_{n\to\infty} ||x_n - T_2 x_n|| = 0$.

Proof. First, we show that $\{x_n\}$ is well-defined. Now, let $f(x)$: $= a_1 x_0 + b_1 S x_0 + c_1 T_1 x + d_1 T_2 x$. Then,

$$||f(x) - f(y)|| \le c_1 ||T_1 x - T_1 y|| + d_1 ||T_2 x - T_2 y|| \le (c_1 + d_1)||x - y|| \le (1 - 2a)||x - y||.$$

By Banach contraction principle, the existence of x_1 is established. Similarly, the existence of $\{x_n\}$ is well-defined.

(i) For each $p \in \Omega$ and $n \in \mathbb{N}$, we have:

$$\begin{aligned}&||x_n - p||\\ &\leq a_n||x_{n-1} - p|| + b_n||Sx_{n-1} - p|| + c_n||T_1x_n - p|| + d_n||T_2x_n - p||\\ &\leq a_n||x_{n-1} - p|| + b_n||x_{n-1} - p|| + (c_n + d_n)||x_n - p||.\end{aligned}$$

This implies that $(1 - c_n - d_n)||x_n - p|| \leq (a_n + b_n)||x_{n-1}-p||$. Hence, $||x_n-p|| \leq ||x_{n-1}-p||$, $\lim_{n\to\infty} ||x_n -p||$ exists, and $\{x_n\}$ is a bounded sequence.

(ii) Take any $p \in \Omega$ and let p be fixed. Suppose that $\lim_{n\to\infty} ||x_n - p|| = d$. Clearly, $\limsup_{n\to\infty} ||T_2x_n - p|| \leq d$, and we have:

$$\begin{aligned}&\lim_{n\to\infty} ||x_n - p||\\ &= \lim_{n\to\infty} ||a_nx_{n-1} + b_nSx_{n-1} + c_nT_1x_n + d_nT_2x_n - p||\\ &= \lim_{n\to\infty} \left\|(1 - d_n)\left[\frac{a_n}{1 - d_n}(x_{n-1} - p) + \frac{b_n}{1 - d_n}(Sx_{n-1} - p) + \frac{c_n}{1 - d_n}(T_1x_n - p)\right] + d_n(T_2x_n - p)\right\|.\end{aligned}$$

Besides,

$$\begin{aligned}&\limsup_{n\to\infty} \left\|\frac{a_n}{1 - d_n}(x_{n-1} - p) + \frac{b_n}{1 - d_n}(Sx_{n-1} - p) + \frac{c_n}{1 - d_n}(T_1x_n - p)\right\|\\ &\leq \limsup_{n\to\infty} \frac{a_n}{1 - d_n}||x_{n-1} - p|| + \frac{b_n}{1 - d_n}||Sx_{n-1} - p|| + \frac{c_n}{1 - d_n}||T_1x_n - p||\\ &\leq \limsup_{n\to\infty} \frac{a_n}{1 - d_n}||x_{n-1} - p|| + \frac{b_n}{1 - d_n}||Sx_{n-1} - p|| + \frac{c_n}{1 - d_n}||T_1x_n - p||\\ &\leq \limsup_{n\to\infty} \frac{a_n + b_n}{1 - d_n}||x_{n-1} - p|| + \frac{c_n}{1 - d_n}||x_n - p||\\ &\leq \limsup_{n\to\infty} \frac{a_n + b_n + c_n}{1 - d_n}||x_{n-1} - p|| = d.\end{aligned}$$

By Lemma 2.3,

$$\lim_{n\to\infty} \left\|\frac{a_n}{1 - d_n}(x_{n-1} - p) + \frac{b_n}{1 - d_n}(Sx_{n-1} - p) + \frac{c_n}{1 - d_n}(T_1x_n - p) - (T_2x_n - p)\right\| = 0.$$

This implies that $\lim_{n\to\infty} ||x_n - T_2x_n|| = 0$. Similarly, $\lim_{n\to\infty} ||x_n - T_1x_n|| = 0$.

Since $\{x_n\}$ is bounded, there exists $r > 0$ such that $2 \sup\{||x_n-p||:n \in \mathbb{N}\} \leq r$.

By Lemma 2.1, there exists a strictly increasing, continuous, and convex function g: $[0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and

$$\begin{aligned}&||x_n - p||^2\\ &\leq a_n||x_{n-1} - p||^2 + b_n||Sx_{n-1} - p||^2 + c_n||T_1x_n - p||^2 + d_n||T_2x_n - p||^2\\ &\quad -a_nb_ng(||x_{n-1} - Sx_{n-1}||)\\ &\leq (a_n + b_n)||x_{n-1} - p||^2 + (c_n + d_n)||x_n - p||^2 - a_nb_ng(||x_{n-1} - Sx_{n-1}||).\end{aligned}$$

This implies that

$$a_n b_n g(||x_{n-1} - Sx_{n-1}||) \leq (a_n + b_2)(||x_{n-1} - p||^2 - ||x_n - p||^2).$$

By the properties of g and $\lim_{n\to\infty} ||x_n - p|| = d$, we get $\lim_{n\to\infty} ||x_n - Sx_n|| = 0$.

Theorem 3.1. Let E be a uniformly convex Banach space with Opial's condition, C be a nonempty closed convex subset of E, and let $T_1, T_2 : C \to C$ be two nonexpansive mappings, and let S: $C \to C$ be a mapping with condition (C). Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences with $0 < a \leq a_n, b_n, c_n, d_n \leq b < 1$ and $a_n + b_n + c_n + d_n = 1$. Suppose that $\Omega := F(S) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Define a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C \text{ chosen arbitrary,} \\ x_n = a_n x_{n-1} + b_n S x_{n-1} + c_n T_1 x_n + d_n T_2 x_n. \end{cases}$$

Then, $x_n \rightharpoonup z$ for some $z \in \Omega$.

Proof. By Lemma 3.1, $\{x_n\}$ is a bounded sequence. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in C$ such that $x_{n_k} \rightharpoonup z$. By Lemmas 2.6, 2.7, and 3.1, we know that $z \in \Omega$. Since E has Opial's condition, it is easy to see that $x_n \rightharpoonup z$.

Hence, the proof is completed.

