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Fixed point of generalized weakly contractive mappings in ordered partial metric spaces

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Abstract

In this article, we prove some fixed point results for generalized weakly contractive mappings defined on a partial metric space. We provide some examples to validate our results. These results unify, generalize and complement various known comparable results from the current literature.

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1 Introduction and preliminaries

In the past years, the extension of the theory of fixed point to generalized structures as cone metrics, partial metric spaces and quasi-metric spaces has received much attention (see, for instance, [1-7] and references therein). Partial metric space is generalized metric space in which each object does not necessarily have to have a zero distance from itself [8]. A motivation behind introducing the concept of a partial metric was to obtain appropriate mathematical models in the theory of computation and, in particular, to give a modified version of the Banach contraction principle, more suitable in this context [8,9]. Salvador and Schellekens [10] have shown that the dual complexity space can be modelled as stable partial monoids. Subsequently, several authors studied the problem of existence and uniqueness of a fixed point for mappings satisfying different contractive conditions (e.g., [1,2,11-18]).

Existence of fixed points in ordered metric spaces has been initiated in 2004 by Ran and Reurings [19], and further studied by Nieto and Lopez [20]. Subsequently, several interesting and valuable results have appeared in this direction [21-28].

The aim of this article is to study the necessary conditions for existence of common fixed points of four maps satisfying generalized weak contractive conditions in the framework of complete partial metric spaces endowed with a partial order. Our results extend and strengthen various known results [8,29-32].

In the sequel, the letters $\mathbb{R}$, $\mathbb{R}^+$, ω and $\mathbb{N}$ will denote the set of real numbers, the set of nonnegative real numbers, the set of nonnegative integer numbers and the set of positive integer numbers, respectively. The usual order on $\mathbb{R}$ (respectively, on $\mathbb{R}^+$) will be indistinctly denoted by $\leq$ or by $\geq$.

Consistent with [8,12], the following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set. A mapping $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold true:

(P_1) $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$;

(P_2) $p(x, x) \leq p(x, y)$;

(P_3) $p(x, y) = p(y, x)$;

(P_4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is then called a partial metric space. Throughout this article, X will denote a partial metric space equipped with a partial metric p unless or otherwise stated.

If $p(x, y) = 0$, then (P_1) and (P_2) imply that $x = y$. But converse does not hold always.

A trivial example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $p(x, y) = \max\{x, y\}$.

Example 1.2. [8] If $X = \{[a, b]: a, b \in \mathbb{R}, a \leq b\}$ then $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$ defines a partial metric p on X.

For some more examples of partial metric spaces, we refer to [12,13,16,17].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon): x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

Observe (see [8, p. 187]) that a sequence $\{x_n\}$ in X converges to a point $x \in X$, with respect to τ_p, if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

If p is a partial metric on X, then the function $p^S : X \times X \to \mathbb{R}^+$ given by $p^S(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, defines a metric on X.

Furthermore, a sequence $\{x_n\}$ converges in (X, p^S) to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x). \tag{1.1}$$

Definition 1.3. [8] Let X be a partial metric space.

(a) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

(b) X is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $\lim_{n\to\infty} p(x, x_n) = p(x, x)$. In this case, we say that the partial metric p is complete.

Lemma 1.4. [8,12] Let X be a partial metric space. Then:

(a) A sequence $\{x_n\}$ in X is a Cauchy sequence in X if and only if it is a Cauchy sequence in metric space (X, p^S).

(b) A partial metric space X is complete if and only if the metric space (X, p^S) is complete.

Definition 1.5. A mapping $f : X - X$ is said to be a weakly contractive if

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)), \quad \text{for all } x, y \in X, \tag{1.2}$$

In 1997, Alber and Guerre-Delabriere [33] proved that weakly contractive mapping defined on a Hilbert space is a Picard operator. Rhoades [34] proved that the corresponding result is also valid when Hilbert space is replaced by a complete metric space. Dutta et al. [35] generalized the weak contractive condition and proved a fixed point theorem for a selfmap, which in turn generalizes Theorem 1 in [34] and the corresponding result in [33].

Recently, Aydi [29] obtained the following result in partial metric spaces.

Theorem 1.6. Let $(X, \leq_X)$ be a partially ordered set and let p be a partial metric on X such that (X, p) is complete. Let $f : X \to X$ be a nondecreasing map with respect to $\leq_X$. Suppose that the following conditions hold: for $y \leq x$, we have

(i)

$$p(fx, fy) \leq p(x, y) - \varphi(p(x, y)), \tag{1.3}$$

where $\phi : [0, +\infty[\to [0, +\infty[$ is a continuous and non-decreasing function such that it is positive in $]0, +\infty[$, $\phi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$;

(ii) there exist $x_0 \in X$ such that $x_0 \leq_X fx_0$;

(iii) f is continuous in (X, p), or;

(iii) if a non-decreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \leq_X \times$ for all n.

Then f has a fixed point $u \in X$. Moreover, $p(u, u) = 0$.

A nonempty subset W of a partially ordered set X is said to be well ordered if every two elements of W are comparable.

2 Fixed point results

In this section, we obtain several fixed point results for selfmaps satisfying generalized weakly contractive conditions defined on an ordered partial metric space, i.e., a (partially) ordered set endowed with a complete partial metric.

We start with the following result.

Theorem 2.1. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exist a complete partial metric p on X and f a nondecreasing selfmap on X. Suppose that for every two elements $x, y \in X$ with $y \preccurlyeq x$, we have

$$\psi(p(fx, fy)) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{2.1}$$

where

$$M(x, y) = \max\{p(x, y), p(fx, x), p(fy, y), \frac{p(x, fy) + p(y, fx)}{2}\},$$

$\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, ψ is continuous and nondecreasing, φ is a lower semicontinuous, and $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq fx_0$ and one of the following two conditions is satisfied:

(a) f is continuous self map on (X, p^S);

(b) for any nondecreasing sequence $\{x_n\}$ in $(X, \preccurlyeq)$ with $\lim_{n\to\infty} p^S(z, x_n) = 0$ it follows $x_n \preccurlyeq z$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Note that if f has a fixed point u, then $p(u, u) = 0$. Indeed, assume that $p(u, u) > 0$. Then from (2.1) with $x = y = u$, we have

$$\psi(p(u,u)) = \psi(p(fu,fu)) \le \psi(M(u,u)) - \phi(M(u,u)), \tag{2.2}$$

where

$$\begin{aligned} M(u,u) &= \max\{p(u,u), p(fu,u), p(fu,u), \frac{p(u,fu)+p(u,fu)}{2}\} \\ &= \max\{p(u,u), p(u,u), p(u,u), \frac{p(u,u)+p(u,u)}{2}\} = p(u,u). \end{aligned}$$

Now we have:

$$\psi(p(u,u)) = \psi(p(fu,fu)) \le \psi(p(u,u)) - \phi(p(u,u)),$$

$\varphi(p(u, u)) \le 0$, a contradiction. Hence $p(u, u) = 0$. Now we shall prove that there exists a nondecreasing sequence $\{x_n\}$ in $(X, \preccurlyeq)$ with $fx_n = x_{n+1}$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. For this, let x_0 be an arbitrary point of X. Since f is nondecreasing, and $x_0 \preccurlyeq fx_0$, we have

$$x_1 = fx_0 \preccurlyeq f^2x_0 \preccurlyeq \ldots \preccurlyeq f^n x_0 \preccurlyeq f^{n+1}x_0 \preccurlyeq \ldots.$$

