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# Multi-valued ($\psi$, $\phi$, $\varepsilon$, $\lambda$)-contraction in probabilistic metric space

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**Abstract**
In this article, we present a new definition of a class of contraction for multi-valued case. Also we prove some fixed point theorems for multivalued ($\psi$, $\phi$, $\varepsilon$, $\lambda$)-contraction mappings in probabilistic metric space.

**Keywords:** probabilistic metric space, ($\psi$, $\varphi$, $\varepsilon$, $\lambda$)-contraction, fixed point

## 1 Introduction

The class of ($\varepsilon$, $\lambda$)-contraction as a subclass of $B$-contraction in probabilistic metric space was introduced by Mihet [1]. He and other researchers achieved to some interesting results about existence of fixed point in probabilistic and fuzzy metric spaces [2-4]. Mihet defined the class of ($\psi$, $\phi$, $\varepsilon$, $\lambda$)-contraction in fuzzy metric spaces [4]. On the other hand, Hadzic et al. extended the concept of contraction to the multi valued case [5]. They introduced multi valued ($\psi$ - $C$)-contraction [6] and obtained fixed point theorem for multi valued contraction [7]. Also Žikić generalized multi valued case of Hick's contraction [8]. We extended ($\phi$ - $k$) - $B$ contraction which introduced by Mihet [9] to multi valued case [10]. Now, we will define the class of ($\psi$, $\phi$, $\varepsilon$, $\lambda$)-contraction in the sense of multi valued and obtain fixed point theorem.

The structure of article is as follows: Section 2 recalls some notions and known results in probabilistic metric spaces and probabilistic contractions. In Section 3, we will prove three theorems for multi valued ($\psi$, $\phi$, $\varepsilon$, $\lambda$)- contraction.

## 2 Preliminaries

We recall some concepts from the books [11-13].

**Definition 2.1**. A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (a $t$-norm) if the following conditions are satisfied:

(1) $T(a, 1) = a$ for every $a \in [0, 1]$;
(2) $T(a, b) = T(b, a)$ for every $a, b \in [0, 1]$;
(3) $a \geq b, c \geq d \Rightarrow T(a, c) \geq T(b, d)$ $a, b, c, d \in [0, 1]$;
(4) $T(T(a, b), c) = T(a, T(b, c))$, $a, b, c \in [0, 1]$.

Basic examples are, $T_L(a, b) = \max\{a + b - 1, 0\}$, $T_P(a, b) = ab$ and $T_M(a, b) = \min\{a, b\}$.

**Definition 2.2**. If $T$ is a $t$-norm and $(x_1, x_2, \ldots, x_n) \in [0, 1]^n$ $(n \geq 1)$, $\top_{i=1}^{\infty} x_i$ is defined recurrently by $\top_{i=1}^{1} x_i = x_1$ and $\top_{i=1}^{n} x_i = T\left(\top_{i=1}^{n-1} x_i, x_n\right)$ for all $n \geq 2$. $T$ can be extended to a countable infinitary operation by defining $\top_{i=1}^{\infty} x_i$ for any sequence $(x_i)_{i \in N*}$ as $\lim_{n\to\infty} \top_{i=1}^{n} x_i$.

**Definition 2.3**. Let $\Delta_+$ be the class of all distribution of functions $F : [0, \infty] \to [0, 1]$ such that:

(1) $F(0) = 0$,

(2) $F$ is a non-decreasing,

(3) $F$ is left continuous mapping on $[0, \infty]$.

$D_+$ is the subset of $\Delta_+$ which $\lim_{x\to\infty} F(x) = 1$.

**Definition 2.4**. The ordered pair $(S, F)$ is said to be a probabilistic metric space if $S$ is a nonempty set and $F : S \times S \to D_+$ ($F(p, q)$ written by $F_{pq}$ for every $(p, q) \in S \times S$) satisfies the following conditions:

(1) $F_{uv}(x) = 1$ for every $x > 0 \Rightarrow u = v$ $(u, v \in S)$,

(2) $F_{uv} = F_{vu}$ for every $u, v \in S$,

(3) $F_{uv}(x) = 1$ and $F_{vw}(y) = 1 \Rightarrow F_{u,w}(x + y) = 1$ for every $u, v, w \in S$, and every $x, y \in R^+$.

