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Generalized uniform spaces, uniformly locally contractive set-valued dynamic systems and fixed points

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Abstract

Motivated by classical Banach contraction principle, Nadler investigated set-valued contractions with respect to Hausdorff distances h in complete metric spaces, Covitz and Nadler (Jr.) investigated set-valued maps which are uniformly locally contractive or contractive with respect to generalized Hausdorff distances H in complete generalized metric spaces and Suzuki investigated set-valued maps which are contractive with respect to distances Q_p in complete metric spaces with τ-distances p. Here, we provide more general results which, in particular, include the mentioned ones above. The concepts of generalized uniform spaces, generalized pseudodistances in these spaces and new distances induced by these generalized pseudodistances are introduced and a new type of sequential completeness which extended the usual sequential completeness is defined. Also, the new two kinds of set-valued dynamic systems which are uniformly locally contractive or contractive with respect to these new distances are studied and conditions guaranteeing the convergence of dynamic processes and the existence of fixed points of these uniformly locally contractive or contractive set-valued dynamic systems are established. In addition, the concept of the generalized locally convex space as a special case of the generalized uniform space is introduced. Examples illustrating ideas, methods, definitions, and results are constructed, and fundamental differences between our results and the well-known ones are given. The results are new in generalized uniform spaces, uniform spaces, generalized locally convex and locally convex spaces and they are new even in generalized metric spaces and in metric spaces.

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Introduction

Let 2^X denotes the family of all nonempty subsets of a space X. Recall that a *set-valued dynamic system* is defined as a pair (X, T), where X is a certain space and T is a set-valued map $T : X \to 2^X$; in particular, a set-valued dynamic system includes the usual dynamic system where T is a single-valued map.

Let (X, T) be a set-valued dynamic system. By $Fix(T)$ and $End(T)$ we denote the sets of all *fixed points* and *endpoints* (or *stationary points*) of T, respectively i.e., $Fix(T) = \{w \in X : w \in T(w)\}$ and $End(T) = \{w \in X : \{w\} = T(w)\}$.

A *dynamic process* or a *trajectory starting at* $w^0 \in X$ or a *motion* of the system (X, T) at w^0 is a sequence $(w^m : m \in \{0\} \cup \mathbb{N})$ defined by $w^m \in T(w^{m-1})$ for $m \in \mathbb{N}$ (see, [1,2]).

If (X, T) is a dynamic system and $w^0 \in X$ then, by $\mathcal{O}(X, T, w^0)$, we denote the set of all dynamic processes of the system (X, T) starting at w^0.

A beautiful Banach's contraction principle [3] has inspired a large body of work over the last 50 years and there are several ways in which one might hope to improve this principle.

Theorem 1 [3]*Let (X, d) be a complete metric space. Let $T : X \to X$ be a single-valued map satisfying the condition*

$$\exists_{\lambda\in[0,1)}\forall_{x,y\in X}\{d(T(x),T(y)) \le \lambda d(x,y)\}. \tag{1}$$

Then: (i) T has a unique fixed point w in X, i.e. $Fix(T) = \{w\}$; and (ii) the sequence $\{T^{[m]}(u)\}$ converges to w for each $u \in X$.

Let (X, d) be a metric space and let $CB(X)$ denote the class of all nonempty closed and bounded subsets of X. If $h : CB(X) \times CB(X) \to [0, \infty)$ represents a Hausdorff metric induced by d, it has the form

$$h(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}, A, B \in CB(X),$$

where $d(x, C) = \inf_{c\in C} d(x, c)$, $x \in X$, $C \in CB(X)$.

A natural question to ask is whether the single-valued dynamic system in this principle can be replaced by the set-valued dynamic system. One of the first results in this direction was established in [4].

Theorem 2 [[4], *Th.* 5] *Let (X, d) be a complete metric space. Assume that the set-valued dynamic system (X, T) satisfying $T : X \to CB(X)$ is (h, λ)-contractive, i.e.,*

$$\exists_{\lambda\in[0,1)}\forall_{x,y\in X}\{h(T(x),T(y)) \le \lambda d(x,y)\}. \tag{2}$$

Then T has a fixed point w in X, i.e. $w \in T(w)$.

There are other important ways of extending the Banach theorem. In particular, many interesting theorems in this setting, proposed by Covitz and Nadler, Jr. [[5], Theorem 1], concern the set-valued dynamic systems in generalized metric spaces.

The concepts of generalized metric spaces and the canonical decompositions of these spaces appeared first in Luxemburg [6] and Jung [7]. Recall that a generalized metric space is a pair (X, d) where X is a nonempty set and $d : X^2 \to [0, \infty]$ satisfies: (a) $\forall_{x,y\in X}\{d(x, y) = 0 \text{ iff } x = y\}$; (b) $\forall_{x,y\in X}\{d(x, y) = d(y, x)\}$; (c) $\forall_{x,y,z\in X}\{[d(x, z) < +\infty \wedge d(y, z) < +\infty] \Rightarrow [d(x, y) < +\infty \wedge d(x, y) \le d(x, z) + d(z, y)]\}$. Some characterizations of these spaces were presented by Jung [7] who proved the essential theorems about decomposition of a generalized metric spaces and discovered the way to obtain generalized (complete) metric spaces. Let $\{(X_\beta, d_\beta) : \beta \in \mathcal{B}\}$, $\mathcal{B}$-index set, be a family of disjoint metric spaces. If $X = \bigcup_{\beta\in\mathcal{B}} X_\beta$ and, for any $x, y \in X$,