Define a sequence $\{x_n\}$ in X with $x_n = f^n x_0$ and so $x_{n+1} = fx_n$ for $n \in \mathbb{N}$. We may assume that $M(x_{n+1}, x_n) > 0$, for all $n \in \mathbb{N}$. If not, then it is clear that $x_k = x_{k+1}$ for some k, so $fx_k = x_{k+1} = x_k$, and thus x_k is a fixed point of f. Now, by taking $M(x_{n+1}, x_n) > 0$ for all $n \in \mathbb{N}$, consider

$$\begin{aligned} \psi(p(x_{n+2},x_{n+1})) &= \psi(p(fx_{n+1},fx_n)) \\ &\le \psi(M(x_{n+1},x_n)) - \phi(M(x_{n+1},x_n)), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} M(x_{n+1},x_n) &= \max\{p(x_{n+1},x_n), p(fx_{n+1},x_{n+1}), p(fx_n,x_n), \\ &\quad \frac{p(x_{n+1},fx_n)+p(x_n,fx_{n+1})}{2}\} \\ &= \max\{p(x_{n+1},x_n), p(x_{n+2},x_{n+1}), p(x_{n+1},x_n), \\ &\quad \frac{p(x_{n+1},x_{n+1})+p(x_n,x_{n+2})}{2}\} \\ &\le \max\{p(x_{n+1},x_n), p(x_{n+2},x_{n+1}), \\ &\quad \frac{p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})}{2}\} \\ &= \max\{p(x_{n+1},x_n), p(x_{n+2},x_{n+1})\}. \end{aligned}$$

Suppose that $\max\{p(x_{k+1}, x_k), p(x_{k+2}, x_{k+1})\} = p(x_{k+2}, x_{k+1})$ for some $k \in \mathbb{N}$.

Then $\psi(p(x_{k+2}, x_{k+1})) \le \psi(p(x_{k+2}, x_{k+1})) - \varphi(M(x_{k+1}, x_k))$ implies $\varphi(M(x_{k+1}, x_k)) \le 0$, a contradiction. Consequently

$$\psi(p(x_{n+2}, x_{n+1})) \leq \psi(p(x_{n+1}, x_n)) - \phi(M(x_{n+1}, x_n))$$
$$\leq \psi(p(x_{n+1}, x_n)),$$

for all $n \in \mathbb{N}$. Since ψ is nondecreasing, so the sequence of positive real numbers $\{p(x_{n+1}, x_n)\}$ is nonincreasing, therefore $\{p(x_{n+1}, x_n)\}$ converges to a $c \geq 0$. Suppose that $c > 0$. Then

$$\psi(p(x_{n+2}, x_{n+1})) \leq \psi(M(x_{n+1}, x_n)) - \phi(M(x_{n+1}, x_n)),$$

and lower semicontinuity of φ gives that

$$\limsup_{n\to\infty} \psi(p(x_{n+2}, x_{n+1})) \leq \limsup_{n\to\infty} \psi(M(x_{n+1}, x_n)) - \liminf_{n\to\infty} \phi(M(x_{n+1}, x_n)),$$

which implies that $\psi(c) \leq \psi(c) - \varphi(c)$, a contradiction. Therefore $c = 0$, i.e., $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$.

Now, we prove that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. If not, then there exists $\varepsilon > 0$ and sequences $\{n_k\}$, $\{m_k\}$ in $\mathbb{N}$, with $n_k > m_k \geq k$, and such that $p(x_{n_k}, x_{m_k}) \geq \varepsilon$ for all $k \in \mathbb{N}$. We can suppose, without loss of generality that $p(x_{n_k}, x_{m_k-1}) < \varepsilon$.

So

$$\varepsilon \leq p(x_{m_k}, x_{n_k}) \leq p(x_{n_k}, x_{m_k-1}) + p(x_{m_k-1}, x_{m_k}) - p(x_{m_k-1}, x_{m_k-1})$$

implies that

$$\lim_{k\to\infty} p(x_{m_k}, x_{n_k}) = \varepsilon. \tag{2.4}$$

Also (2.4) and inequality $p(x_{m_k}, x_{n_k}) \leq p(x_{m_k}, x_{m_k-1}) + p(x_{m_k-1}, x_{n_k}) - p(x_{m_k-1}, x_{m_k-1})$ gives that $\varepsilon \leq \lim_{k\to\infty} p(x_{m_k-1}, x_{n_k})$, while (2.4) and inequality $p(x_{m_k-1}, x_{n_k}) \leq p(x_{m_k-1}, x_{m_k}) + p(x_{m_k}, x_{n_k}) - p(x_{m_k}, x_{m_k})$ yields $\lim_{k\to\infty} p(x_{m_k-1}, x_{n_k}) \leq \varepsilon$, and hence

$$\lim_{k\to\infty} p(x_{m_k-1}, x_{n_k}) = \varepsilon. \tag{2.5}$$

Also (2.5) and inequality $p(x_{m_k-1}, x_{n_k}) \leq p(x_{m_k-1}, x_{n_k+1}) + p(x_{n_k+1}, x_{n_k}) - p(x_{n_k+1}, x_{n_k+1})$ implies that $\varepsilon \leq \lim_{k\to\infty} p(x_{m_k-1}, x_{n_k+1})$, while inequality $p(x_{m_k-1}, x_{n_k+1}) \leq p(x_{m_k-1}, x_{n_k}) + p(x_{n_k}, x_{n_k+1}) - p(x_{n_k}, x_{n_k})$ yields $\lim_{k\to\infty} p(x_{m_k-1}, x_{n_k+1}) \leq \varepsilon$, and

hence

$$\lim_{k\to\infty} p(x_{m_k-1}, x_{n_k+1}) = \varepsilon. \tag{2.6}$$

Finally $p(x_{m_k}, x_{n_k}) \leq p(x_{m_k}, x_{n_k+1}) + p(x_{n_k+1}, x_{n_k}) - p(x_{n_k+1}, x_{n_k+1})$ gives that $\varepsilon < \lim_{k\to\infty} p(x_{m_k}, x_{n_k+1})$, and the inequality $p(x_{m_k}, x_{n_k+1}) \leq p(x_{m_k}, x_{n_k}) + p(x_{n_k}, x_{n_k+1}) - p(x_{n_k}, x_{n_k})$ gives $\lim_{k\to\infty} p(x_{m_k}, x_{n_k+1}) \leq \varepsilon$, and hence

$$\lim_{k\to\infty} p(x_{m_k}, x_{n_k+1}) = \varepsilon. \tag{2.7}$$

As

$$\begin{aligned} M(x_{n_k}, x_{m_k-1}) &= \max\{p(x_{n_k}, x_{m_k-1}), p(fx_{n_k}, x_{n_k}), \\ &\quad p(fx_{m_k-1}, x_{m_k-1}), \frac{p(x_{n_k}, fx_{m_k-1}) + p(x_{m_k-1}, fx_{n_k})}{2}\} \\ &= \max\{p(x_{n_k}, x_{m_k-1}), p(x_{n_k+1}, x_{n_k}), \\ &\quad p(x_{m_k}, x_{m_k-1}), \frac{p(x_{n_k}, x_{m_k}) + p(x_{m_k-1}, x_{n_k+1})}{2}\}, \end{aligned}$$

therefore $\lim_{k\to\infty} M(x_{n_k}, x_{m_k-1}) = \max\{\varepsilon, 0, 0, \varepsilon\} = \varepsilon$. From (2.1), we obtain

$$\psi(p(x_{n_k+1}, x_{m_k})) = \psi(p(fx_{n_k}, fx_{m_k-1})) \le \psi(M(x_{n_k}, x_{m_k-1})) - \phi(M(x_{n_k}, x_{m_k-1})).$$