A Menger space is a triple $(S, F, T)$ where $(S, F)$ is a probabilistic metric space, $T$ is a triangular norm (abbreviated $t$-norm) and the following inequality holds $F_{uv}(x + y) \geq T(F_{uw}(x), F_{wv}(y))$ for every $u, v, w \in S$, and every $x, y \in R^+$.

**Definition 2.5**. Let $\phi : (0, 1) \to (0, 1)$ be a mapping, we say that the $t$-norm $T$ is $\phi$-convergent if

$$\forall\delta \in (0, 1)\forall\lambda \in (0, 1)\exists s = s(\delta, \lambda) \in \mathrm{N}\ \top_{i=1}^{n}(1 - \varphi^{s+i}(\delta)) > 1 - \lambda, \forall n \geq 1.$$

**Definition 2.6**. A sequence $(x_n)_{n\in \mathrm{N}}$ is called a convergent sequence to $x \in S$ if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists $N = N(\varepsilon, \lambda) \in \mathrm{N}$ such that $F_{x_n x}(\varepsilon) > 1 - \lambda$, $\forall n \geq N$.

**Definition 2.7**. A sequence $(x_n)_{n\in \mathrm{N}}$ is called a Cauchy sequence if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists $N = N(\varepsilon, \lambda) \in N$ such that $F_{x_n x_{n+m}}(\varepsilon) > 1 - \lambda$, $\forall n \geq N\ \forall m \in \mathbb{N}$.

We also have

$$x_n \to^F x \Leftrightarrow F_{x_n x}(t) \to 1\ \forall t > 0.$$

A probabilistic metric space $(S, F, T)$ is called sequentially complete if every Cauchy sequence is convergent.

In the following, $2^S$ denotes the class of all nonempty subsets of the set $S$ and $C(S)$ is the class of all nonempty closed (in the $F$-topology) subsets of $S$.

**Definition 2.8** [14]. Let $F$ be a probabilistic distance on $S$ and $M \in 2^S$. A mapping f: $S \to 2^S$ is called continuous if for every $\varepsilon > 0$ there exists $\delta > 0$, such that

$$F_{uv}(\delta) > 1 - \delta \Rightarrow \forall x \in fu\, \exists y \in fv\ : F_{xy}(\varepsilon) > 1 - \varepsilon.$$

**Theorem 2.1** [14]. Let $(S, F, T)$ be a complete Menger space, $\sup_{0\leq t<1} T(t, t) = 1$ and $f : S \to C(S)$ be a continuous mapping. If there exist a sequence $(t_n)_{n\in \mathrm{N}} \subset (0, \infty)$ with $\sum_1^\infty t_n < \infty$ and a sequence $(x_n)_{n\in \mathrm{N}} \subset S$ with the properties:

$$x_{n+1} \in f x_n \text{ for all } n \text{ and } \lim_{n\to\infty} \top_{i=1}^{\infty} g_{n+i-1} = 1,$$

Where $g_n := F_{x_n x_{n+1}}(t_n)$, then $f$ has a fixed point.

The concept of ($\psi$, $\phi$, $\varepsilon$, $\lambda$) - $B$ contraction has been introduced by Mihet [15]. We will consider comparison functions from the class $\varphi$ of all mapping $\phi : (0, 1) \to (0, 1)$ with the properties:

(1) $\phi$ is an increasing bijection;

(2) $\phi(\lambda) < \lambda \ \forall \lambda \in (0, 1)$.

Since every such a comparison mapping is continuous, it is easy to see that if $\phi \in \varphi$, then $\lim_{n\to\infty}\phi^n(\lambda) = 0 \ \forall\lambda \in (0, 1)$.

**Definition 2.9**[15]. Let ($X$, $M$, *) be a fuzzy Metric space. $\psi$ be a map from $(0, \infty)$ to $(0, \infty)$ and $\phi$ be a map from (0, 1) to (0, 1). A mapping f: $X \to X$ is called ($\psi$, $\phi$, $\varepsilon$, $\lambda$)-contraction if for any $x, y \in X$, $\varepsilon > 0$ and $\lambda \in (0, 1)$.

$$M(x, y, \varepsilon) > 1 - \lambda \Rightarrow M(f(x), f(y), \psi(\varepsilon)) > 1 - \varphi(\lambda).$$

If $\psi$ is of the form of $\psi(\varepsilon) = k\varepsilon$ ($k \in (0, 1)$), one obtains the contractive mapping considered in [3].