$$d(x,y) = \begin{cases} d_\beta(x,y) \text{ if } x,y \in X_\beta, \beta \in \mathcal{B} \\ +\infty \quad \text{ if } x \in X_{\beta_1}, y \in X_{\beta_2}, \beta_1, \beta_2 \in \mathcal{B}, \beta_1 \neq \beta_2 \end{cases}$$

then (X, d) is a generalized metric space. Moreover, if for each $\beta \in \mathcal{B}$, (X_β, d_β) is complete then (X, d) is a generalized complete metric space. Also, in generalized metric spaces (X, d) he introduced the following equivalence relation on X:

$$x \sim y \text{ iff } d(x,y) < +\infty, x, y \in X.$$

Therefore, X is decomposed uniquely into (disjoint) equivalence classes $\{X_\beta : \beta \in \mathcal{B}\}$, which is called a *canonical decomposition*. We may read these results as follows.

Theorem 3 [7]*Let* (X, d) *be a generalized metric space, let* $X = \bigcup_{\beta\in\mathcal{B}} X_\beta$ *be the canonical decomposition and let* $\forall_{\beta\in\mathcal{B}} \{d_\beta = d|_{X_\beta \times X_\beta}\}$. *Then:* (**I**) *For each* $\beta \in \mathcal{B}$, (X_β, d_β) *is a metric space;* (**II**) *For any* $\beta_1, \beta_2 \in \mathcal{B}$, *with* $\beta_1 \neq \beta_2$, $d(x, y) = +\infty$ *for any* $x \in X_{\beta_1}$ *and* $y \in X_{\beta_2}$; *and* (**III**) (X, d) *is a generalized complete metric space iff, for each* $\beta \in \mathcal{B}$, (X_β, d_β) *is a complete metric space.*

Before presenting the results of Covitz and Nadler, Jr. [5] we recall some notations.

Definition 1 Let (X, d) be a generalized metric space.

(**a**) We say that a nonempty subset Y of X is *closed* in X if $Y = Cl(Y)$ where $Cl(Y)$, the *closure* of Y in X, denote the set of all $x \in X$ for which there exists a sequence $(x_m : m \in \mathbb{N})$ in Y which is d-convergent to x.

(**b**) The class of all nonempty closed subsets of X is denoted by $C(X)$, i.e. $C(X) = \{Y : Y \in 2^X \wedge Y = Cl(Y)\}$.

(**c**) A generalized Hausdorff distance $H : C(X) \times C(X) \to [0, \infty]$ induced by d is defined by: for each $A, B \in C(X)$,

$$H(A,B) = \begin{cases} \inf\{\varepsilon > 0 : A \subset N(\varepsilon, B) \wedge B \subset N(\varepsilon, A)\} & \text{if is finite} \\ +\infty & \text{otherwise} \end{cases}$$

where, for each $E \in C(X)$ and $\varepsilon > 0$, $N(\varepsilon, E) = \{x \in X : \exists_{e\in E} \{d(x, e) < \varepsilon\}\}$.

Theorem 4 [[5], *Theorem 1*] *Let* (X, d) *be a generalized complete metric space and let* $w^0 \in X$. *Assume that a set-valued dynamic system* (X, T) *satisfying* $T : X \to C(X)$ *is* $(H, \varepsilon, \lambda)$*-uniformly locally contractive, i.e.*

$$\exists_{\varepsilon\in(0,\infty]} \exists_{\lambda\in[0,1)} \forall_{x,y\in X} \{d(x,y) < \varepsilon \Rightarrow H(T(x), T(y)) \leq \lambda d(x,y)\}.$$

Then the following alternative holds: either

(**A**) $\forall_{(w^m : m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)} \forall_{m\in\mathbb{N}} \{d(w^{m-1}, w^m) \geq \varepsilon\}$; *or*

(**B**) $\exists_{(w^m : m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)} \exists_{w\in X} \{w \in Fix(T) \wedge \lim_{m\to\infty} w^m = w\}$.

It is not hard to see that each (H, λ)-contractive set-valued dynamic system defined below is, for each $\varepsilon \in (0, +\infty)$, $(H, \varepsilon, \lambda)$-uniformly locally contractive.

Theorem 5 [[5], *Corollary 1*] *Let* (X, d) *be a generalized complete metric space and let* $w^0 \in X$. *Assume that the set-valued dynamic system* (X, T) *satisfying* $T : X \to C(X)$ *is* (H, λ)*-contractive, i.e.,*

$$\exists_{\lambda\in[0,1)} \forall_{x,y\in X} \{H(T(x), T(y)) \leq \lambda d(x,y)\} \text{ whenever } d(x,y) < \infty. \tag{3}$$

Then the following alternative holds: either

(**A**) $\forall_{(w^m : m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)} \forall_{m\in\mathbb{N}} \{d(w^{m-1}, w^m) = \infty\}$; *or*

(**B**) $\exists_{(w^m: m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)}\exists_{w\in X}\left\{w\in Fix(T)\wedge\lim_{m\to\infty}w^m=w\right\}$.

The following follows from Theorem 5 and generalize Nadler's Theorem 2.

Theorem 6 [[5], *Corollary 3*] *Let* (X, d) *be a complete metric space and let* $w^0 \in X$. *Assume that a set-valued dynamic system* (X, T) *satisfying* $T : X \to C(X)$ *is* (h, λ)*-contractive, i.e.*

$$\exists_{\lambda\in[0,1)}\forall_{x,y\in X}\{h(T(x),T(y))\le\lambda d(x,y)\}. \tag{4}$$

Then $\exists_{(w^m: m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)}\exists_{w\in X}\left\{w\in Fix(T)\wedge\lim_{m\to\infty}w^m=w\right\}$.