Remark 3.1. The conclusion of Theorem 3.1 is still true if S: $C \to C$ is a quasi-nonexpansive mapping, and $I - S$ is demiclosed at zero, that is, $x_n \rightharpoonup x$ and $(I - S)x_n \rightharpoonup 0$ implies that $(I - S)x = 0$.

In Theorem 3.1, if $S = I$, then we get the following result. Hence, Theorem 3.1 generalizes Theorem 4 in [5].

Corollary 3.1. [5] Let E be a uniformly convex Banach space with Opial's condition, C be a nonempty closed convex subset of E, and let $T_1, T_2 : C \to C$ be two nonexpansive mappings. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $0 < a \leq a_n, b_n, c_n \leq b < 1$ and $a_n + b_n + c_n = 1$. Suppose that $\Omega := F(T_1) \cap F(T_2) \neq \emptyset$.

Define a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C \text{ chosen arbitrary,} \\ x_n = a_n x_{n-1} + b_n T_1 x_n + c_n T_2 x_n. \end{cases}$$

Then, $x_n \rightharpoonup z$ for some $z \in \Omega$.

Besides, it is easy to get the following result from Theorem 3.1.

Corollary 3.2. Let E be a uniformly convex Banach space with Opial's condition, C be a nonempty closed convex subset of E, and let S: $C \to C$ be a mapping with condition (C). Let $\{a_n\}$ be a sequence with $0 < a \leq a_n \leq b < 1$. Suppose that $F(S) \neq \emptyset$. Define a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C \text{ chosen arbitrary,} \\ x_n = a_n x_{n-1} + (1 - a_n) S x_{n-1}. \end{cases}$$

Then, $x_n \rightharpoonup z$ for some $z \in F(S)$.

Proof. Let $T_1 = T_2 = I$, where I is the identity mapping. For each $n \in \mathbb{N}$, we know that

$$x_n = \frac{a_n}{2} x_{n-1} + \frac{1 - a_n}{2} S x_{n-1} + \frac{1}{4} T_1 x_n + \frac{1}{4} T_2 x_n.$$

By Theorem 3.1, it is easy to get the conclusion.

Theorem 3.2. Let E be a uniformly convex Banach space with Opial's condition, C be a nonempty closed convex subset of E, and let T_1, $T_2 : C \to C$ be two nonexpansive mappings, and let S: $C \to C$ be a mapping with condition (C). Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences with $0 < a \leq a_n, b_n, c_n, d_n \leq b < 1$ and $a_n + b_n + c_n = 1$. Suppose that $\Omega := F(S) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Define a sequence $\{x_n\}$ *by*

$$\begin{cases} x_0 \in C \text{ chosen arbitrary,} \\ y_n = a_n x_{n-1} + b_n T_1 y_n + c_n T_2 y_n, \\ x_n = d_n y_n + (1 - d_n) S y_n. \end{cases}$$

Then, $x_n \rightharpoonup z$ for some $z \in \Omega$.

Proof. Following the same argument as in Lemma 3.1, we know that $\{y_n\}$ is well-defined. Take any $w \in \Omega$ and let w be fixed. Then, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} ||y_n - w|| &= ||a_n x_{n-1} + b_n T_1 y_n + c_n T_2 y_n - w|| \\ &\leq a_n ||x_{n-1} - w|| + b_n ||T_1 y_n - w|| + c_n ||T_2 y_n - w|| \\ &\leq a_n ||x_{n-1} - w|| + (b_n + c_n) ||y_n - w||. \end{aligned}$$

This implies that $||y_n - w|| \leq ||x_{n-1} - w||$ for each $n \in \mathbb{N}$. Besides, we also have

$$\begin{aligned} ||x_n - w|| &= ||d_n y_n + (1 - d_n) S y_n - w|| \\ &\leq d_n ||y_n - w|| + (1 - d_n) ||S y_n - w|| \\ &\leq ||y_n - w||. \end{aligned}$$

Hence, $||x_n - w|| \leq ||y_n - w|| \leq ||x_{n-1} - w||$ for each $n \in \mathbb{N}$. So, $\lim_{n\to\infty} ||x_n - w||$ and $\lim_{n\to\infty} ||y_n - w||$ exist, and $\{x_n\}$, $\{y_n\}$ are bounded sequences.

Suppose that $\lim_{n\to\infty} ||x_n - w|| = \lim_{n\to\infty} ||y_n - w|| = d$. Clearly, $\limsup_{n\to\infty} ||T_2 y_n - w|| \leq d$, and we have

$$\begin{aligned} &\lim_{n\to\infty} ||y_n - w|| \\ &= \lim_{n\to\infty} ||a_n x_{n-1} + b_n T_1 y_n + c_n T_2 y_n - w|| \\ &= \lim_{n\to\infty} \left\| (1 - c_n) \left[\frac{a_n}{1 - c_n} (x_{n-1} - w) + \frac{b_n}{1 - c_n} (T_1 y_n - w) \right] + c_n (T_2 y_n - w) \right\|. \end{aligned}$$

Besides,

$$\begin{aligned} &\limsup_{n\to\infty} \left\| \frac{a_n}{1 - c_n} (x_{n-1} - w) + \frac{b_n}{1 - c_n} (T_1 y_n - w) \right\| \\ &\leq \limsup_{n\to\infty} \frac{a_n}{1 - c_n} ||x_{n-1} - w|| + \frac{b_n}{1 - c_n} ||T_1 y_n - w|| \\ &\leq \limsup_{n\to\infty} \frac{a_n}{1 - c_n} ||x_{n-1} - w|| + \frac{b_n}{1 - c_n} ||y_n - w|| \\ &\leq \limsup_{n\to\infty} ||x_{n-1} - w|| = d. \end{aligned}$$

By Lemma 2.3,

$$\lim_{n\to\infty} \left\| \frac{a_n}{1 - c_n} (x_{n-1} - w) + \frac{b_n}{1 - c_n} (T_1 y_n - w) - (T_2 y_n - w) \right\| = 0.$$

This implies that $\lim_{n\to\infty} ||y_n - T_2 y_n|| = 0$. Similarly, $\lim_{n\to\infty} ||y_n - T_1 y_n|| = 0$.