Taking upper limit as $k \to \infty$ implies that $\psi(\varepsilon) \le \psi(\varepsilon) - \varphi(\varepsilon)$, which is a contradiction as $\varepsilon > 0$. Thus, we obtain that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, i.e., $\{x_n\}$ is a Cauchy sequence in (X, p), and hence in the metric space (X, p^S) by Lemma 1.4. Finally, we prove that f has a fixed point. Indeed, since (X, p) is complete, then from Lemma 1.4, (X, p^S) is also complete, so the sequence $\{x_n\}$ is convergent in the metric space (X, p^S). Therefore, there exists $u \in X$ such that $\lim_{n\to\infty} p^S(u, x_n) = 0$, equivalently,

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, u) = p(u, u) = 0, \tag{2.8}$$

because $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. If f is continuous self map on (X, p^S), then it is clear that $fu = u$. If f is not continuous, we have, by our hypothesis, that $x_n \preccurlyeq u$ for all $n \in \mathbb{N}$, because $\{x_n\}$ is a nondecreasing sequence with $\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, u) = p(u, u) = 0,$. Now from the following inequalities

$$\begin{aligned} p(fu, u) &\le M(u, x_n) \\ &= \max\{p(u, x_n), p(fu, u), p(fx_n, x_n), \\ &\quad \frac{p(u, fx_n) + p(x_n, fu)}{2}\} \\ &= \max\{p(u, x_n), p(fu, u), p(x_{n+1}, x_n), \\ &\quad \frac{p(u, x_{n+1}) + p(x_n, fu)}{2}\} \\ &\le \max\{p(u, x_n), p(fu, u), p(x_{n+1}, x_n), \\ &\quad \frac{p(x_{n+1}, u) + p(x_n, u) + p(u, fu) - p(u, u)}{2}\}, \end{aligned}$$

we deduce, taking limit as $n \to \infty$, that

$$\lim_{n\to\infty} M(u, x_n) = p(fu, u).$$

Hence,

$$\psi(p(fu, fx_{n+1})) \le \psi(M(u, x_n)) - \phi(M(u, x_n)). \tag{2.9}$$

On taking upper limit as $n \to \infty$, we have

$$\psi(p(fu, u)) \le \psi(p(fu, u)) - \phi(p(fu, u)),$$

and $fu = u$. Finally, suppose that set of fixed points of f is well ordered. We prove that fixed point of f is unique. Assume on contrary that $fv = v$ and $fw = w$ but $v \neq w$. Hence

$$\psi(p(v,w)) = \psi(p(fv,fw)) \leq \psi(M(v,w)) - \phi(M(v,w)), \tag{2.10}$$

where

$$\begin{aligned} M(v,w) &= \max\{p(v,w), p(fv,v), p(fw,w), \frac{p(v,fw)+p(w,fv)}{2}\} \\ &= \max\{p(v,w), p(v,v), p(w,w), \frac{p(v,w)+p(w,v)}{2}\} \\ &= p(v,w), \end{aligned}$$

that is, by (2.10),

$$\psi(p(v,w)) = \psi(p(fv,fw)) \leq \psi(p(v,w)) - \phi(p(v,w)),$$

a contradiction because $p(v, w) > 0$. Hence $v = w$. Conversely, if f has only one fixed point then the set of fixed point of f being singleton is well ordered. □

Consistent with the terminology in [36], we denote Υ the set of all functions ϕ: $\mathbb{R}^+ \to \mathbb{R}^+$, where ϕ is a Lebesgue integrable mapping with finite integral on each compact subset of $\mathbb{R}^+$, nonnegative, and for each $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t)dt > 0$ (see also, [37]). As a consequence of Theorem 2.1, we obtain following fixed point result for a mapping satisfying contractive conditions of integral type in a complete partial metric space X.

Corollary 2.2. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exist a complete partial metric p on X and f a nondecreasing selfmap on X. Suppose that for every two elements $x, y \in X$ with $y \preccurlyeq x$, we have

$$\int_0^{\psi(p(fx,fy))} \varphi(t)dt \leq \int_0^{\psi(M(x,y))} \varphi(t)dt - \int_0^{\phi(M(x,y))} \varphi(t)dt, \tag{2.11}$$

where $\phi \in \Upsilon$,

$$M(x,y) = \max\{p(x,y), p(fx,x), p(fy,y), \frac{p(x,fy)+p(y,fx)}{2}\},$$

$\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, ψ is continuous and nondecreasing, φ is a lower semicontinuous, and $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq fx_0$ and one of the following two conditions is satisfied:

(a) f is continuous self map on (X, p^S);

(b) for any nondecreasing sequence $\{x_n\}$ in $(X, \preccurlyeq)$ with $\lim_{n\to\infty} p^S(z, x_n) = 0$ it follows $x_n \preccurlyeq z$ for all $n \in \mathbb{N}$,

then f has a fixed point.

Proof. Define Ψ: $[0, \infty) \to [0, \infty)$ by $\Psi(x) = \int_0^x \varphi(t)dt$, then from (2.11), we have

$$\Psi(\psi(p(fx,fy))) \leq \Psi(\psi(M(x,y))) - \Psi(\phi(M(x,y))), \tag{2.12}$$

which can be written as

$$\psi_1(p(fx, fy)) \leq \psi_1(M(x, y)) - \phi_1(M(x, y)), \tag{2.13}$$

where $\psi_1 = \Psi \circ \psi$ and $\varphi_1 = \Psi \circ \varphi$. Clearly, ψ_1, $\varphi_1 : \mathbb{R}^+ \to \mathbb{R}^+$, ψ_1 is continuous and nondecreasing, φ_1 is a lower semicontinuous, and $\psi_1(t) = \varphi_1(t) = 0$ if and only if $t = 0$. Hence by Theorem 2.1, f has a fixed point. □

If we take $\psi(t) = t$ in Theorem 2.1, we have the following corollary.

Corollary 2.3. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exist a complete partial metric p on X and f a nondecreasing selfmap on X. Suppose that for every two elements $x, y \in X$ with $y \preccurlyeq x$, we have

$$p(fx, fy) \leq M(x, y) - \phi(M(x, y)), \tag{2.14}$$

where

$$M(x, y) = \max\left\{p(x, y), p(fx, x), p(fy, y), \frac{p(x, fy) + p(y, fx)}{2}\right\},$$

$\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous and $\varphi(t) = 0$ if and only if $t = 0$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq fx_0$ and one of the following two conditions is satisfied:

(a) f is continuous self map on (X, p^S);

(b) for any nondecreasing sequence $\{x_n\}$ in $(X, \preccurlyeq)$ with $\lim_{n\to\infty} p^S(z, x_n) = 0$ it follows $x_n \preccurlyeq z$ for all $n \in \infty$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

If we take $\varphi(t) = (1 - k)t$ for $k \in [0, 1)$ in Corollary 2.3, we have the following corollary.