## 3 Main results

In this section we will generalize the Definition 2.9 to multi valued case in probabilistic metric spaces.

**Definition 3.1**. Let $S$ be a nonempty set, $\phi \in \varphi$, $\psi$ be a map from $(0, \infty)$ to $(0, \infty)$ and $F$ be a probabilistic distance on $S$. A mapping $f : S \to 2^S$ is called a multi-valued ($\psi$, $\phi$, $\varepsilon$, $\lambda$)-contraction if for every $x, y \in S$, $\varepsilon > 0$ and for all $\lambda \in (0, 1)$ the following implication holds:

$$F_{xy}(\varepsilon) > 1 - \lambda \Rightarrow \forall p \in fx \, \exists q \in fy : \ F_{pq}(\psi(\varepsilon)) > 1 - \varphi(\lambda).$$

Now, we need to define some conditions on the $t$-norm $T$ or on the contraction mapping in order to be able to prove fixed point theorem. These two conditions are parallel. If one of them holds, Theorem 3.1 will obtain.

**Definition 3.2**[11]. Let ($S$, $F$) be a probabilistic metric space, $M$ a nonempty subset of $S$ and $f : M \to 2^S$ - $\{\varnothing\}$, a mapping $f$ is weakly demicompact if for every sequence $(p_n)_{n\in \mathrm{N}}$ from $M$ such that $p_{n+1} \in fp_n$, for every $n \in \mathrm{N}$ and $\lim F_{p_{n+1},p_n}(\varepsilon) = 1$, for every $\varepsilon > 0$, there exists a convergent subsequence $(p_{n_j})_{j\in\mathrm{N}}$.

The other condition is mentioned in the Theorem 3.1.

**Theorem 3.1**. Let ($S$, $F$, $T$) be a complete Menger space with $\sup_{0 \le a < 1} T(a, a) = 1$, $M \in C(S)$ and $f : M \to C(M)$ be a multi-valued ($\psi$, $\phi$, $\varepsilon$, $\lambda$)-contraction, where the series $\Sigma\psi^n(\varepsilon)$ is convergent for every $\varepsilon > 0$ and $\phi \in \varphi$. Let there exists $x_0 \in M$ and $x_1 \in fx_0$ such that $F_{x_0x_1} \in D_+$. If $f$ is weakly demicompact or

$$\lim_{n\to\infty} \top_{i=1}^{\infty}(1 - \varphi^{n+i-1}(\varepsilon)) = 1 \text{ for every } \varepsilon > 0 \tag{1}$$

then there exists at least one element $x \in M$ such that $x \in fx$.

**Proof**. Since there exists $x_0 \in M$ and $x_1 \in fx_0$ such that $F_{x_0x_1} \in D_+$, hence for every $\lambda \in (0, 1)$ there exists $\varepsilon > 0$ such that $F_{x_0x_1} > 1 - \lambda$. The mapping $f$ is a ($\psi$, $\phi$, $\varepsilon$, $\lambda$)-contraction and therefore there exists $x_2 \in fx_1$ such that

$$F_{x_2x_1}(\psi(\varepsilon)) > 1 - \varphi(\lambda)$$

Continuing in this way we obtain a sequence $(x_n)_{n\in N}$ from $M$ such that for every $n \geq 2$, $x_n \in fx_{n-1}$and

$$F_{x_n,x_{n-1}}(\psi^{n-1}(\varepsilon)) > 1 - \varphi^{n-1}(\lambda). \tag{2}$$

Since the series $\Sigma\psi^n(\varepsilon)$ is convergent we have $\lim_{n\to\infty}\psi^n(\varepsilon) = 0$ and by assumption $\phi \in \varphi$, so $\lim_{n\to\infty}\phi^n(\lambda) = 0$. We infer for every $\varepsilon_0 > 0$ that

$$\lim_{n\to\infty} F_{x_n x_{n-1}}(\varepsilon_0) = 1. \tag{3}$$

Indeed, if $\varepsilon_0 > 0$ and $\lambda_0 \in (0, 1)$ are given, and $n_0 = n_0(\varepsilon_0, \lambda_0)$ is enough large such that for every $n \geq n_0$, $\psi^n(\varepsilon) \leq \varepsilon_0$ and $\phi^n(\lambda) \leq \lambda_0$ then

$$F_{x_{n+1}x_n}(\varepsilon_0) \geq F_{x_{n+1}x_n}(\psi^n(\varepsilon)) > 1 - \varphi^n(\lambda) > 1 - \lambda_0 \text{ for every } n \geq n_0.$$