Since $\{x_n\}$ and $\{y_n\}$ are bounded sequences, there exists $r > 0$ such that

$$2\sup\{||x_n||, ||y_n||, ||x_n - w||, ||y_n - w|| : n \in \mathbb{N}\} \leq r.$$

By Lemma 2.1, there exists a strictly increasing, continuous, and convex function g: $[0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and

$$||d_n y_n + (1-d_n)Sy_n - w||^2 \leq d_n||y_n - w||^2 + (1-d_n)||Sy_n - w||^2 - d_n(1-d_n)g(||y_n - Sy_n||).$$

This implies that

$$d_n(1-d_n)g(||y_n - Sy_n||) \leq ||y_n - w||^2 - ||x_n - w||^2.$$

Since $\lim_{n\to\infty} ||x_n - w|| = \lim_{n\to\infty} ||y_n - w|| = d$, and the properties of g, we get $\lim_{n\to\infty} ||y_n - Sy_n|| = 0$. Besides,

$$||x_n - y_n|| = ||d_n y_n + (1-d_n)Sy_n - y_n|| = (1-d_n)||y_n - Sy_n||.$$

Hence, $\lim_{n\to\infty} ||x_n - y_n|| = 0$. Finally, following the same argument as in the proof of Theorem 3.1, we know that $x_n \rightharpoonup z$ for some $z \in \Omega$.

Next, we give the following examples for Theorems 3.1 and 3.2.

Example 3.1. Let $E = \mathbb{R}$, $C: = [0,3]$, $T_1x = T_2x = x$, and let $S: C \to C$ be the same as in [1]:

$$Sx := \begin{cases} 0 \text{ if } x \neq 3, \\ 1 \text{ if } x = 3. \end{cases}$$

For each n, let $a_n = b_n = c_n = d_n = \frac{1}{4}$. Let $x_0 = 1$. Then, for the sequence $\{x_n\}$, in Theorem 3.1, we know that $x_n = \frac{1}{2^n}$ for all $n \in \mathbb{N}$, and $x_n \to 0$, and 0 is a common fixed point of S, T_1, and T_2.

Example 3.2. Let E, C, T_1, T_2, S be the same as in Example 3.1. For each n, let $a_n = b_n = c_n = \frac{1}{3}$, and $d_n = \frac{1}{2}$. Let $x_0 = 1$. Then, for the sequence $\{x_n\}$ in Theorem 3.1, we know that $x_n = \frac{1}{2^n}$ for all $n \in \mathbb{N}$, and $x_n \to 0$, and 0 is a common fixed point of S, T_1, and T_2.

Example 3.3. Let E, C, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, and let S: $C \to C$ be the same as in Example 3.1. Let $T_1x = T_2x = 0$ for each $x \in C$. Then, for the sequence $\{x_n\}$ in Theorem 3.1, we know that $x_n = \frac{1}{4^n}$ for all $n \in \mathbb{N}$.

Example 3.4. Let E, C, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, and let S: $C \to C$ be the same as in Example 3.2. Let $T_1x = T_2x = 0$ for each $x \in C$. Then, for the sequence $\{x_n\}$ in Theorem 3.2, we know that $x_n = \frac{1}{6^n}$ for all $n \in \mathbb{N}$.

Remark 3.2.

(i) For the rate of convergence, by Examples 3.3 and 3.4, we know that the iteration process in Theorem 3.2 may be faster than the iteration process in Theorem 3.1.

But, the times of iteration process for Theorem 3.2 is much than ones in Theorem 3.1.

(ii) The conclusion of Theorem 3.2 is still true if S: $C \to C$ is a quasi-nonexpansive mapping, and I - S is demiclosed at zero, that is, $x_n \rightharpoonup x$ and $(I - S)x_n \to 0$ implies that $(I - S)x = 0$.

(iii) Corollaries 3.1 and 3.2 are special cases of Theorem 3.2.

Definition 3.1. [18] Let C be a nonempty subset of a Banach space E. A mapping T: $C \to E$ satisfy condition (E) if there exists $\mu \geq 1$ such that for all $x, y \in C$,

$$||x - Ty|| \leq \mu||x - Tx|| + ||x - y||.$$

By Lemma 7 in [1], we know that if T satisfies condition (C), then T satisfies condition (E). But, the converse may be false [18, Example 1]. Furthermore, we also observe the following result.

Lemma 3.2. [18] Let C be a nonempty subset of a Banach space E. Let T: $C \to E$ be a mapping. Assume that:

(i) $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ and $x_n \rightharpoonup x$;
(ii) T satisfies condition (E);
(iii) E has Opial condition.

Then, $Tx = x$.

By Lemma 3.2, if S satisfies condition (E), then the conclusions of Theorems 3.1 and 3.2 are still true. Hence, we can use the following condition to replace condition (C) in Theorems 3.1 and 3.2 by Proposition 19 in [19].

Definition 3.2. [19] Let T be a mapping on a subset C of a Banach space E.

Then, T is said to satisfy (SKC)-condition if

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \Rightarrow ||Tx - Ty|| \leq N(x, y),$$

where $N(x, y) := \max\{||x - y||, \frac{1}{2}(||x - Tx|| + ||Ty - y||), \frac{1}{2}(||Tx - y|| + ||x - Ty||)\}$ for all $x, y \in C$.

4 Strong convergence theorems (I)

Let C be a nonempty closed convex subset of a Banach space E. A point p in C is said to be an asymptotic fixed point of a mapping T: $C \to C$ if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}$, $||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping $T: C \to C$ is called relatively nonexpansive [20] if $F(T) \neq \emptyset, \hat{F}(T) = F(T)$, and $\varphi(p, Tx) \leq \varphi(p, x)$ for all $x \in C$ and $p \in F(T)$. Note that every identity mapping is a relatively nonexpansive mapping.

Lemma 4.1. [21] Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let T: $C \to C$ be a relatively nonexpansive mapping. Then, $F(T)$ is a closed and convex subset of C.

The following property is motivated by the property (Q_4) in [22].

Definition 4.1. Let E be a Banach space. Then, we say that E satisfies condition (Q) if for each $x, y, z_1, z_2 \in E$ *and* $t \in [0,1]$,

$$||x - z_i|| \leq ||y - z_i||, i = 1, 2 \Rightarrow ||x - (tz_1 + (1 - t)z_2|| \leq ||y - (tz_1 + (1 - t)z_2)||.$$

Remark 4.1. If H is a Hilbert space, then H satisfies condition (Q).