Corollary 2.4. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exist a complete partial metric p on X and f a nondecreasing selfmap on X. Suppose that for every two elements $x, y \in X$ with $y \preccurlyeq x$, we have

$$p(fx, fy) \leq kM(x, y), \tag{2.15}$$

where

$$M(x, y) = \max\left\{p(x, y), p(fx, x), p(fy, y), \frac{p(x, fy) + p(y, fx)}{2}\right\},$$

and $k \in [0, 1)$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq fx_0$ and one of the following two conditions is satisfied:

(a) f is continuous self map on (X, p^S);

(b) for any nondecreasing sequence $\{x_n\}$ in $(X, \preccurlyeq)$ with $\lim_{n\to\infty} p^S(z, x_n) = 0$ it follows $x_n \preccurlyeq z$ for all $n \in \infty$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Corollary 2.5. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exist a complete partial metric p on X and f a nondecreasing selfmap on X. Suppose that for every two elements $x, y \in X$ with $y \preccurlyeq x$, we have

$$\psi(p(fx, fy)) \leq p(x, y) - \phi(p(x, y)), \tag{2.16}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous, and $\varphi(t) = 0$ if and only if $t = 0$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq fx_0$ and one of the following two conditions is satisfied:

(a) f is continuous self map on (X, p^S);

(b) for any nondecreasing sequence $\{x_n\}$ in $(X, \preccurlyeq)$ with $\lim_{n\to\infty} p^S(z, x_n) = 0$ it follows $x_n \preccurlyeq z$ for all $n \in \infty$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Theorem 2.6. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exist a complete partial metric p on X and f a nondecreasing selfmap on X. Suppose that for $x, y \in X$ with $y \preccurlyeq x$, we have

$$\psi(p(fx, fy)) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{2.17}$$

Where

$$M(x, y) = a_1 p(x, y) + a_2 p(fx, x) + a_3 p(fy, y) + a_4 p(fy, x) + a_5 p(fx, y),$$

a_1, $a_2 > 0$, $a_i \geq 0$ for $i = 3, 4, 5$, and, if $a_4 \geq a_5$, then $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, and if $a_4 < a_5$, then $a_1 + a_2 + a_3 + a_4 + 2a_5 < 1$ and $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, ψ is a continuous and nondecreasing, φ is a lower semicontinuous, and $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq fx_0$ and one of the following two conditions is satisfied:

(a) f is continuous self map on (X, p^S);

(b) for any nondecreasing sequence $\{x_n\}$ in $(X, \preccurlyeq)$ with $\lim_{n\to\infty} p^S(z, x_n) = 0$ it follows $x_n \preccurlyeq z$ for all $n \in \infty$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. If f has a fixed point u, then $p(u, u) = 0$. Assume that $p(u, u) > 0$. Then from (2.17) with $x = y = u$, we have

$$\psi(p(u, u)) = \psi(p(fu, fu)) \leq \psi(M(u, u)) - \phi(M(u, u)), \tag{2.18}$$

where

$$\begin{aligned} M(u, u) &= a_1 p(u, u) + a_2 p(fu, u) + a_3 p(fu, u) + a_4 p(fu, u) + a_5 p(fu, u) \\ &= (a_1 + a_2 + a_3 + a_4 + a_5) p(u, u), \end{aligned}$$

that is

$$\psi(p(u, u)) \leq \psi((a_1 + a_2 + a_3 + a_4 + a_5) p(u, u)) - \phi((a_1 + a_2 + a_3 + a_4 + a_5) p(u, u)),$$

$\varphi((a_1 + a_2 + a_3 + a_4 + a_5)p(u, u)) \leq 0$, a contradiction. Hence $p(u, u) = 0$. Now let x_0 be an arbitrary point of X: If $fx_0 = x_0$, then the proof is finished, so we assume that $fx_0 \neq x_0$. As in Theorem 2.1, define a sequence $\{x_n\}$ in X with $x_n = f^n x_0$ and so $x_{n+1} = fx_n$ for $n \in \mathbb{N}$. If $M(x_{k+1}, x_k) = 0$ for some k, then x_k is a fixed point of f, as in the proof of Theorem 2.1. Thus we can suppose that

$$M(x_{k+1}, x_k) > 0, \quad \text{for all } k \in \mathbb{N}. \tag{2.19}$$

Now form (2.17), consider

$$\psi(p(x_{n+2}, x_{n+1})) = \psi(p(fx_{n+1}, fx_n)) \leq \psi(M(x_{n+1}, x_n)) - \phi(M(x_{n+1}, x_n)), \tag{2.20}$$

where

$$\begin{aligned} M(x_{n+1}, x_n) &= a_1 p(x_{n+1}, x_n) + a_2 p(fx_{n+1}, x_{n+1}) + a_3 p(fx_n, x_n) \\ &\quad + a_4 p(fx_n, x_{n+1}) + a_5 p(fx_{n+1}, x_n) \\ &= a_1 p(x_{n+1}, x_n) + a_2 p(x_{n+2}, x_{n+1}) + a_3 p(x_{n+1}, x_n) \\ &\quad + a_4 p(x_{n+1}, x_{n+1}) + a_5 p(x_{n+2}, x_n) \\ &\leq (a_1 + a_3) p(x_{n+1}, x_n) + a_2 p(x_{n+2}, x_{n+1}) + a_4 p(x_{n+1}, x_{n+1}) \\ &\quad + a_5 [p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n) - p(x_{n+1}, x_{n+1})] \\ &= (a_1 + a_3 + a_5) p(x_{n+1}, x_n) + (a_2 + a_5) p(x_{n+2}, x_{n+1}) \\ &\quad + (a_4 - a_5) p(x_{n+1}, x_{n+1}). \end{aligned}$$