Recall that the investigations of fixed points of maps in complete generalized metric spaces appeared for the first time in Diaz and Margolis [8] and Margolis [9].

Another natural problem is to extend the Nadler's [[4], Th. 5] theorem to set-valued dynamic systems which are contractive with respect to more general distances. In complete metric spaces, this line of research was pioneered by Suzuki [10], who developed many crucial technical tools.

Definition 2 [11] Let (X, d) be a metric space. A map $p : X \times X \to [0, \infty)$ is called a *τ-distance* on X if there exists a map $\eta : X \times [0, \infty) \to [0, \infty)$ and the following conditions hold: (**S1**) $\forall_{x,y,z\in X}\{p(x, z) \le p(x, y) + p(y, z)\}$; (**S2**) $\forall_{x\in X}\forall_{t>0}\{\eta(x, 0) = 0 \wedge \eta(x, t) \ge t\}$ and η is concave and continuous in its second variable; (**S3**) $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty}\sup_{m\ge n}\eta(z_n, p(z_n, x_m)) = 0$ imply that $\forall_{w\in X}\{p(w, x) \le \liminf_{n\to\infty} p(w, x_n)\}$; (**S4**) $\lim_{n\to\infty}\sup_{m\ge n} p(x_n, y_m)) = 0$ and $\lim_{n\to\infty}\eta(x_n, t_n) = 0$ imply that $\lim_{n\to\infty}\eta(y_n, t_n) = 0$; and (**S5**) $\lim_{n\to\infty}\eta(z_n, p(z_n, x_n)) = 0$ and $\lim_{n\to\infty}\eta(z_n, p(z_n, y_n)) = 0$ imply that $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Theorem 7 [[10], *Theorem 3.7*] *Let* (X, d) *be a complete metric space and let p be a τ-distance on X. Let a set-valued dynamic system* (X, T) *satisfying* $T : X \to C(X)$ *be* (Q_p, λ)*-contractive, i.e.*

$$\exists_{\lambda\in[0,1)}\forall_{x,y\in X}\{Q_p(T(x),T(y))\le\lambda p(x,y)\} \tag{5}$$

where $Q_p(A, B) = \sup_{a\in A}\inf_{b\in B} p(a, b)$. *Then there exists* $w \in X$ *such that* $w \in T(w)$ *and* $p(w, w) = 0$.

Remark 1 Let us observe that this beautiful Suzuki's theorem include Covitz-Nadler's Theorem 6. Indeed, first we see that each metric d is τ-distance (cf. [11]) and next we see that each (h, λ)-contractive set-valued dynamic system (X, T) satisfying $T : X \to C(X)$ is (Q_d, λ)-contractive; in fact, $Q_d \le h$ on $C(X)$ (cf. [12]). Moreover, there exist (Q_d, λ)-contractive set valued dynamic systems (X, T) satisfying $T : X \to C(X)$ which are not (h, λ)-contractive.

It is worth noticing that a number of authors introduce the new various concepts of set-valued contractions of Nadler type in complete metric spaces, study the problem concerning the existence of fixed points for such contractions and obtain the various generalizations of Nadler's result which are different from the mentioned above; see, e.g., Takahashi [13], Jachymski [[14], Theorem 5], Feng and Liu [12], Zhong et al. [15], Mizoguchi and Takahashi [16], Eldred et al. [17], Suzuki [18], Kaneko [19], Reich [20,21], Quantina and Kamran [22], Suzuki and Takahashi [23], Al-Homidan et al. [24], Latif and Al-Mezel [25], Frigon [26], Klim and Wardowski [27], Ćirić [28] and Pathak and Shahzad [29].

The above are some of the reasons why in nonlinear analysis the study of uniformly locally contractive and contractive set-valued dynamic systems play a particularly important part in the fixed point theory and its applications.

Let us notice that in the proofs of the results of [3-29], among other things, the following assumptions and observations are essential: (**O1**) *The completeness of metric and generalized metric spaces is necessary*; (**O2**) *In Theorems 1, 2 and 4-7, the maps* $T : (X, d) \to (X, d)$, $T : (X, d) \to (CB(X), h)$, $T : (X, d) \to (C(X), H)$ *and* $T : (X, p) \to (C(X), Q_p)$ *are investigated and the conditions (1)-(5) imply that these maps between spaces* (X, d), (X, p), $(CB(X), h)$, $(C(X), H)$ *and* $(C(X), Q_p)$, *respectively, are continuous*; (**O3**) *By Theorems 1, 2 and 4-7, for each* $w \in Fix(T)$ *the following equalities* $d(w, w) = 0$, $h(T(w), T(w)) = 0$, $H(T(w), T(w)) = 0$, $Q_p(T(w), T(w)) = 0$ *and* $p(w, w) = 0$ *hold, respectively*; (**O4**) *The distances* h, H, *and* Q_p *are defined only on the spaces* $CB(X)$ *or* $C(X)$, *respectively.*

Also, let us observe that in [30-36] we studied some families of generalized pseudodistances in uniform spaces and generalized quasipseudodistances in quasigauge spaces which generalize: *metrics, distances* of Tataru [37], *w-distances* of Kada et al. [38], *τ-distances* of Suzuki [11] and *τ-functions* of Lin and Du [39] in metric spaces and *distances* of Vályi [40] in uniform spaces.