Theorem 4.1. Let E be a uniformly convex and uniformly smooth Banach space with condition (Q), and let C be a nonempty closed convex subset of E, and let $T_1, T_2 : C \to C$ be two nonexpansive mappings, and let S: $C \to C$ be a relatively nonexpansive mapping. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences in (0,1) with and $a_n + b_n + c_n = 1$. Suppose that $\Omega := F(S) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Define a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C \text{ chosen arbitrary and } C_0 = D_0 = C, \\ y_n = a_n x_{n-1} + b_n T_1 y_n + c_n T_2 y_n, \\ z_n = J^{-1}(d_n J y_n + (1 - d_n) JS y_n), \\ C_n = \{z \in C_{n-1} : \phi(z, z_n) \leq \phi(z, y_n)\}, \\ D_n = \{z \in D_{n-1} : ||y_n - z|| \leq ||x_{n-1} - z||\}, \\ x_n = \Pi_{C_n \cap D_n} x_0. \end{cases}$$

Assume that $\liminf_{n\to\infty} b_n > 0$, $\liminf_{n\to\infty} c_n > 0$, and $\liminf_{n\to\infty} d_n(1 - d_n) > 0$. Then, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \Pi_\Omega x_0$.

Proof. Following the same argument as in Lemma 3.1, we know that $\{y_n\}$ is well-defined.

Clearly, C_0 and D_0 are nonempty closed convex subsets of C, and C_n is a closed subset of C for every $n \in \mathbb{N}$. Since $\varphi(z, z_n) \leq \varphi(z, y_n)$ is equivalent to

$$2\langle z, Jy_n - Jz_n\rangle \leq ||y_n||^2 - ||z_n||^2,$$

it is easy to see that C_n is a convex set for each $n \in \mathbb{N}$. Besides, by condition (Q), it is easy to see that D_n is a nonempty closed convex subset of C.

Next, we want to show that $\Omega \subseteq C_n \cap D_n$ for each $n \in \mathbb{N} \cup \{0\}$. Clearly, $\Omega \subseteq C_0$. Suppose that $\Omega \subseteq C_{n-1}$. Let $w \in \Omega$. Then, $w \in F(S)$ and

$$\begin{aligned} \phi(w, z_n) &= \phi(w, J^{-1}(d_n J y_n + (1 - d_n) JS y_n)) \\ &\leq d_n \phi(w, y_n) + (1 - d_n)\phi(w, S y_n) \\ &\leq d_n \phi(w, y_n) + (1 - d_n)\phi(w, y_n) = \phi(w, y_n). \end{aligned}$$

So, $\Omega \subseteq C_n$. By induction, $\Omega \subseteq C_n$ for each $n \in \mathbb{N} \cup \{0\}$.

Clearly, $\Omega \subseteq D_0$. Suppose that $\Omega \subseteq D_{n-1}$. Let $w \in \Omega$. Then, $w \in F(T_1) \cap F(T_2)$ and

$$\begin{aligned} ||y_n - w|| &\leq a_n ||x_{n-1} - w|| + b_n ||T_1 y_n - w|| + c_n ||T_2 y_n - w|| \\ &\leq a_n ||x_{n-1} - w|| + b_n ||y_n - w|| + c_n ||y_n - w||. \end{aligned}$$

This implies that $||y_n - w|| \leq ||x_{n-1} - w||$ and $w \in D_n$. By induction, $\Omega \subseteq D_n$ for each $n \in \mathbb{N} \cup \{0\}$. So, $\Omega \subseteq C_n \cap D_n$ for each $n \in \mathbb{N} \cup \{0\}$.

Since $x_n = \Pi_{C_n \cap D_n} x_0$,

$$\phi(x_n, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0)$$

for each $w \in \Omega$. Therefore, $\{\varphi(x_n, x_0)\}$ is a bounded sequence. Furthermore, $\{x_n\}$ is a bounded sequence.

By Lemma 2.5, $x_n = \Pi_{C_n \cap D_n} x_0$, and $x_{n+1} = \Pi_{C_{n+1} \cap D_{n+1}} x_0$,

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n \cap D_n} x_0) \le \phi(x_{n+1}, x_0) - \phi(x_n, x_0).$$

Hence, $\varphi(x_n, x_0) \le \varphi(x_{n+1}, x_0)$, $\lim_{n\to\infty} \varphi(x_n, x_0)$ exists, and $\lim_{n\to\infty} \varphi(x_{n+1}, x_n) = 0$. By Lemma 2.4, $\lim_{n\to\infty}$, $||x_{n+1} - x_n|| = 0$. Since $x_n \in D_n$, we know that $||y_n - x_n|| \le ||x_{n-1} - x_n||$ and $\lim_{n\to\infty} ||x_n - y_n|| = 0$. Furthermore, $\lim_{n\to\infty} \varphi(x_n, y_n) = 0$. Since $x_n \in C_n$, it is easy to see that $\lim_{n\to\infty} \varphi(x_n, z_n) = 0$. Hence, $\lim_{n\to\infty} ||x_n - z_n|| = 0$.

Take any $w \in \Omega$ and let w be fixed. Let r: = 2sup$\{||x_n||, ||x_n - w||, ||y_n||, ||y_n - w||: n \in \mathbb{N}\}$. By Lemma 2.1, there exists a strictly increasing, continuous, and convex function g: $[0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and

$$\begin{aligned}
&||y_n - w||^2 \\
&\le a_n||x_{n-1} - w||^2 + b_n||T_1y_n - w||^2 + c_n||T_2y_n - w||^2 - a_nb_ng(||x_{n-1} - T_1y_n||) \\
&\le a_n||x_{n-1} - w||^2 + b_n||y_n - w||^2 + c_n||y_n - w||^2 - a_nb_ng(||x_{n-1} - T_1y_n||).
\end{aligned}$$

This implies that

$$b_ng(||x_{n-1} - T_1y_n||) \le ||x_{n-1} - y_n||(||x_{n-1} - w|| + ||y_n - w||).$$

So, $\lim_{n\to\infty} b_n\, g(||x_{n-1} - T_1y_n||) = 0$. By (ii), $\lim_{n\to\infty} ||x_{n-1} - T_1y_n|| = 0$. Furthermore, $\lim_{n\to\infty} ||y_n - T_1y_n|| = 0$. Similarly, $\lim_{n\to\infty} ||y_n - T_2y_n|| = 0$.