If $f$ is weakly demicompact (3) implies that there exists a convergent subsequence $(x_{n_k})_{k\in N}$.

Suppose that (1) holds and prove that $(x_n)_{n\in N}$ is a Cauchy sequence. This means that for every $\varepsilon_1 > 0$ and every $\lambda_1 \in (0, 1)$ there exists $n_1(\varepsilon_1, \lambda_1) \in N$ such that

$$F_{x_{n+p}x_n}(\varepsilon_1) > 1 - \lambda_1 \tag{4}$$

for every $n_1 \geq n_1(\varepsilon_1, \lambda_1)$ and every $p \in N$.

Let $n_2(\varepsilon_1) \in N$ such that $\sum_{n\geq n_2(\varepsilon_1)}\psi^n(\varepsilon) < \varepsilon_1$. Since $\sum_{n=1}^{\infty}\psi^n(\varepsilon)$ is convergent series such a natural number $n_2(\varepsilon_1)$ exists. Hence for every $p \in N$ and every $n \geq n_2(\varepsilon_1)$ we have that

$$F_{x_{n+p+1},x_n}(\varepsilon_1) \geq \top_{i=1}^{p+1} F_{x_{n+i},x_{n+i-1}}(\psi^{n+i-1}(\varepsilon)),$$

and (2) implies that

$$F_{x_{n+p+1},x_n}(\varepsilon_1) \geq \top_{i=1}^{p+1}(1 - \varphi^{n+i-1}(\lambda))$$

for every $n \geq n_2(\varepsilon_1)$ and every $p \in N$.

For every $p \in N$ and $n \geq n_2(\varepsilon_1)$

$$\top_{i=1}^{p+1}(1 - \varphi^{n+i-1}(\lambda)) \geq \top_{i=1}^{\infty}(1 - \varphi^{n+i-1}(\lambda))$$

and therefore for every $p \in N$ and $n \geq n_2(\varepsilon_1)$,

$$F_{x_{n+p+1},x_n}(\varepsilon_1) \geq \top_{i=1}^{\infty}(1 - \varphi^{n+i-1}(\lambda)). \tag{5}$$

From (1) it follows that there exists $n_3(\lambda_1) \in N$ such that

$$\top_{i=1}^{\infty}(1 - \varphi^{n+i-1}(\lambda)) > 1 - \lambda_1 \tag{6}$$

for every $n \geq n_3(\lambda_1)$. The conditions (5) and (6) imply that (4) holds for $n_1(\varepsilon_1, \lambda_1) = \max(n_2(\varepsilon_1), n_3(\lambda_1))$ and every $p \in N$. This means that $(x_n)_{n\in N}$ is a Cauchy sequence and since $S$ is complete there exists $\lim_{n\to\infty}x_n$. Hence in both cases there exists $(x_{n_k})_{k\in N}$ such that

$$\lim_{k\to\infty} x_{n_k} = x.$$

It remains to prove that $x \in fx$. Since $fx = \overline{fx}$ it is enough to prove that $x \in \overline{fx}$ i.e., for every $\varepsilon_2 > 0$ and $\lambda_2 \in (0, 1)$ there exists $b_{\varepsilon_2,\lambda_2} \in fx$ such that

$$F_{x,b_{\varepsilon_2,\lambda_2}}(\varepsilon_2) > 1 - \lambda_2. \quad (7)$$

Since $\sup_{x<1} T(x, x) = 1$ for $\lambda_2 \in (0, 1)$ there exists $\delta(\lambda_2) \in (0, 1)$ such that $T(1 - \delta(\lambda_2), 1 - \delta(\lambda_2)) > 1 - \lambda_2$.