Motivated by the comments and observations stated above our main interest of this article is the following:

Question 1 Are there spaces X, new distances on X which are more general than d, h, H, p and Q_p, and set-valued dynamic systems (X, T) which are uniformly locally contractive or contractive with respect to new distances, such that the analogous assertions as in Theorems 1, 2 and 4-7 hold but, unfortunately: (**M1**) *Spaces* X (*metric, generalized metric and more general) are not necessarily complete*; (**M2**) *If new distances we replaced by* d, h, H, p *or* Q_p *then maps* T *are not necessarily continuous in the sense defined by inequalities (1)-(5), respectively*; (**M3**) *For* T, $w \in Fix(T)$ *and for new distances the properties in (O3) do not necessarily hold in such generality*; (**M4**) *The new distances are defined on* 2^X, *and thus not only on* $CB(X)$ *or* $C(X)$ *as in (O4)*?

Our purpose in this article is to answer our question in the affirmative and providing the illustrating examples. More precisely, inspired by ideas of Diaz and Margolis [8], Margolis [9], Luxemburg [6], Jung [7], Nadler [[4], Th. 5], Covitz and Nadler [5] and Suzuki [10] and the above comments and observations, the concepts of the families $\mathcal{D} = \{d_\alpha : X \times X \to [0, \infty], \ \alpha \in \mathcal{A}\}$ ($\mathcal{A}$-index set) of generalized pseudometrics on a nonempty set X and the generalized uniform spaces $(X, \mathcal{D})$ are introduced, the classes $\mathbb{L}_{(X,\mathcal{D})}$ of $\mathcal{L}$-families of generalized pseudodistances in $(X, \mathcal{D})$ are defined and, in $(X, \mathcal{D})$, a new type of $\mathcal{L}$-sequentially completeness with respect to $\mathcal{L}$-families (which extend the usual sequentially completeness in uniform and locally convex spaces and completeness in metric and generalized metric spaces) are studied (see the following section). Moreover, some partial quasiordered space $K^{\mathcal{A}}$ is defined (see Section "Partial quasiordered space $K^{\mathcal{A}}$") and, using $K^{\mathcal{A}}$, $\mathbb{H}^{\mathcal{L}}_{(i)}$-distances on 2^X ($i \in \{1, 2\}$) with respect to $\mathcal{L}$-families are introduced (see Section "$\mathbb{H}^{\mathcal{L}}_{(i)}$-distances on 2^X, $i \in \{1,2\}$"). Also, we introduce the definitions of $\left(\mathbb{H}^{\mathcal{L}}_{(i)}, \Upsilon, \Lambda\right)$ -uniformly locally contractive and $\left(\mathbb{H}^{\mathcal{L}}_{(i)}, \Lambda\right)$-contractive set-valued dynamic systems (X, T) ($i \in \{1, 2\}$) satisfying $T : X \to 2^X$ (see

Section "$\left(\mathbb{H}^{\mathcal{L}}_{(i)}, \Upsilon, \Lambda\right)$-uniformly locally contractive and $\left(\mathbb{H}^{\mathcal{L}}_{(i)}, \Lambda\right)$-contractive set-valued dynamic systems (X, T), $i \in \{1, 2\}$") and, for $w^0 \in X$, we establish the conditions guaranteeing the convergence of dynamic processes $\mathcal{O}\left(X, T, w^0\right)$ and the existence of fixed points for such contractions and, additionally, a special case when $T : X \to C(X)$ and $\mathcal{L} = \mathcal{D}$ is studied (see Sections 6-8). Also the concept of the generalized locally convex space as a special case of the generalized uniform space is introduced (see Section "Generalized locally convex spaces $(X, \mathcal{P})$"). By generality of spaces and $\mathcal{L}$-families, our results, in particular, include and essentially generalize Theorems 1, 2 and 4-7. The examples illustrating ideas, methods and results are constructed and comparisons of our results with the results of Nadler [[4], Th. 5], Covitz and Nadler [5] and Suzuki [10] are given (see Sections 10-13). Finally, a natural question is formulated (see Section "Concluding remarks"). The results are new in generalized uniform spaces, uniform spaces, generalized locally convex and locally convex spaces and are new even in generalized metric spaces and in metric spaces.

Generalized uniform spaces $(X, \mathcal{D})$ and the class $\mathbb{L}_{(X,\mathcal{D})}$ of $\mathcal{L}$-families of generalized pseudodistances on $(X, \mathcal{D})$

The following terminologies will be much used.

Definition 3 Let X be a nonempty set. (**a**) The family

$$\mathcal{D} = \{d_\alpha : X \times X \to [0, \infty], \alpha \in \mathcal{A}\}, \mathcal{A} \text{ - index set,}$$

is said to be a *$\mathcal{D}$-family of generalized pseudometrics on X* (*$\mathcal{D}$-family on X*, for short) if the following three conditions hold:

($\mathcal{D}$1) $\forall_{\alpha \in \mathcal{A}} \forall_{x \in X} \left\{d_\alpha(x, x) = 0\right\}$;

($\mathcal{D}$2) $\forall_{\alpha \in \mathcal{A}} \forall_{x,y \in X} \left\{d_\alpha(x, y) = d_\alpha(y, x)\right\}$; and

($\mathcal{D}$3) If $\alpha \in \mathcal{A}$ and $x, y, z \in X$ and if $d_\alpha(x, z)$ and $d_\alpha(y, z)$ are finite, then $d_\alpha(x, y)$ is finite and $d_\alpha(x, y) \leq d_\alpha(x, z) + d_\alpha(z, y)$.