By Lemma 2.2, there exists a strictly increasing, continuous, and convex function g': $[0, 2r] \to [0, \infty)$ such that $g'(0) = 0$ and

$$\begin{aligned}
\phi(w, z_n) &\le d_n\phi(w, y_n) + (1 - d_n)\phi(w, Sy_n) - d_n(1 - d_n)g\prime(||Jy_n - JSy_n||) \\
&\le \phi(w, y_n) - d_n(1 - d_n)g\prime(||Jy_n - JSy_n||).
\end{aligned}$$

Hence,

$$\begin{aligned}
&d_n(1 - d_n)g\prime(||Jy_n - JSy_n||) \\
&\le \phi(w, y_n) - \phi(w, z_n) \\
&= (||w||^2 + ||y_n||^2 - 2\langle w, Jy_n\rangle) - (||w||^2 + ||z_n||^2 - 2\langle w, Jz_n\rangle) \\
&= ||y_n||^2 - ||z_n||^2 + 2\langle w, Jz_n - Jy_n\rangle \\
&= ||y_n - z_n||(||y_n|| + ||z_n||) + 2||w|| \cdot ||Jz_n - Jy_n||.
\end{aligned}$$

By Remark 2.1, $\lim_{n\to\infty} d_n(1 - d_n)g(||Jy_n - JSy_n||) = 0$. By assumptions and the properties of g, $\lim_{n\to\infty} ||Jy_n - JSy_n|| = 0$. Furthermore, $\lim_{n\to\infty} ||y_n - Sy_n|| = 0$.

Since $\{y_n\}$ is a bounded sequence, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and $\bar{x} \in C$ such that $y_{n_k} \rightharpoonup \bar{x}$. By Lemma 2.6, $\bar{x} \in F(T_1) \cap F(T_2)$. Besides, since S is a relatively nonexpansive mapping, $\bar{x} \in \hat{F}(S) = F(S)$. So, $\bar{x} \in \Omega$.

Finally, we want to show that $y_n \to \Pi_\Omega x_0$. Let $q = \Pi_\Omega x_0$. Then, $q \in \Omega \subseteq C_n \cap D_n$ for each $n \in \mathbb{N}$. So,

$$\phi(x_n, x_0) = \min_{y \in C_n \cap D_n} \phi(y, x_0) \le \phi(q, x_0).$$

On the other hand, from weakly lower semicontinuity of the norm and $\lim_{n\to\infty} ||x_n - y_n|| = 0$, we have

$$\begin{aligned}\phi(\bar{x}, x_0) &= ||\bar{x}||^2 - 2\langle \bar{x}, Jx_0\rangle + ||x_0||^2 \\ &\le \liminf_{n\to\infty}(||y_{n_k}||^2 - 2\langle y_{n_k}, Jx_0\rangle + ||x_0||^2) \\ &= \liminf_{n\to\infty}(||x_{n_k}||^2 - 2\langle x_{n_k}, Jx_0\rangle + ||x_0||^2) \\ &\le \liminf_{n\to\infty}\phi(x_{n_k}, x_0) \\ &\le \limsup_{n\to\infty}\phi(x_{n_k}, x_0) \le \phi(q, x_0).\end{aligned}$$

Since $q = \Pi_\Omega x_0, \bar{x} = q$. Hence, $\lim_{n\to\infty}\phi(x_{n_k}, x_0) = \phi(\bar{x}, x_0)$. So, we have $\lim_{n\to\infty}||x_{n_k}|| = ||\bar{x}||$. Using the Kadec-Klee property of E, we obtain that $\lim_{k\to\infty}x_{n_k} = q = \Pi_\Omega x_0$.

Furthermore, for each weakly convergence subsequence $\{x_{n_m}\}$ of $\{x_n\}$, we know that $\lim_{m\to\infty}x_{n_m} = q = \Pi_\Omega y_1$ by following the same argument as the above conclusion. Therefore,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \Pi_\Omega x_0.$$

Hence, the proof is completed.

Remark 4.2. Since nonspreading mappings with fixed points in a strictly convex Banach space with a uniformly Gateaux differentiable norm are relatively nonex-pansive mappings [[23], Theorem 3.3], we know that the conclusion of Theorem 4.1 is still true if S is replaced by a nonspreading mapping.

Next, we give an easy example for Theorem 4.1.

Example 4.1. Let $E = \mathbb{R}$, $C: = [0,3]$, $T_1x = T_2x = x$, and let $S: C \to C$ be the as in [1]:

$$Sx := \begin{cases} 0 \text{ if } x \neq 3, \\ 1 \text{ if } x = 3. \end{cases}$$

For each n, let $a_n = b_n = c_n = \frac{1}{3}$ and $d_n = \frac{1}{2}$. Let $x_0 = 1$. Hence, we have

(a) $y_n = x_{n\text{-}1}$ for each $n \in \mathbb{N}$;

(b) $z_n = \frac{1}{2}y_n$ for each $n \in \mathbb{N}$;

(c) $C_n := \{z \in C_{n-1} : |z - z_n| \le |z - y_n|\} = 0\left[0, \frac{y_n + zn}{2}\right]$;

(d) $D_n : = \{z \in D_{n\text{-}1} : |z - y_n| \le |z - x_{n\text{-}1}|\} = [0,3]$;

(e) $x_n = \frac{1}{2}(y_n + z_n) = \frac{1}{2}\left(x_{n-1} + \frac{1}{2}x_{n-1}\right) = \frac{3}{4}x_{n-1}$.

By (e) and $x_0 = 1$, we know that $x_n = \left(\frac{3}{4}\right)^n$ for each $n \in \mathbb{N} \cup \{0\}$, $\lim_{n\to\infty} x_n = 0$, and 0 is a common fixed point of S, T_1, and T_2.

The following results are special cases of Theorem 4.1.

Corollary 4.1. Let E be a uniformly convex and uniformly smooth Banach space with condition (Q), and let C be a nonempty closed convex subset of E, and let T_1, $T_2 : C \to C$ be two nonexpansive mappings. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences in (0,1) with and $a_n + b_n + c_n = 1$. Suppose that $\Omega := F(T_1) \cap F(T_2) \neq \emptyset$. Define a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C \text{ chosen arbitrary and } D_0 = C, \\ y_n = a_n x_{n-1} + b_n T_1 y_n + c_n T_2 y_n, \\ D_n = \{z \in D_{n-1} : ||y_n - z|| \leq ||x_{n-1} - z||\}, \\ x_n = \Pi_{D_n} x_0. \end{cases}$$

Assume that $\liminf_{n\to\infty} b_n > 0$, $\liminf_{n\to\infty} c_n > 0$. Then, $\lim_{n\to\infty} x_n = \lim_{n\to\infty}, y_n = \Pi_\Omega x_0$.