If $\delta'(\lambda_2)$ is such that

$$T(1 - \delta'(\lambda_2), 1 - \delta'(\lambda_2)) > 1 - \delta(\lambda_2)$$

and $\delta''(\lambda_2) = \min(\delta(\lambda_2), \delta'(\lambda_2))$ we have that

$$\begin{aligned} T(1 - \delta''(\lambda_2), T((1 - \delta''(\lambda_2), 1 - \delta''(\lambda_2))) &\geq T(1 - \delta(\lambda_2), T((1 - \delta'(\lambda_2), 1 - \delta(\lambda_2))) \\ &\geq T(1 - \delta(\lambda_2), 1 - \delta(\lambda_2)) \\ &> 1 - \lambda_2. \end{aligned}$$

Since $\lim_{k\to\infty} x_{n_k} = x$ there exists $k_1 \in N$ such that $F_{x,x_{n_k}}\left(\frac{\varepsilon}{3}\right) > 1 - \delta''(\lambda_2)$ for every $k \geq k_1$. Let $k_2 \in N$ such that

$$F_{x_{n_k},x_{n_k+1}}\left(\frac{\varepsilon_2}{3}\right) > 1 - \delta''(\lambda_2) \text{ for every } k \geq k_2.$$

The existence of such a $k_2$ follows by (3). Let $\varepsilon \in R_+$ be such that $\psi(\varepsilon) < \frac{\varepsilon_2}{3}$ and $k_3 \in N$ such that $F_{x_{n_k},x}(\varepsilon) > 1 - \delta''(\lambda_2)$ for every $k \geq k_3$. Since $f$ is a $(\psi, \phi, \varepsilon, \lambda)$-contraction there exists $b_{\varepsilon_2,\lambda_2,k} \in fx$ such that

$$F_{x_{n_{k+1}},b_{\varepsilon_2,\lambda_2,k}}(\psi(\varepsilon)) > 1 - \varphi(\delta''(\lambda_2)) \text{ for every } k \geq k_3.$$

Therefore for every $k \geq k_3$

$$\begin{aligned} F_{x_{n_{k+1}},b_{\varepsilon_2,\lambda_2,k}}\left(\frac{\varepsilon_2}{2}\right) &\geq F_{x_{n_k+1},b_{\varepsilon_2,\lambda_2,k}}(\psi(\varepsilon)) \\ &> 1 - \varphi(\delta''(\lambda_2)) \\ &> 1 - \delta''(\lambda_2) \end{aligned}$$

If $k \geq \max(k_1, k_2, k_3)$ we have

$$\begin{aligned} F_{x,b_{\varepsilon_2,\lambda_2,k}}(\varepsilon_2) &\geq T\left(F_{x,x_{n_k}}\left(\frac{\varepsilon_2}{3}\right), T\left(F_{x_{n_k},x_{n_k+1}}\left(\frac{\varepsilon_2}{3}\right), F_{x_{n_k+1},b_{\varepsilon_2,\lambda_2,k}}\left(\frac{\varepsilon_2}{3}\right)\right)\right) \\ & T(1 - \delta''(\lambda_2), T(1 - \delta''(\lambda_2), 1 - \delta''(\lambda_2))) \\ &> 1 - \lambda_2 \end{aligned}$$

and (7) is proved for $b_{\varepsilon_2,\lambda_2} = b_{\varepsilon_2,\lambda_2,k}$, $k \geq \max(k_1, k_2, k_3)$. Hence $x \in \overline{fx} = fx$, which means $x$ is a fixed point of the mapping $f$.

Now, suppose that instead of $\Sigma\psi^n(\varepsilon)$ be convergent series, $\psi$ is increasing bijection.

**Theorem 3.2**. Let $(S, F, T)$ be a complete Menger space with $\sup_{0 \leq a < 1} T(a, a) = 1$ and $f: S \to C(S)$ be a multi-valued $(\psi, \phi, \varepsilon, \lambda)$- contraction.

If there exist $p \in S$ and $q \in fp$ such that $F_{pq} \in D_+$, $\psi$ is increasing bijection and $\lim_{n\to\infty} \top_{i=1}^{\infty}(1 - \varphi^{n+i-1}(\lambda)) = 1$, for every $\lambda \in (0, 1)$, then, $f$ has a fixed point.