(**b**) If $\mathcal{D}$ is $\mathcal{D}$-family, then the pair $(X, \mathcal{D})$ is called a *generalized uniform space.*

(**c**) Let $(X, \mathcal{D})$ be a generalized uniform space. A $\mathcal{D}$-family $\mathcal{D}$ is said to be *separating* if

($\mathcal{D}$4) $\forall_{x,y \in X} \left\{x \neq y \Rightarrow \exists_{\alpha \in \mathcal{A}} \left\{0 < d_\alpha(x, y)\right\}\right\}$ ..

(**d**) If a $\mathcal{D}$-family $\mathcal{D}$ is separating, then the pair $(X, \mathcal{D})$ is called a *Hausdorff generalized uniform space.*

(**e**) Let $(X, \mathcal{D})$ be a generalized uniform space and let $(x_m : m \in \mathbb{N})$ be a sequence in X. We say that $(x_m : m \in \mathbb{N})$ is *$\mathcal{D}$-Cauchy sequence in X* if $\forall_{\alpha \in \mathcal{A}} \left\{\lim_{n \to \infty} \sup_{m > n} d_\alpha(x_n, x_m) = 0\right\}$. We say that $(x_m : m \in \mathbb{N})$ is *$\mathcal{D}$ -convergent in X* if there is an $x \in X$ such that $\forall_{a \in \mathcal{A}} \left\{\lim_{m \to \infty} d_\alpha(x_m, x) = 0\right\}$ $\left(\forall_{a \in \mathcal{A}} \left\{\lim_{m \to \infty} x_m = x\right\}\right.$, for short).

(**f**) If every $\mathcal{D}$-Cauchy sequence in X is $\mathcal{D}$-convergent sequence in X, then a pair $(X, \mathcal{D})$ is called a *$\mathcal{D}$-sequentially complete generalized uniform space.*

Definition 4 Let X be a nonempty set. The family

$$\mathcal{Q} = \{q_\alpha : X \times X \to [0, \infty], \alpha \in \mathcal{A}\}, \mathcal{A} \text{ - index set,}$$

is said to be a *$\mathcal{Q}$ -family of generalized quasi pseudometrics on X* (*$\mathcal{Q}$ -family on X*, for short) if the following two conditions hold:

$(\mathcal{Q}1)$ $\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\left\{q_\alpha(x,x)=0\right\}$;

$(\mathcal{Q}2)$ If $\alpha \in \mathcal{A}$ and $x, y, z \in X$ and if $q_\alpha(x, z)$ and $q_\alpha(z, y)$ are finite, then $q_\alpha(x, y)$ is finite and $q_\alpha(x, y) \le q_\alpha(x, z) + q_\alpha(z, y)$.

Definition 5 Let $(X, \mathcal{D})$ be a generalized uniform space.

(**a**) The family

$$\mathcal{L} = \{L_\alpha : X \times X \to [0, \infty], \alpha \in \mathcal{A}\}, \mathcal{A}\text{ - index set,}$$

is said to be a *$\mathcal{L}$-family of generalized pseudodistances on X* (*$\mathcal{L}$-family on X*, for short) if the following two conditions hold:

$(\mathcal{L}1)$ If $\alpha \in \mathcal{A}$ and $x, y, z \in X$ and if $L_\alpha(x, z)$ and $L_\alpha(z, y)$ are finite, then $L_\alpha(x, y)$ is finite and $L_\alpha(x, y) \le L_\alpha(x, z) + L_\alpha(z, y)$; and

$(\mathcal{L}2)$ For any sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X such that

$$\forall_{\alpha\in\mathcal{A}}\{\lim_{n\to\infty}\sup_{m>n} L_\alpha(x_n, x_m) = 0\} \tag{6}$$

and

$$\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} L_\alpha(x_m, y_m) = 0\}, \tag{7}$$

the following holds

$$\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} d_\alpha(x_m, y_m) = 0\}. \tag{8}$$

(**b**) Let $\mathbb{L}_{(X,\mathcal{D})}$ be a class defined as follows

$$\mathbb{L}_{(X,\mathcal{D})} = \{\mathcal{L} : \mathcal{L} \text{ is } \mathcal{L}\text{ - family on } X\}.$$

Remark 2 Let $(X, \mathcal{D})$ be a generalized uniform space. (**i**) $\mathbb{L}_{(X,\mathcal{D})} \neq \emptyset$ since $\mathcal{D} \in \mathbb{L}_{(X,\mathcal{D})}$. (**ii**) $\mathbb{L}_{(X,\mathcal{D})} \neq \{\mathcal{D}\}$; see Sections 10-13.

Definition 6 Let $(X, \mathcal{D})$ be a generalized uniform space, let $\mathcal{L} \in \mathbb{L}_{(X,\mathcal{D})}$ and let $(x_m : m \in \mathbb{N})$ be a sequence in X.

(**a**) We say that $(x_m : m \in \mathbb{N})$ is *$\mathcal{L}$-Cauchy in X* if $\forall_{\alpha\in\mathcal{A}}\left\{\lim_{n\to\infty}\sup_{m>n}L_\alpha(x_n, x_m) = 0\right\}$.

(**b**) We say that $(x_m : m \in \mathbb{N})$ is *$\mathcal{L}$-convergent in X* if there exists $x \in X$ such that $\forall_{a\in\mathcal{A}}\left\{\lim_{m\to\infty}L_\alpha(x_m, x) = 0\right\}$.