We claim that

$$p(x_{n+2}, x_{n+1}) \leq p(x_{n+1}, x_n), \tag{2.21}$$

for all $n \in \mathbb{N}$. Suppose that it is not true, that is, $p(x_{k+2}, x_{k+1}) > p(x_{k+1}, x_k)$ for some $k \in \mathbb{N}$. Now, since $x_k \preccurlyeq x_{k+1}$, also if $a_4 \geq a_5$, then we have

$$\begin{aligned} \psi(p(x_{k+2}, x_{k+1})) &\leq \psi(M(x_{k+1}, x_k)) - \phi(M(x_{k+1}, x_k)) \\ &\leq \psi((a_1 + a_3 + a_5) p(x_{k+1}, x_k) + (a_2 + a_5) p(x_{k+2}, x_{k+1}) \\ &\quad + (a_4 - a_5) p(x_{k+1}, x_{k+1})) - \phi(M(x_{k+1}, x_k)) \\ &\leq \psi((a_1 + a_2 + a_3 + a_4 + a_5) p(x_{k+2}, x_{k+1})) \\ &\quad - \phi(M(x_{k+1}, x_k)) \\ &\leq \psi(p(x_{k+1}, x_{k+2})) - \phi(M(x_{k+1}, x_k)), \end{aligned}$$

which implies that $\varphi(M(x_{k+1}, x_k) \leq 0$, by the property of φ, we have $M(x_{k+1}, x_k) = 0$, a contradiction. If $a_4 < a_5$, then

$$\begin{aligned} \psi(p(x_{k+2}, x_{k+1})) &\leq \psi(M(x_{k+1}, x_k)) - \phi(M(x_{k+1}, x_k)) \\ &\leq \psi((a_1 + a_3 + a_5) p(x_{k+1}, x_k) + (a_2 + a_5) p(x_{k+2}, x_{k+1}) \\ &\quad + (a_4 - a_5) p(x_{k+1}, x_{k+1})) - \phi(M(x_{k+1}, x_k)) \\ &\leq \psi((a_1 + a_2 + a_3 + a_4 + 2a_5) p(x_{k+2}, x_{k+1}) \\ &\quad - \phi(M(x_{k+1}, x_k)) \\ &\leq \psi(p(x_{k+1}, x_{k+2})) - \phi(M(x_{k+1}, x_k)), \end{aligned}$$

implies that $\varphi(M(x_{k+1}, x_k) \leq 0$, a contradiction. Hence $p(x_{n+2}, x_{n+1}) \leq p(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$ and so the sequence of positive real numbers $\{p(x_{n+1}, x_n)\}$ is nonincreasing. Thus $\{p(x_{n+1}, x_n)\}$ converges to a $c \geq 0$. Suppose that $c > 0$. Now, lower semicontinuity of φ gives that

$$\begin{aligned}\limsup_{n\to\infty}\psi(p(x_{n+2},x_{n+1})) &\le \limsup_{n\to\infty}\psi(M(x_{n+1},x_n)) - \liminf_{n\to\infty}\phi(M(x_{n+1},x_n))\\ &\le \limsup_{n\to\infty}\psi((a_1+a_3+a_5)p(x_n,x_{n+1})\\ &\quad + (a_2+a_5)p(x_{n+2},x_{n+1}) + (a_4-a_5)p(x_{n+1},x_{n+1}))\\ &\quad - \liminf_{n\to\infty}\phi(M(x_{n+1},x_n)),\end{aligned}$$

which implies that

$$\psi(c) \le \psi(c) - \liminf_{n\to\infty}\phi(M(x_{n+1},x_n)),$$

a contradiction since $a_1 > 0$. Therefore $c = 0$, i.e., $\lim_{n\to\infty} p(x_{n+1},x_n) = 0$. Now we show that the sequence $\{x_n\}$ is a Cauchy sequence in (X, p). Indeed, we first prove that $\lim_{n,m\to\infty} p(x_n,x_m) = 0$. Assume the contrary. Then there exists $\varepsilon > 0$ and sequences $\{n_k\}$, $\{m_k\}$ in ∞, with $n_k > m_k \ge k$, and such that $p(x_{n_k},x_{m_k}) \ge \varepsilon$ for all $k \in \mathbb{N}$. We can suppose, without loss of generality, that $p(x_{n_k},x_{m_k-1}) < \varepsilon$.

Follows the similar argument as in Theorem 2.1, we have $\lim_{k\to\infty} p(x_{m_k},x_{n_k}) = \varepsilon$, $\lim_{k\to\infty} p(x_{m_k-1},x_{n_k+1}) = \varepsilon$, $\lim_{k\to\infty} p(x_{m_k-1},x_{n_k+1}) = \varepsilon$ and $\lim_{k\to\infty} p(x_{m_k},x_{n_k+1}) = \varepsilon$. As

$$\begin{aligned}M(x_{n_k},x_{m_k-1}) &= a_1 p(x_{n_k},x_{m_k-1}) + a_2 p(fx_{n_k},x_{n_k}) + a_3 p(fx_{m_k-1},x_{m_k-1})\\ &\quad + a_4 p(fx_{m_k-1},x_{n_k}) + a_5 p(fx_{n_k},x_{m_k-1})\\ &= a_1 p(x_{n_k},x_{m_k-1}) + a_2 p(x_{n_k+1},x_{n_k}) + a_3 p(x_{m_k},x_{m_k-1})\\ &\quad + a_4 p(x_{m_k},x_{n_k}) + a_5 p(x_{n_k+1},x_{m_k-1}),\end{aligned}$$

thus $\lim_{k\to\infty} M(x_{n_k},x_{m_k-1}) = (a_1+a_4+a_5)\varepsilon \le \varepsilon$. From (2.17), we obtain

$$\begin{aligned}\psi(p(x_{n_k+1},x_{m_k})) &= \psi(p(fx_{n_k},fx_{m_k-1}))\\ &\le \psi(M(x_{n_k},x_{m_k-1})) - \phi(M(x_{n_k},x_{m_k-1})).\end{aligned} \tag{2.22}$$

Taking upper limit as $k \to \infty$ implies that

$$\psi(\varepsilon) \le \psi(\varepsilon) - \phi((a_1+a_4+a_5)\varepsilon),$$

which is a contradiction as $\varepsilon > 0$. Thus, we obtain that $\lim_{n,m\to\infty} p(x_n,x_m) = 0$, i.e., $\{x_n\}$ is a Cauchy sequence in (X, p), and thus in the metric space (X, p^S) by Lemma 1.4. Since (X, p) is complete, then from Lemma 1.4, (X, p^S) is also complete, so the sequence $\{x_n\}$ is convergent in the metric space (X, p^S). Therefore, there exists $u \in X$ such that $\lim_{n\to\infty} p^S(u,x_n) = 0$, equivalently,

$$\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,u) = p(u,u) = 0, \tag{2.23}$$

because $\lim_{n,m\to\infty} p(x_n,x_m) = 0$. If f is continuous selfmap on (X, p^S), then it is clear that $fu = u$. If f is not continuous, we have, by our hypothesis, that $x_n \preccurlyeq u$ for all $n \in \mathbb{N}$, because $\{x_n\}$ is a nondecreasing sequence with $\lim_{n\to\infty} p^S(u,x_n) = 0$. Now from the following inequality

$$\begin{aligned}&M(u,x_n)\\&=a_1p(u,x_n)+a_2p(fu,u)+a_3p(fx_n,x_n)\\&\quad+a_4p(fx_n,u)+a_5p(fu,x_n)\\&=a_1p(u,x_n)+a_2p(fu,u)+a_3p(x_{n+1},x_n)\\&\quad+a_4p(x_{n+1},u)+a_5p(fu,x_n),\end{aligned}$$

we deduce, taking limit as $n\to\infty$, that $\lim_{n\to\infty}M(u,x_n)=(a_2+a_5)p(fu,u)$. Hence,

$$\begin{aligned}\psi(p(fu,u))&=\limsup_{n\to\infty}\psi(p(fu,fx_{n+1}))\\&\le\limsup_{n\to\infty}[\psi(M(u,x_n))-\phi(M(u,x_n))]\\&=\psi((a_2+a_5)p(fu,u))-\phi((a_2+a_5)p(fu,u))\\&\le\psi(p(fu,u))-\phi((a_2+a_5)p(fu,u)).\end{aligned}$$