Corollary 4.2. Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E, and let S: $C \to C$ be a relatively nonexpansive mapping. Let $\{d_n\}$ be a sequence in (0,1). Suppose that $F(S) \neq \emptyset$. Define a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C \text{ chosen arbitrary and } C_0 = C, \\ z_n = J^{-1}(d_n J x_{n-1} + (1 - d_n) J S x_{n-1}), \\ C_n = \{z \in C_{n-1} : \phi(z, z_n) \leq \phi(z, x_{n-1})\}, \\ x_n = \Pi_{C_n} x_0. \end{cases}$$

Assume that $\liminf_{n\to\infty} d_n(1 - d_n) > 0$. Then, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \Pi_{F(s)} x_0$.

Remark 4.3. Corollary 4.2 is a generalization of Theorem 4.1 in [24]. But, it is a special case of Theorem 3.1 in [25].

5 Strong convergence theorems (II)

In this section, we need the following important lemmas.

Lemma 5.1. [26] Let E be a reflexive Banach space and f: $E \to \mathbb{R} \cup \{+\infty\}$ be a convex and lower semicontinuous function. Let C be a nonempty bounded and closed convex subset of E. Then, the function f attains its minimum on C. That is, there exists $x^* \in C$ such that $f(x^*) \leq f(x)$ for all $x \in C$.

Lemma 5.2. In a Banach space E, there holds the inequality

$$||x + y||^2 \leq ||x||^2 + 2\langle y, j(x + y)\rangle, \quad x, y \in E,$$

where $j(x+y) \in J(x+y)$.

Lemma 5.3. [27] Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm. Let $\{x_n\}$ be a bounded sequence of E and let μ_n be a Banach limit and $z \in C$. Then,

$$\mu_n ||x_n - z||^2 = \min_{y \in C} \mu_n ||x_n - y||^2 \Leftrightarrow \mu_n \langle y - z, J(x_n - z)\rangle \leq 0 \text{ for all } y \in C.$$

Lemma 5.4. [28] Let α be a real number and $(x_0, x_1,...) \in \ell^2$ such that $\mu_n x_n \leq \alpha$ for all Banach μ_n. If $\limsup_{n\to\infty} (x_{n+1} - x_n) \leq 0$, then $\limsup_{n\to\infty} x_n \leq \alpha$.

Lemma 5.5. [29] Assume that $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative real numbers such that $a_{n+1} < (1 - \gamma_n)a_n + \delta_n$, $n \in \mathbb{N}$, where $\{\gamma_n\} \subseteq (0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that (i) $\sum_{n=1}^{\infty} \gamma_n = \infty$; (ii) $\limsup_{n\to\infty} \frac{\delta n}{\gamma n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$. Then, $\lim_{n\to\infty} a_n = 0$.

Theorem 5.1. Let E be a uniformly convex and uniformly smooth Banach space with Opial's condition, C be a nonempty closed convex subset of E, and let T_1, $T_2 : C \to C$ be two nonexpansive mappings, and let S: $C \to C$ be a mapping with condition (C). Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences in (a, b) for some $0 < a, b < 1$ with $a_n + b_n + c_n =$

1. Let $\{d_n\}$ be a sequence in [0,1]. Suppose that $\Omega := F(S) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Define a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C \text{ chosen arbitrary,} \\ y_n = a_n x_{n-1} + b_n T_1 y_n + c_n T_2 y_n, \\ x_n = d_n x_0 + (1 - d_n) S y_n. \end{cases}$$

Assume that:

(i) $\lim_{n\to\infty} d_n = 0$, $\sum_{n=1}^{\infty} d_n = \infty$, and $\lim_{n\to\infty} \frac{|d_{n+1} - d_n|}{d_n} = 0$;

(ii) $\lim_{n\to\infty} (a_{n+1} - a_n) = \lim_{n\to\infty} (b_{n+1} - b_n) = \lim_{n\to\infty} (c_{n+1} - c_n) = 0$.

Then, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \bar{x}$ for some $\bar{x} \in \Omega$.

Proof. Following the same argument as in Lemma 3.1, we know that $\{y_n\}$ is well-defined. Take any $w \in \Omega := F(S) \cap F(T_1) \cap F(T_2)$ and let w be fixed. Then, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} &\|y_n - w\| \\ &\le a_n \|x_{n-1} - w\| + b_n \|T_1 y_n - w\| + c_n \|T_2 y_n - w\| \\ &\le a_n \|x_{n-1} - w\| + (b_n + c_n) \|y_n - w\|. \end{aligned}$$

This implies that $||y_n - w|| \le ||x_{n-1} - w||$ for each $n \in \mathbb{N}$. Next, we have

$$\begin{aligned} &||x_n - w|| \\ &\le d_n||x_0 - w|| + (1 - d_n)||Sy_n - w|| \\ &\le d_n||x_0 - w|| + (1 - d_n)||y_n - w|| \\ &\le d_n||x_0 - w|| + (1 - d_n)||x_{n-1} - w|| \\ &\quad \vdots \\ &\le \max\{||x_0 - w||, ||x_1 - w||\}. \end{aligned}$$

Then, $\{x_n\}$ is a bounded sequence. Furthermore, $\{y_n\}$, $\{Sy_n\}$, $\{T_1 y_n\}$, $\{T_2 y_n\}$ are bounded sequences. Define M as

$$M := \sup\{||x_n||, ||y_n||, ||T_1 y_n||, ||T_2 y_n||, ||Sy_n||, ||x_n - w||, ||y_n - w|| : n \in \mathbb{N}\}.$$

Besides, we know that

$$\begin{aligned} &\limsup_{n\to\infty} ||x_n - w|| \\ &\le \limsup_{n\to\infty} (d_n||x_0 - w|| + (1 - d_n)||y_n - w||) \\ &\le \limsup_{n\to\infty} d_n||x_0 - w|| + \limsup_{n\to\infty} ||y_n - w|| \\ &\le \limsup_{n\leftarrow\infty} ||x_{n-1} - w||. \end{aligned}$$