**Proof**. Let $\varepsilon > 0$ be given and $\delta \in (0, 1)$ be such that $\delta < \min\{\varepsilon, \psi^{-1}(\varepsilon)\}$ or $\psi(\delta) < \varepsilon$ since $\psi$ is increasing bijection. If $F_{uv}(\delta) > 1-\delta$ then, due to $(\psi, \phi, \varepsilon, \lambda)$- contraction for each $x \in fu$ we can find $y \in fv$ such that $F_{xy}(\psi(\delta)) > 1 - \phi(\delta)$, from where we obtain that $F_{xy}(\varepsilon) > F_{xy}(\psi(\delta)) > 1 - \phi(\delta) > 1 - \delta > 1 - \varepsilon$. So $f$ is continuous. Next, let $p_0 = p$ and $p_1 = q$ be in $fp_0$. Since $F_{pq} \in D_+$, hence for every $\lambda \in (0, 1)$ there exist $\varepsilon > 0$ such that $F_{pq}(\varepsilon) > 1 - \lambda$, namely $F_{p_0p_1}(\varepsilon) > 1 - \lambda$.

Using the contraction relation we can find $p_2 \in fp_1$ such that $F_{p_1p_2}(\psi(\varepsilon)) > 1 - \varphi(\lambda)$, and by induction, $p_n$ such that $p_n \in fp_{n-1}$and $F_{p_{n-1}p_n}(\psi^{n-1}(\varepsilon)) > 1 - \varphi^{n-1}(\lambda)$ for all $n \geq 1$. Defining $t_n = \psi^n(\varepsilon)$, we have $g_j = F_{p_jp_{j+1}}(t_j) \geq 1 - \varphi^j(\lambda), \forall j$, so $\lim_{n\to\infty} \top_{i=1}^{\infty} g_{n+i-1} \geq \lim_{n\to\infty} \top_{i=1}^{\infty}(1 - \varphi^{n+i-1}(\lambda)) = 1$.

On the other hand the sequence $(p_n)$ is a Cauchy sequense, that is:

$$\forall \varepsilon > 0\, \exists n_0 = n_0(\varepsilon) \in N : F_{p_np_{n+m}}(\varepsilon) > 1 - \epsilon, \forall n \geq n_0, \forall m \in N.$$

Suppose that $\varepsilon > 0$, then:

$$\lim_{n\to\infty} \top_{i=1}^{\infty} g_{n+i+1} = 1 \Rightarrow \exists n_1 = n_1(\varepsilon) \in N \ : \ \top_{i=1}^{m} g_{n+i-1} > 1 - \varepsilon, \quad \forall n \geq n_1, \quad \forall m \in N.$$

Since the series $\sum_{n=1}^{\infty} t_n$ is convergent, there exists $n_2 (= n_2(\varepsilon))$ such that $\sum_{n=n_2}^{\infty} t_n < \varepsilon$.

Let $n_0 = \max\{n_1, n_2\}$, then for all $n \geq n_0$ and $m \in N$ we have:

$$\begin{aligned} F_{p_np_{n+m}}(\varepsilon) &\geq F_{p_np_{n+m}}\left(\sum_{i=n}^{n+m-1} t_i\right) \geq \top_{i=1}^{m} F_{p_{n+i-1}p_{n+1}}(t_{n+i-1}) \\ &= \top_{i=1}^{m} g_{n+i-1} > 1 - \varepsilon, \end{aligned}$$

as desired.

Now we can apply Theorem 2.1 to find a fixed point of $f$. The theorem is proved. □

When $\psi$ is increasing bijection and $\lim_{n\to\infty}\psi^n(\lambda)$ be zero, by using demicompact contraction we have another theorem.

**Theorem 3.3**. Let $(S, F, T)$ be a complete Menger space, $T$ a $t$-norm such that $\sup_{0 \leq a < 1} T(a, a) = 1$, $M$ a non-empty and closed subset of $S$, $f : M \to C(M)$ be a multi-valued $(\psi, \phi, \varepsilon, \lambda)$- contraction and also weakly demicompact. If there exist $x_0 \in M$ and $x_1 \in fx_0$ such that $F_{x_0x_1} \in D_+$, $\psi$ is increasing bijection and $\lim_{n\to\infty}\psi(\lambda) = 0$ then, $f$ has a fixed point.