implies $p(fu,u)=0$ and $fu=u$. Finally, suppose that set of fixed points of f is well ordered. We prove that fixed point of f is unique. Assume on contrary that $fu=u$ and $fv=v$ but $u\neq v$. Hence

$$\psi(p(u,v))=\psi(p(fu,fv))\le\psi(M(u,v))-\phi(M(u,v)),\tag{2.24}$$

where

$$\begin{aligned}M(u,v)&=a_1p(u,v)+a_2p(fu,u)+a_3p(fv,v)+a_4p(fv,u)+a_5p(fu,v)\\&=a_1p(u,v)+a_2p(u,u)+a_3p(v,v)+a_4p(v,u)+a_5p(u,v)\\&=(a_1+a_4+a_5)p(u,v),\end{aligned}$$

that is, by (2.24),

$$\begin{aligned}\psi(p(u,v))&=\psi(p(fu,fv))\\&\le\psi(a_1+a_4+a_5)p(u,v))-\phi(a_1+a_4+a_5)p(u,v))\\&\le\psi(p(u,v))-\phi(a_1+a_4+a_5)p(u,v)),\end{aligned}$$

we arrive at a contradiction because $a_1p(u,v)>0$. Hence $u=v$. Conversely, if f has only one fixed point then the set of fixed point of f being singleton is well ordered. □

Following similar arguments to those given in Corollary 2.2, we obtain following corollary as an application of Theorem 2.6.

Corollary 2.7. Let $(X,\preccurlyeq)$ be a partially ordered set such that there exist a complete partial metric p on X and f a nondecreasing selfmap on X. Suppose that for every two elements $x,y\in X$ with $y\preccurlyeq x$, we have

$$\int_0^{\psi(p(fx,fy))}\varphi(t)dt\le\int_0^{\psi(M(x,y))}\varphi(t)dt-\int_0^{\phi(M(x,y))}\varphi(t)dt,\tag{2.25}$$

where $\phi\in\Upsilon$,

$$M(x,y)=a_1p(x,y)+a_2p(fx,x)+a_3p(fy,y)+a_4p(fy,x)+a_5p(fx,y),$$

a_1, $a_2>0$, $a_i\ge0$ for $i=3,4,5$, and, if $a_4\ge a_5$, then $a_1+a_2+a_3+a_4+a_5<1$, and if $a_4<a_5$, then $a_1+a_2+a_3+a_4+2a_5<1$ with $\psi,\varphi:\mathbb{R}^+\to\mathbb{R}^+$, ψ is continuous and nondecreasing, φ is a lower semicontinuous, and $\psi(t)=\varphi(t)=0$ if and only if $t=0$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq fx_0$ and one of the following two conditions is satisfied:

(a) f is continuous self map on (X, p^S);

(b) for any nondecreasing sequence $\{x_n\}$ in $(X, \preccurlyeq)$ with $\lim_{n\to\infty} p^S(z, x_n) = 0$ it follows $x_n \preccurlyeq z$ for all $n \in \mathbb{N}$,

then f has a fixed point.

If we take $\psi(t) = t$ in Theorem 2.6, we have following corollary.

Corollary 2.8. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exist a complete partial metric p on X and f a nondecreasing selfmap on X. Suppose that for $x, y \in X$ with $y \preccurlyeq x$,

$$p(fx, fy) \le M(x, y) - \phi(M(x, y)), \tag{2.26}$$

where

$$M(x, y) = a_1 p(x, y) + a_2 p(fx, x) + a_3 p(fy, y) + a_4 p(fx, y) + a_5 p(fy, x),$$

a_1, a_2 >0, $a_i \ge 0$ for $i = 3, 4, 5$, and, if $a_4 \ge a_5$, then $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, and if $a_4 < a_5$, then $a_1 + a_2 + a_3 + a_4 + 2a_5 < 1$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous with $\varphi(t) = 0$ if and only if $t = 0$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq fx_0$ and one of the following two conditions is satisfied:

(a) f is continuous self map on (X, p^S);

(b) for any nondecreasing sequence $\{x_n\}$ in $(X, \preccurlyeq)$ with $\lim_{n\to\infty} p^S(z, x_n) = 0$ it follows $x_n \preccurlyeq z$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Corollary 2.9. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exist a complete partial metric p on X and f a nondecreasing selfmap on X. Suppose that for $x, y \in X$ with $y \preccurlyeq x$, we have

$$p(fx, fy) \le M(x, y), \tag{2.27}$$

where

$$M(x, y) = a_1 p(x, y) + a_2 p(fx, x) + a_3 p(fy, y) + a_4 p(fx, y) + a_5 p(fy, x),$$

a_1, a_2 >0, $a_i \ge 0$ for $i = 3, 4, 5$, and, if $a_4 \ge a_5$, then $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, and if $a_4 < a_5$, then $a_1 + a_2 + a_3 + a_4 + 2a_5 < 1$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq fx_0$ and one of the following two conditions is satisfied:

(a) f is continuous self map on (X, p^S);

(b) for any nondecreasing sequence $\{x_n\}$ in $(X, \preccurlyeq)$ with $\lim_{n\to\infty} p^S(z, x_n) = 0$ it follows $x_n \preccurlyeq z$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Corollary 2.10. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exist a complete partial metric p on X and f a nondecreasing selfmap on X. Suppose that for $x, y \in X$ with $y \preccurlyeq x$, we have

$$p(fx,\ fy) \leq \alpha p(x,\ y) + \beta[p(fx,\ x) + p(fy,\ y)] + \gamma[p(fx,\ y) + p(fy,\ x)]) \tag{2.28}$$

where $\alpha, \beta, \gamma > 0$ and $\alpha + 2\beta + 2\gamma < 1$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq fx_0$ and one of the following two conditions is satisfied:

(a) f is continuous self map on (X, p^S);

(b) for any nondecreasing sequence $\{x_n\}$ in $(X, \preccurlyeq)$ with $\lim_{n\to\infty} p^S(z, x_n) = 0$ it follows $x_n \preccurlyeq z$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

We conclude the article with some examples which illustrate the obtained results.