This implies that

$$\limsup_{n\to\infty} ||x_n - w|| = \limsup_{n\to\infty} ||y_n - w||.$$

By Lemma 2.1, there exists a strictly increasing, continuous, and convex function g: $[0, 2M] \to \mathbb{R}$ such that

$$\begin{aligned}
&||y_n - w||^2 \\
&\le a_n||x_{n-1} - w||^2 + b_n||T_1y_n - w||^2 + c_n||T_2y_n - w||^2 - a_nb_n||x_{n-1} - T_1y_n||^2 \\
&\le a_n||x_{n-1} - w||^2 + b_n||y_n - w||^2 + c_n||y_n - w||^2 - a_nb_n||x_{n-1} - T_1y_n||^2.
\end{aligned}$$

Then,

$$\begin{aligned}
&||y_n - w||^2 \\
\le\ & ||y_n - w||^2 + a||x_{n-1} - T_1y_n||^2 \\
\le\ & ||y_n - w||^2 + b_n||x_{n-1} - T_1y_n||^2 \\
\le\ & ||x_{n-1} - w||^2.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} ||x_{n-1} - T_1y_n|| = 0.$$

Similar, we have

$$\lim_{n\to\infty} ||x_{n-1} - T_2y_n|| = 0.$$

By (i),

$$\lim_{n\to\infty} ||x_n - Sy_n|| = \lim_{n\to\infty} d_n||x_0 - Sy_n|| = 0$$

and

$$\begin{aligned}
&||x_{n+1} - x_n|| \\
&= ||d_{n+1}x_0 + (1 - d_{n+1})Sy_{n+1} - d_nx_0 - (1 - d_n)Sy_n|| \\
&\le\ ||d_{n+1}x_0 + (1 - d_{n+1})Sy_{n+1} - d_nx_0 - (1 - d_n)Sy_{n+1}|| \\
&\quad + ||d_nx_0 + (1 - d_n)Sy_{n+1} - d_nx_0 - (1 - d_n)Sy_n|| \\
&\le\ |d_{n+1} - d_n| \cdot ||x_0|| + |d_{n+1} - d_n| \cdot ||Sy_{n+1}|| + (1 - d_n) \cdot ||Sy_{n+1} - Sy_n|| \\
&\le |d_{n+1} - d_n| \cdot ||x_0|| + |d_{n+1} - d_n| \cdot ||Sy_{n+1}|| + (1 - d_n) \cdot ||x_{n+1} - d_{n+1}x_0 - x_n + d_nx_0|| \\
&\le 2M \cdot |d_{n+1} - d_n| + (1 - d_n) \cdot (||x_{n+1} - x_n|| + |d_{n+1} - d_n| \cdot ||x_0||).
\end{aligned}$$

So,

$$||x_{n+1} - x_n|| \le \frac{3M \cdot |d_{n+1} - d_n|}{d_n}.$$

By (i),

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0.$$

Furthermore,

$$\lim_{n\to\infty} ||Sy_{n+1} - Sy_n|| = 0.$$

Next, we have

$$
\begin{aligned}
&\|y_{n+1} - y_n\| \\
&= \|(a_{n+1}x_n + b_{n+1}T_1y_{n+1} + c_{n+1}T_2y_{n+1}) - (a_n\, x_{n-1} + b_n\, T_1\, y_n + c_nT_2y_n)\| \\
&\le \|(a_{n+1}x_n + b_{n+1}T_1y_{n+1} + c_{n+1}T_2y_{n+1}) - (a_nx_n + b_nT_1y_{n+1} + c_nT_2y_{n+1})\| \\
&\quad + \|(a_nx_n + b_nT_1y_{n+1} + c_nT_2y_{n+1}) - (a_n\, x_{n-1} + b_n\, T_1\, y_n + c_nT_2y_n)\| \\
&\le |a_{n+1} - a_n| \cdot \|x_n\| + |b_{n+1} - b_n| \cdot \|T_1y_{n+1}\| + |c_{n+1} - c_n| \cdot \|T_2y_{n+1})\| \\
&\quad + a_n\|x_n - x_{n-1}\| + b_n\|T_1y_{n+1} - T_1\, y_n\| + c_n\|T_2y_{n+1} - T_2y_n\| \\
&\le M \cdot (|a_{n+1} - a_n| + |b_{n+1} - b_n| + |c_{n+1} - c_n|) \\
&\quad + a_n\|x_n - x_{n-1}\| + b_n\|y_{n+1} - y_n\| + c_n\|y_{n+1} - y_n\|.
\end{aligned}
$$

This implies that

$$\|y_{n+1} - y_n\| \le \frac{M \cdot (|a_{n+1} - a_n| + |b_{n+1} - b_n| + |c_{n+1} - c_n|)}{a_n} + \|x_n - x_{n-1}\|.$$

So,

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0.$$

Besides,

$$\lim_{n\to\infty} \|y_n - x_{n-1}\| = \lim_{n\to\infty} \|b_n(T_1y_n - x_{n-1}) + c_n(T_2y_n - x_{n-1})\| = 0$$

and

$$\lim_{n\to\infty} \|y_n - Sy_n\| = \lim_{n\to\infty} \|y_n - T_1y_n\| = \lim_{n\to\infty} \|y_n - T_2y_n\| = 0.$$

Let ϕ: $C \to \mathbb{R}$ be defined by $\phi(u)$: $= \mu_n\|x_n - u\|$ for each $u \in C$. Clearly, ϕ is convex and continuous. Taking $p \in \Omega$ and defining a subset D of C by

$$D := \{x \in C : \|x - p\| \le r\},$$

where r: $= \max\{\|x_0 - p\|, \|x_1 - p\|\}$. Then, D is a nonempty closed bounded convex subset of C and $\{x_n\} \subseteq D$. By Lemma 5.1,

$$C_{\min} := \{z \in D : \varphi(z) := \min_{y\in D}\varphi(y)\} \neq \emptyset.$$

Obviously, $C_{\min}$ is a bounded closed convex subset. Following the property of Banach limit μ_n, for all $z \in C_{\min}$, we have

$$
\begin{aligned}
\varphi(Sz) &= \mu_n\|x_n - Sz\|^2 \\
&\le \mu_n(\|x_n - y_n\| + \|y_n - Sz\|)^2 \\
&\le \mu_n(\|x_n - y_n\| + 3\|y_n - Sy_n\| + \|y_n - z\|)^2 \\
&= \mu_n\|y_n - z\|^2 \\
&\le \mu_n(\|y_n - x_n\| + \|x_n - z\|)^2 \\
&\le \mu_n\|x_n - z\|^2.
\end{aligned}
$$