**Proof**. We can construct a sequence $(p_n)_{n\in N}$ from $M$, such that $p_1 = x_1 \in fx_0$, $p_{n+1} \in fp_n$. Given $t > 0$ and $\lambda \in (0, 1)$, we will show that

$$\lim_{n\to\infty} F_{p_{n+1}p_n}(t) = 1. \tag{11}$$

Indeed, since $F_{x_0x_1} \in D_+$, hence for every $\xi > 0$ there exist $\eta > 0$ such that $F_{x_0x_1}(\eta) > 1 - \xi$, and by induction $F_{p_{n-1}p_n}(\psi^n(\eta)) > 1 - \varphi^n(\xi)$ for all $n \in$ N. By choosing $n$ such that $\psi^n(\eta) < t$ and $\phi^n(\xi) < \lambda$, we obtain

$$F_{p_{n+1}p_n}(t) > 1 - \lambda.$$

Since $t$ and $\lambda$ are arbitrary, the proof of (1) is complete.

By Definition 3.2, there exists a subsequence $(p_{n_j})_{j \in \mathrm{N}}$ such that $\lim_{j \to \infty} p_{n_j}$ exists. We shall prove that $x = \lim_{j \to \infty} p_{n_j}$ is a fixed point of $f$. Since $fx$ is closed, $fx = \overline{fx}$, and therefore, it remains to prove that $x = \overline{fx}$, i.e., for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exist $b(\varepsilon, \lambda) \in fx$, such that $F_{x,b(\varepsilon,\lambda)}(\varepsilon) > 1 - \lambda$. From the condition $\sup_{0 \leq a < 1} T(a, a) = 1$ it follows that there exists $\eta(\lambda) \in (0, 1)$ such that

$$u > 1 - \eta(\lambda) \Rightarrow T(u, u) > 1 - \lambda.$$

Let $j_1(\varepsilon, \lambda) \in \mathrm{N}$ be such that

$$F_{p_{n_j},x}\left(\psi^{-1}\left(\frac{\varepsilon}{2}\right)\right) > 1 - \frac{\eta(\lambda)}{2} \text{for every } j \geq j_1(\varepsilon, \lambda).$$

Since $x = \lim_{j \to \infty} p_{n_j}$, such a number $j_1(\varepsilon, \lambda)$ exists. Since $f$ is $(\psi, \phi, \varepsilon, \lambda)$-contraction and $\psi$ is increasing bijection, for $p_{n_j+1} \in f p_{n_j}$ there exists $b_j(\varepsilon) \in fx$ such that

$$F_{p_{n_j+1},b_{j(\varepsilon)}}\left(\frac{\varepsilon}{2}\right) > 1 - \varphi\left(\frac{\eta(\lambda)}{2}\right) > 1 - \frac{\eta(\lambda)}{2} \text{ for every } j \geq j_1(\varepsilon, \lambda).$$

From (1) it follows that $\lim_{j \to \infty} p_{n_j+1} = x$ and therefore, there exists $j_2(\varepsilon, \lambda) \in \mathrm{N}$ such that $F_{x,p_{n_j+1}}\left(\frac{\varepsilon}{2}\right) > 1 - \frac{\eta(\lambda)}{2}$ for every $j \geq j_2(\varepsilon, \lambda)$. Let $j_3(\varepsilon, \lambda) = \max\{j_1(\varepsilon, \lambda), j_2(\varepsilon, \lambda)\}$. Then, for every $j \geq j_3(\varepsilon, \lambda)$ we have $F_{x,b_j(\varepsilon)}(\varepsilon) \geq T\left(F_{x,p_{n_j+1}}\left(\frac{\varepsilon}{2}\right), F_{p_{n_j+1},b_{j(\varepsilon)}}\left(\frac{\varepsilon}{2}\right)\right) > 1 - \lambda$. Hence, if $j > j_3(\varepsilon, \lambda)$, then, we can choose $b(\varepsilon, \lambda) = b_j(\varepsilon) \in fx$. The proof is complete. □

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**Authors' contributions**
PA defined the definitions and wrote the introduction, preliminaries and abstract. AB proved the theorems. AB has approved the final manuscript. Also PA has verified the final manuscript

**Competing interests**
The authors declare that they have no competing interests.

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