(**c**) We say that $(X, \mathcal{D})$ is *$\mathcal{L}$-sequentially complete* if each $\mathcal{L}$-Cauchy sequence in X is $\mathcal{L}$-convergent in X.

In the following remark, we list some basic properties of $\mathcal{L}$-families.

Remark 3 Let $(X, \mathcal{D})$ be a generalized uniform space and let $\mathcal{L} \in \mathbb{L}_{(X,\mathcal{D})}$. (**i**) If $\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\left\{L_\alpha(x,x)=0\right\}$, then $\mathcal{L}$ is a $\mathcal{Q}$ -family on X; examples of $\mathcal{L} \in \mathbb{L}_{(X,\mathcal{D})}$ which are not $\mathcal{Q}$ -families on X are given in Section "Examples of the decompositions of the generalized uniform spaces". (**ii**) There exist $\mathcal{L}$-sequentially complete spaces which are not $\mathcal{D}$-sequentially complete; see Example 15. (**iii**) If $(x_m : m \in \mathbb{N})$ in X is $\mathcal{L}$-convergent in X, then its limit point is not necessary unique; see Example 1.

Example 1 Let $(\mathbb{R}, |\cdot|)$ be a metric space. Define the family of $\mathcal{L} = \left\{L : \mathbb{R} \times \mathbb{R} \to [0, \infty]\right\}$ to be

$$L(x,y) = \begin{cases} 0 \text{ if } x \le y \\ 1 \text{ if } x > y \end{cases} x, y \in \mathbb{R}.$$

It is obvious that $\mathcal{L}$ is $\mathcal{L}$-family on $\mathbb{R}$ and the sequence $(1/m : m \in \mathbb{N})$ is $\mathcal{L}$-convergent to each point $w \in (0, +\infty)$.

One can prove the following proposition:

Proposition 1 *Let* $(X, \mathcal{D})$ *be a Hausdorff generalized uniform space and let* $\mathcal{L} \in \mathbb{L}_{(X,\mathcal{D})}$.

(**I**) *If* $x \neq y$, $x, y \in X$, *then* $\exists_{\alpha\in\mathcal{A}} \left\{L_\alpha(x,y) > 0 \vee L_\alpha(y,x) > 0\right\}$.

(**II**) *If* $(X, \mathcal{D})$ *is* $\mathcal{L}$*-sequentially complete and if* $(x_m : m \in \mathbb{N})$ *is* $\mathcal{L}$*-Cauchy sequence in* X, *then* $(x_m : m \in \mathbb{N})$ *is* $\mathcal{D}$*-convergent in* X.

Proof. (**I**)) Assume that there are $x \neq y$, $x, y \in X$, such that $\forall_{\alpha\in\mathcal{A}} \left\{L_\alpha(x,y) = L_\alpha(y,x) = 0\right\}$. Then, $\forall_{\alpha\in\mathcal{A}} \left\{L_\alpha(x,x) = 0\right\}$, since, by using $(\mathcal{L}1)$, it follows that $\forall_{\alpha\in\mathcal{A}} \left\{L_\alpha(x,x) \le L_\alpha(x,y) + L_\alpha(y,x) = 0\right\}$. Defining the sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X by $x_m = x$ and $y_m = y$ for $m \in \mathbb{N}$, and observing that $\forall_{\alpha\in\mathcal{A}} \left\{L_\alpha(x,y) = L_\alpha(y,x) - L_\alpha(x,x) = 0\right\}$, this implies that (6) and (7) for these sequences hold. Then, by $(\mathcal{L}2)$, (8) holds, so it is $\forall_{\alpha\in\mathcal{A}} \left\{d_\alpha(x,y) = 0\right\}$. On the other hand, $\mathcal{D}$ is separating, so, since $x \neq y$, it is $\exists_{\alpha\in\mathcal{A}} \left\{d_\alpha(x,y) \neq 0\right\}$. This leads to a contradiction.

(**II**) Since $\forall_{\alpha\in\mathcal{A}} \left\{\lim_{n\to\infty}\sup_{m>n}L_\alpha(x_n,x_m) = 0\right\}$, by Definition 6(c), this proves the existence of $x \in X$ such that $\forall_{a\in\mathcal{A}} \left\{\lim_{m\to\infty}L_\alpha(x_m,x) = 0\right\}$. We can apply $(\mathcal{L}2)$ to sequences $(x_m : m \in \mathbb{N})$ and $(y_m = x/ : m \in \mathbb{N})$ and then we find that $\forall_{\alpha\in\mathcal{A}} \left\{\lim_{m\to\infty}d_\alpha\left(x_m,y_m\right) = \lim_{m\to\infty}d_\alpha(x_m,x) = 0\right\}$. The uniqueness of the point of x follows from the fact that $\mathcal{D}$ is separating. □