Then, $Sz \in C_{\min}$. By Theorem 4 in [1], there exists $\bar{x} \in C_{\min}$ such that $S\bar{x} = \bar{x}$. By Lemma 5.3,

$$\mu_n\langle y-\bar{x}, J(x_n-\bar{x})\rangle \leq 0 \text{ for all } y \in C.$$

Take any $y \in C$ and let y be fixed. Since $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$, then it follows from the norm-weak* uniformly continuity of the duality mapping J that

$$\lim_{n\to\infty}\left(\langle y-\bar{x}, J(x_{n+1}-\bar{x})\rangle - \langle y-\bar{x}, J(x_n-\bar{x})\rangle\right) = 0.$$

By Lemma 5.4,

$$\lim_{n\to\infty}\langle y-\bar{x}, J(x_n-\bar{x})\rangle \leq 0 \text{ for all } y \in C.$$

By Lemma 5.2,

$$\begin{aligned}
&||x_n-\bar{x}||^2 \\
&= ||d_n(x_0-\bar{x}) + (1-d_n)(Sy_n-\bar{x})||^2 \\
&\leq (1-d_n)^2||Sy_n-\bar{x}||^2 + 2d_n\langle x_0-\bar{x}, J(x_n-\bar{x})\rangle \\
&\leq (1-d_n)^2||y_n-\bar{x}||^2 + 2d_n\langle x_0-\bar{x}, J(x_n-\bar{x})\rangle \\
&\leq (1-d_n)||x_{n-1}-\bar{x}||^2 + 2d_n\langle x_0-\bar{x}, J(x_n-\bar{x})\rangle.
\end{aligned}$$

By Lemma 5.5, $\lim_{n\to\infty}||x_n-\bar{x}|| = 0$. Furthermore, since T_1 and T_2 are nonexpansive mappings, we know that $\bar{x}$ is also a fixed point of T_1 and T_2. Therefore, the proof is completed.

The following is a special case of Theorem 5.1 when T_1 and T_2 are identity mappings.

Theorem 5.2. Let E be a uniformly convex and uniformly smooth Banach space with Opial's condition, C be a nonempty closed convex subset of E, and let S: $C \to C$ be a mapping with condition (C). Let $\{d_n\}$ be a sequence in (0,1). Suppose that $F(S) \neq \emptyset$. Define a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C \text{ chosen arbitrary,} \\ x_n = d_n x_0 + (1-d_n)Sx_{n-1}, n \in \mathbb{N}. \end{cases}$$

Assume that $\lim_{n\to\infty}d_n = 0, \sum_{n=1}^{\infty} d_n = \infty$, and $\lim_{n\to\infty}\dfrac{|d_{n+1}-d_n|}{d_n} = 0.$. Then, $\lim_{n\to\infty}x_n = \lim_{n\to\infty}y_n = \bar{x}$ for some $\bar{x} \in F(S)$.

6 Application

Let E be a reflexive, strictly convex, and smooth Banach space and *let* $A \subseteq E \times E^*$ be a set-valued mapping with range $R(A)$: = $\{x^* \in E^*: x^* \in Ax\}$ and domain $D(A) = \{x \in E : Ax \neq \emptyset\}$. Then, the mapping A is said to be monotone if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever (x, x^*), $(y, y^*) \in A$. It is also said to be maximal monotone if A is monotone and there is no monotone operator from E into E^* whose graph properly contains the graph of A. It is known that if $A \subseteq E \times E^*$ is maximal monotone, then $A^{-1}0$ is closed and convex.

Lemma 6.1. [30] Let E be a reflexive, strictly convex, and smooth Banach space and let $A \subseteq E \times E^*$ be a monotone operator. Then, A is maximal monotone if and only if $R(J + rA) = E^*$ for all $r > 0$.

By Lemma 6.1, for every $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that $Jx \in Jx_r + rAx_r$. Hence, define a single valued mapping $J_r : E \to D(A)$ by $J_r x = x_r$, that is, $J_r = (J + rA)^{-1} J$ and such J_r is called the relative resolvent of A. We know that $A^{-1}0 = F(J_r)$ for all $r > 0$ [8].

Lemma 6.2. [21] Let E be a uniformly convex and uniformly smooth Banach space and let $A \subseteq E \times E^*$ be a maximal monotone operator. Let J_r be the relative resolvent of A, where $r > 0$. If $A^{-1}0$ is nonempty, then J_r is a relatively nonexpansive mapping on E.

By Theorem 4.1 and Lemma 6.2, it is easy to get the following result.

Theorem 6.1. Let E be a uniformly convex and uniformly smooth Banach space with property (Q), and let C be a nonempty closed convex subset of E, and let $T_1, T_2 : C \to C$ be two nonexpansive mappings, and let $A \subseteq E \times E^*$ be a maximal monotone operator. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences in (0,1) with and $a_n + b_n + c_n = 1$. Suppose that $\Omega := A^{-1}0 \cap F(T_1) \cap F(T_2) \neq \emptyset$. Define a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C \text{ chosen arbitrary and } C_0 = D_0 = C, \\ y_n = a_n x_{n-1} + b_n T_1 y_n + c_n T_2 y_n, \\ z_n = J^{-1}(d_n J y_n + (1-d_n) J J_r y_n), \\ C_n = \{z \in C_{n-1} : \phi(z, z_n) \le \phi(z, y_n)\}, \\ D_n = \{z \in D_{n-1} : \|y_n - z\| \le \|x_{n-1} - z\|\}, \\ x_n = \Pi_{C_n \cap D_n} x_0. \end{cases}$$

Assume that $\liminf_{n\to\infty} b_n > 0$, $\liminf_{n\to\infty} c_n > 0$, and $\liminf_{n\to\infty} d_n(1 - d_n) > 0$. Then, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \Pi_\Omega x_0$.

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L-JL responsible for problem resign, coordinator, discussion, revise the important part, and submit. C-SC is responsible for the important results of this article, discuss, and draft. Z-TY is responsible for discussion and the applications. All authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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