Example 2.11. Let $X = [0, 1]$ be endowed with usual order and let p be the complete partial metric on X defined by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Let $f : X \to X$ be defined by $fx = 2x/3$. Define $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi(t) = 3t \text{ and } \phi(t) = \begin{cases} t/2, & \text{if } t \in [0, 1], \\ \dfrac{e^{1-t}}{2}, & \text{if } t > 1. \end{cases}$$

Clearly ψ is continuous and nondecreasing, φ is a lower semicontinuous, and $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. We show that condition (2.1) is satisfied. If $x, y \in [0, 1]$ with $y \preccurlyeq x$, then we have

$$\begin{aligned}\psi(p(fx, fy)) &= \psi\left(\max\left\{\frac{2x}{3}, \frac{2y}{3}\right\}\right) = \psi\left(\frac{2x}{3}\right) = 2x \leq \frac{5}{2}x \\ &= \frac{5}{2}\max\left\{\frac{2x}{3}, x\right\} = \frac{5}{2}p(fx, x) \leq \frac{5}{2}M(x, y) \\ &= 3M(x, y) - \frac{M(x, y)}{2} \\ &= \psi(M(x, y)) - \phi(M(x, y)).\end{aligned}$$

Thus f satisfies all the conditions of Theorem 2.1. Moreover, 0 is the unique fixed point of f. □

Example 2.12. Let $X = \mathbb{R}$ be endowed with usual order. Let $p : X \times X \to \mathbb{R}^+$ be defined by $p(x, y) = |x - y|$ if $x, y \in [0, 1)$, and $p(x, y) = \max\{x, y\}$ otherwise. It is easy to verify that p is complete partial metric on X. Now let $f : X \to X$ be given by

$$fx = \begin{cases} \frac{1}{2}, \text{ if } x < 1, \\ \frac{x}{2}, \text{ if } 1 \leq x < 2, \\ 1, \text{ otherwise.} \end{cases}$$

Clearly, f is continuous on (X, p^S). Define $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi(t) = 2t \text{ and } \phi(t) = t/6 \text{ for all } t \in \mathbb{R}^+.$$

Obviously, ψ is a continuous and nondecreasing, φ is a lower semicontinuous, and $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Now, we shall show that f satisfies condition (2.17) of Theorem 2.6, for $a_1 = a_2 = 3/10$ and $a_3 = a_4 = a_5 = 1/10$. We shall distinguish six cases with $y \preccurlyeq x$.

(1) If $x, y \in [0, 1)$, then (2.17) is satisfied as $p(fx, fy) = p(\frac{1}{2}, \frac{1}{2}) = 0$,

(2) if $y \in [0, 1)$, $x \in [1, 2)$, then $p(fx, fy) = \frac{1}{2}(x - 1)$ and $p(x, y) = x$, $p(fx, x) = x$, $p(fy, y) = \left|y - \frac{1}{2}\right|$, $p(fy, x) = x$, $p(fx, y) = \left|\frac{x}{2} - y\right|$. Therefore

$$\begin{aligned}\psi(p(fx, fy)) &= \psi\left(\frac{1}{2}(x-1)\right) = x - 1 \le \frac{11}{6}\left(\frac{3}{10}x + \frac{3}{10}x\right)\\ &\le \frac{11}{6}\left[\frac{3}{10}x + \frac{3}{10}x + \frac{1}{10}|y - 1/2| + \frac{1}{10}x + \frac{1}{10}|x/2 - y|\right]\\ &= \frac{11}{6}[a_1 p(x, y) + a_2 p(fx, x) + a_3 p(fy, y)\\ &\quad + a_4 p(fy, x) + a_5 p(fx, y)]\\ &= \frac{11}{6}M(x, y) = 2M(x, y) - \frac{1}{6}M(x, y)\\ &= \psi(M(x, y)) - \phi(M(x, y)).\end{aligned}$$

(3) For $y \in [0, 1)$ and $x \geq 2$, then $p(fx, fy) = 1$ and $p(x, y) = x$, $p(fx, x) = x$, $p(fy, y) = \left|y - \frac{1}{2}\right|$, $p(fy, x) = x$, $p(fx, y) = 1$. Therefore

$$\begin{aligned}\psi(p(fx, fy)) &= \psi(1) = 2 \le \frac{11}{6}\left(\frac{3}{10}x + \frac{3}{10}x\right)\\ &\le \frac{11}{6}\left[\frac{3}{10}x + \frac{3}{10}x + \frac{1}{10}|y - 1/2| + \frac{1}{10}x + \frac{1}{10}\right]\\ &= \frac{11}{6}[a_1 p(x, y) + a_2 p(fx, x) + a_3 p(fy, y)\\ &\quad + a_4 p(fy, x) + a_5 p(fx, y)]\\ &= \frac{11}{6}M(x, y) = 2M(x, y) - \frac{1}{6}M(x, y)\\ &= \psi(M(x, y)) - \phi(M(x, y)).\end{aligned}$$

(4) If $x, y \in [1, 2)$, then $p(fx, fy) = p(x/2, y/2) = \frac{1}{2}(x - y)$ and $p(x, y) = x$, $p(fx, x) = x$, $p(fy, y) = y$, $p(fy, x) = x$, $p(fx, y) = y$. Therefore

$$\begin{aligned}\psi(p(fx, fy)) &= \psi\left(\frac{1}{2}(x-y)\right) = (x - y) \le \frac{11}{6}\left(\frac{3}{10}x + \frac{3}{10}x\right)\\ &\le \frac{11}{6}\left[\frac{3}{10}x + \frac{3}{10}x + \frac{1}{10}y + \frac{1}{10}x + \frac{1}{10}y\right]\\ &= \frac{11}{6}[a_1 p(x, y) + a_2 p(fx, x) + a_3 p(fy, y)\\ &+ a_4 p(fy, x) + a_5 p(fx, y)]\\ &= \frac{11}{6}M(x, y) = 2M(x, y) - \frac{1}{6}M(x, y)\\ &= \psi(M(x, y)) - \phi(M(x, y)).\end{aligned}$$

(5) If $y \in [1, 2)$, $x \geq 2$, then $p(fx, fy) = p(1, y/2) = 1 - y/2$ and $p(x, y) = x$, $p(fx, x) = x$, $p(fy, y) = y$, $p(fy, x) = x$, $p(fx, y) = y$. Therefore

$$\begin{aligned}
\psi(p(fx, fy)) &= \psi(1 - y/2) = 2 - y \leq \frac{11}{6}\left(\frac{3}{10}x + \frac{3}{10}x\right) \\
&\leq \frac{11}{6}\left[\frac{3}{10}x + \frac{3}{10}x + \frac{1}{10}y + \frac{1}{10}x + \frac{1}{10}y\right] \\
&= \frac{11}{6}[a_1 p(x, y) + a_2 p(fx, x) + a_3 p(fy, y) \\
&\quad + a_4 p(fy, x) + a_5 p(fx, y)] \\
&= \frac{11}{6}M(x, y) = 2M(x, y) - \frac{1}{6}M(x, y) \\
&= \psi(M(x, y)) - \phi(M(x, y)).
\end{aligned}$$

(6) If $x, y \geq 2$, then $p(fx, fy) = p(1, 1) = 1$ and $p(x, y) = x$, $p(fx, x) = x$, $p(fy, y) = y$, $p(fy, x) = x$, $p(fx, y) = y$. Therefore

$$\begin{aligned}
\psi(p(fx, fy)) &= \psi(1) = 2 \leq \frac{11}{6}\left(\frac{3}{10}x + \frac{3}{10}x\right) \\
&\leq \frac{11}{6}\left[\frac{3}{10}x + \frac{3}{10}x + \frac{1}{10}y + \frac{1}{10}x + \frac{1}{10}y\right] \\
&= \frac{11}{6}[a_1 p(x, y) + a_2 p(fx, x) + a_3 p(fy, y) \\
&\quad + a_4 p(fy, x) + a_5 p(fx, y)] \\
&= \frac{11}{6}M(x, y) = 2M(x, y) - \frac{1}{6}M(x, y) \\
&= \psi(M(x, y)) - \phi(M(x, y)).
\end{aligned}$$

Thus all the axioms of Theorem 2.6 are satisfied. Moreover, 1/2 is the unique fixed point of f. □

3 A homotopy result

Let Φ be denote the set of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that φ is a nondecreasing, lower semicontinuous, and $\varphi(t) = 0$ if and only if $t = 0$.