Partial quasiordered space $K^{\mathcal{A}}$

Proposition 2 *Let* $K^{\mathcal{A}}$ *be a set of elements* $\Theta = (\eta_\alpha : \alpha \in \mathcal{A})$ *defined by the formula*

$$K^{\mathcal{A}} = \{\Theta = (\eta_\alpha : \alpha \in \mathcal{A}) : \forall_{\alpha\in\mathcal{A}}\{\eta_\alpha \in [-\infty,\infty]\}\}, \mathcal{A} \text{ - index set,}$$

and let $\forall_{\Theta=(\eta_\alpha:\alpha\in\mathcal{A})\in K^{\mathcal{A}}}\forall_{\alpha\in\mathcal{A}} \left\{[\Theta]_\alpha = \eta_\alpha\right\}$. *The relation* $\preccurlyeq_{K^{\mathcal{A}}}$ *on* $K^{\mathcal{A}}$ *defined by*

$$\forall_{\Theta=(\eta_\alpha:\alpha\in\mathcal{A}),\Omega=(\omega_\alpha:\alpha\in\mathcal{A})\in K^{\mathcal{A}}}\{\Theta\preccurlyeq_{K^{\mathcal{A}}}\Omega \Leftrightarrow \forall_{\alpha\in\mathcal{A}}\{\eta_\alpha = [\Theta]_\alpha \le [\Omega]_\alpha = \omega_\alpha\}\}$$

is a partial quasiordered on $K^{\mathcal{A}}$ *and the pair* $\left(K^{\mathcal{A}}, \preccurlyeq_{K^{\mathcal{A}}}\right)$ *is a partial quasiordered space.*

Proof. For all $\Theta \in K^{\mathcal{A}}$ the condition $\Theta\preccurlyeq_{K^{\mathcal{A}}}\Theta$ holds. For all $\Theta, \Omega, \Upsilon \in K^{\mathcal{A}}$, the conditions $\Theta\preccurlyeq_{K^{\mathcal{A}}}\Omega$ and $\Omega\preccurlyeq_{K^{\mathcal{A}}}\Upsilon$ imply $\Theta\preccurlyeq_{K^{\mathcal{A}}}\Upsilon$. For all $\Theta, \Omega \in K^{\mathcal{A}}$, the conditions $\Theta\preccurlyeq_{K^{\mathcal{A}}}\Omega$ and $\Omega\preccurlyeq_{K^{\mathcal{A}}}\Theta$ imply $\Theta = \Omega$. □

Notation. The following notation is fixed throughout the article:

$\Theta_0 = (\eta_\alpha = 0 : \alpha \in \mathcal{A})$;

$\Theta_{+\infty} = (\eta_\alpha = +\infty : \alpha \in \mathcal{A})$;

$K^{\mathcal{A}}_{0,+\infty} = \{\Theta \in K^{\mathcal{A}} : \Theta_0\preccurlyeq_{K^{\mathcal{A}}}\Theta \wedge \Theta\preccurlyeq_{K^{\mathcal{A}}}\Theta_{+\infty}\}$;

$K^{\mathcal{A}}_{+\infty} = \{\Theta = (\eta_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}} : \forall_{\alpha\in\mathcal{A}}\{\eta_\alpha \in (0, +\infty)\}\}$.

In the sequel, if $\Theta, \Omega \in K^{\mathcal{A}}$, then $\Theta \prec_{K^{\mathcal{A}}} \Omega$ will stand for $\Theta \preccurlyeq_{K^{\mathcal{A}}} \Omega$ and $\Theta \neq \Omega$.

Definition 7 Let $S^{\mathcal{A}}$ be a nonempty subset of $K^{\mathcal{A}}$. We say that $\mathbb{I}_{S^{\mathcal{A}}} = \mathbb{INF}(S^{\mathcal{A}}) \in K^{\mathcal{A}}$ is a *infimum of* $S^{\mathcal{A}}$ if the following two conditions hold:

($\mathbb{I}$1) $\forall_{\Theta \in S^{\mathcal{A}}}\{\mathbb{I}_{S^{\mathcal{A}}} \preccurlyeq_{K^{\mathcal{A}}} \Theta\}$;

($\mathbb{I}$2) $\forall_{\Omega \in K^{\mathcal{A}}}\{\{\mathbb{I}_{S^{\mathcal{A}}} \prec_{K^{\mathcal{A}}} \Omega\} \Rightarrow \exists_{\Theta \in S^{\mathcal{A}}}\{\Theta \prec_{K^{\mathcal{A}}} \Omega\}$.

Example 2 Let $\mathcal{A} = \{1, 2, 3\}$ and let $K^{\mathcal{A}} = \{\Theta = (\eta_1, \eta_2, \eta_3) : \forall_{\alpha \in \mathcal{A}}\{\eta_\alpha \in [-\infty, \infty]\}\}$. If $S_1^{\mathcal{A}} = \{(3, 5, 7), (4, 1, 8)\}$ then $S_1^{\mathcal{A}} \subset K^{\mathcal{A}}$ and $\mathbb{INF}(S_1^{\mathcal{A}})$ does not exist since (3, 5, 7) and (4, 1, 8) are not comparable. If $S_2^{\mathcal{A}} = \{(3, 5, 7), (4, 6, 8)\}$ then $S_2^{\mathcal{A}} \subset K^{\mathcal{A}}$ and $\mathbb{INF}(S_2^{\mathcal{A}}) = (3, 5, 7)$.

$\mathbb{H}_{(i)}^{\mathcal{L}}$-distances on 2^X, i ⌊ {1, 2}

Definition 8 Let $(X, \mathcal{D})$ be a Hausdorff generalized uniform space and let $\mathcal{L} \in \mathbb{L}_{(X,\mathcal{D})}$.