Now, we state a following homotopy result.

Theorem 3.1. Let $(X, \preccurlyeq)$ be a partially ordered set such that there exist a complete partial metric p on X, U be an open subset of X and V be a closed subset of X with $U \subset V$. Let $H : V \times [0, 1] \to X$ be a given mapping such that $H(., \lambda)$: $V \to X$ is nondecreasing and continuous for each $\lambda \in [0, 1]$. If following conditions hold:

(a) $x \neq H(x, \lambda)$ for every $x \in V \backslash U$ and $\lambda \in [0, 1]$.

(b) For every $x, y \in V$, either $H(y, \lambda) \preccurlyeq H(x, \lambda)$ or $H(x, \lambda) \preccurlyeq H(y, \lambda)$ for each $\lambda \in [0, 1]$.

(c) For all comparable elements x, y in V, there exist $\varphi \in \Phi$ such that

$$p(H(x, \lambda), H(y, \lambda))) \leq p(x, y) - \phi(p(x, y)),$$

holds for each $\lambda \in [0, 1]$.

(d) There exists $L \geq 0$, such that

$$p(H(x, s), H(x, t)) \leq L|s - t|$$

for each $x \in V$ and $s, t \in [0, 1]$.

Then $H(., 0)$ has a fixed point in U if and only if $H(., 1)$ has a fixed point in U provided that there exist $x_0 \in X$ with $x_0 \preccurlyeq H(x_0, \lambda)$ for each $\lambda \in [0, 1]$.

Proof. Consider

$$A = \{\lambda \in [0,1] : x = H(x,\lambda) \text{for some } x \in U\}.$$

Now, if $H(., 0)$ has a fixed point in U, then $0 \in A$, so A is nonempty. Now we show that A is both open and closed in $[0, 1]$, which by connectedness of $[0, 1]$ will imply that $A = [0, 1]$. First, we show that A is closed in $[0, 1]$.

For this, let $\{\lambda_n\}$ be a sequence in A such that $\lambda_n \to \lambda$ for some $\lambda \in [0, 1]$ as $n \to \infty$. Since $\lambda_n \in A$ for $n = 1, 2, 3,...,$ there exists $x_n \in U$ such that $x_n = H(x_n, \lambda_n)$. Let $n, m \in \mathbb{N}$. For each $\lambda_n \in [0, 1]$, $x_m \in U$, using property (b) and by nondecreasness of $H(., \lambda)$ for each $\lambda \in [0, 1]$, we obtain that x_n and x_m are comparable. Now

$$\begin{aligned} p(x_n, x_m) &= p(H(x_n, \lambda_n), H(x_m, \lambda_m)) \\ &\leq p(H(x_n, \lambda_n), H(x_m, \lambda_n)) + p(H(x_m, \lambda_n), H(x_m, \lambda_m)) \\ &\quad - p(H(x_m, \lambda_n), H(x_m, \lambda_n)) \\ &\leq p(H(x_n, \lambda_n), H(x_m, \lambda_n)) + p(H(x_m, \lambda_n), H(x_m, \lambda_m)) \\ &\leq p(x_n, x_m) - \phi(p(x_n, x_m)) + L|\lambda_n - \lambda_m|, \end{aligned}$$

that is

$$\phi(p(x_n, x_m)) \leq L|\lambda_n - \lambda_m|.$$

Since $\lambda_n \to \lambda$ as $n \to \infty$, so that on taking the upper limit as $n \to \infty$ we obtain that $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, that is, $\{x_n\}$ is a Cauchy sequence in (X, p). Since (X, p) is complete, there exist x in V such that $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = 0$. As $x_n \in U$ and $x \in V$, using property (b) and the nondecreasness of $H(., \lambda)$ for each $\lambda \in [0, 1]$, x and x_n are comparable. So we have

$$\begin{aligned} p(H(x, \lambda), x_n) &= p(H(x, \lambda), H(x_n, \lambda_n)) \\ &\leq p(H(x, \lambda), H(x_n, \lambda)) + p(H(x_n, \lambda), H(x_n, \lambda_n)) \\ &\quad - p(H(x_n, \lambda), H(x_n, \lambda)) \\ &\leq p(x, x_n) - \phi(p(x_n, x)) + L|\lambda - \lambda_n| \end{aligned}$$

which on taking the upper limit as $n \to \infty$ implies

$$\limsup_{n\to\infty} p(x_n, H(x, \lambda)) \leq \limsup_{n\to\infty} p(x_n, x) - \liminf_{n\to\infty} \phi(p(x_n, x)) + \limsup_{n\to\infty} L|\lambda - \lambda_n|$$

Hence $\lim_{n\to\infty} p(x_n, H(x, \lambda)) = 0$ and

$$\lim_{n\to\infty} p(x_n, H(x, \lambda)) = p(x, H(x, \lambda)) = 0.$$

Thus $\lambda \in A$ and A is closed in $[0, 1]$. Next we show that A is an open in $[0, 1]$. Let $\lambda_0 \in A$. Then there exists $x_0 \in U$ such that $x_0 = H(x_0, \lambda_0)$. Since U is open, there exists $r > 0$ such that $B_p(x_0, r) \subseteq U$. Now, take $\delta = \inf\{p(x_0, x): x \in V \backslash U\}$. Then we have, $r = \delta - p(x_0, x_0) > 0$. Fix $\varepsilon > 0$ with $\varepsilon \leq \frac{\phi(\delta)}{L}$. Let $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ and

$x \in \overline{B_p(x_0, r)} = \{x \in X : p(x, x_0) \leq r + p(x_0, x_0)\}$. Since $H(., \lambda)$ is nondecreasing for each $\lambda \in [0, 1]$, property (b) implies that x and x_0 are comparable and

$$\begin{aligned} p(H(x,\lambda),x_0) &= p(H(x,\lambda),H(x_0,\lambda_0)) \\ &\leq p(H(x,\lambda),H(x_0,\lambda)) + p(H(x_0,\lambda),H(x_0,\lambda_0)) \\ &\quad - p(H(x_0,\lambda),H(x_0,\lambda)) \\ &\leq p(x,x_0) - \phi(p(x,x_0)) + L|\lambda-\lambda_0| \\ &\leq r + p(x_0,x_0) - \phi(p(x,x_0)) + L|\lambda-\lambda_0| \\ &\leq r + p(x_0,x_0) - \phi(\delta) + L\varepsilon \\ &\leq r + p(x_0,x_0). \end{aligned}$$

Thus for each fixed $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, $H(., \lambda) : \overline{B_p(x_0, r)} \to \overline{B_p(x_0, r)}$.

Taking $V = \overline{U}$ and applying Corollary 2.5, we obtain that $H(., \lambda)$ has a fixed point in $\overline{U}$. But this fixed point must be in U in the presence of assumption (a). Thus $\lambda \in A$ for any $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, therefore A in open in [0, 1]. Similarly, the reverse implication follows. □

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Authors' contributions
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Competing interests
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