(**a**) For $C \in 2^X$ and $\Theta = (\eta_\alpha : \alpha \in \mathcal{A}) \in K_{+\infty}^{\mathcal{A}}$, let us denote

$$U_{\mathcal{L}}(\Theta, C) = \{u \in X : \exists_{c \in C}\forall_{\alpha \in \mathcal{A}}\{L_\alpha(u, c) < \eta_\alpha\}\}. \tag{9}$$

(**b**) For A, $B \in 2^X$ let us denote:

$$\mathcal{H}_{(1)}^{\mathcal{L}}(A, B) = \{\Theta \in K_{+\infty}^{\mathcal{A}} : A \subset U_{\mathcal{L}}(\Theta, B)\}, \tag{10}$$

$$\mathcal{H}_{(2)}^{\mathcal{L}}(A, B) = \{\Theta \in K_{+\infty}^{\mathcal{A}} : A \subset U_{\mathcal{L}}(\Theta, B) \wedge B \subset U_{\mathcal{L}}(\Theta, A)\}. \tag{11}$$

(**c**) Let $i \in \{1, 2\}$. The map $\mathbb{H}_{(i)}^{\mathcal{L}} : 2^X \times 2^X \to K_{0,+\infty}^{\mathcal{A}}$ of the form

$$\mathbb{H}_{(i)}^{\mathcal{L}}(A, B) = \begin{cases} \mathbb{INF}(\mathcal{H}_{(i)}^{\mathcal{L}}(A, B)) & \text{if } \mathbb{INF}(\mathcal{H}_{(i)}^{\mathcal{L}}(A, B)) \text{ exists and} \\ & \forall_{\alpha \in \mathcal{A}}\{[\mathbb{INF}(\mathcal{H}_{(i)}^{\mathcal{L}}(A, B))]_\alpha < +\infty\} \\ & \\ & \text{if } \mathbb{INF}(\mathcal{H}_{(i)}^{\mathcal{L}}(A, B)) \text{ does not exist or} \\ \Theta_{+\infty} & \text{if } \mathbb{INF}(\mathcal{H}_{(i)}^{\mathcal{L}}(A, B)) \text{ exists and} \\ & \exists_{\alpha \in \mathcal{A}}\{[\mathbb{INF}(\mathcal{H}_{(i)}^{\mathcal{L}}(A, B))]_\alpha = +\infty\} \end{cases},$$

A, $B \in 2^X$, is called a $\mathbb{H}_{(i)}^{\mathcal{L}}$ *-distance on* 2^X *generated by* $\mathcal{L}$ ($\mathbb{H}_{(i)}^{\mathcal{L}}$ *-distance on* 2^X, for short).

Remark 4 For each A, $B \in 2^X$, $\mathbb{H}_{(1)}^{\mathcal{L}}(A, B) \preccurlyeq_{K^{\mathcal{A}}} \mathbb{H}_{(2)}^{\mathcal{L}}(A, B)$.

$\left(\mathbb{H}_{(i)}^{\mathcal{L}}, \Upsilon, \Lambda\right)$-uniformly locally contractive and $\left(\mathbb{H}_{(i)}^{\mathcal{L}}, \Lambda\right)$-contractive set-valued dynamic systems (*X*, *T*), i ⌊ {1, 2}

Definition 9 Let $(X, \mathcal{D})$ be a Hausdorff generalized uniform space, let $\mathcal{L} \in \mathbb{L}_{(X,\mathcal{D})}$ and let $i \in \{1,2\}$.

(**a**) Let $\mathbb{H}_{(i)}^{\mathcal{L}}$ be a $\mathbb{H}_{(i)}^{\mathcal{L}}$-distance on 2^X and let $\Upsilon = (\varepsilon_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}}$ and $\Lambda = (\lambda_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}}$ be such that $\forall_{\alpha \in \mathcal{A}}\{\varepsilon_\alpha \in (0, \infty) \wedge \lambda_\alpha \in [0, 1)\}$. We say that a set-valued dynamic system (X, T), $T : X \to 2^X$, is $\left(\mathbb{H}_{(i)}^{\mathcal{L}}, \Upsilon, \Lambda\right)$*-uniformly locally contractive on* X if

$$\forall_{\alpha \in \mathcal{A}}\forall_{x,y \in X}\{L_\alpha(x, y) < \varepsilon_\alpha \Rightarrow [\mathbb{H}_{(i)}^{\mathcal{L}}(T(x), T(y))]_\alpha \leq \lambda_\alpha L_\alpha(x, y)\}. \tag{12}$$

(**b**) Let $\mathbb{H}^{\mathcal{L}}_{(i)}$ be a $\mathbb{H}^{\mathcal{L}}_{(i)}$-distance on 2^X and let $\Lambda = (\lambda_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}}$ be such that $\forall_{\alpha\in\mathcal{A}}\{\lambda_\alpha \in [0, 1)\}$. We say that a set-valued dynamic system (X, T), $T : X \to 2^X$, is $\left(\mathbb{H}^{\mathcal{L}}_{(i)}, \Lambda\right)$ *- contractive on X* if

$$\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in X}\{[\mathbb{H}^{\mathcal{L}}_{(i)}(T(x), T(y))]_\alpha \le \lambda_\alpha L_\alpha(x, y)\}. \tag{13}$$

Remark 5 Let $(X, \mathcal{D})$ be a Hausdorff generalized uniform space, let $\mathcal{L} \in \mathbb{L}_{(X,\mathcal{D})}$ and let $\Upsilon = (\varepsilon_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}}$ and $\Lambda = (\lambda_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}}$ be such that $\forall_{\alpha\in\mathcal{A}}\{\varepsilon_\alpha \in (0, \infty) \wedge \lambda_\alpha \in [0, 1)\}$.

(**i**) If (X, T), $T : X \to 2^X$, is $(\mathbb{H}^{\mathcal{L}}_{(2)}, \Upsilon, \Lambda)$-uniformly locally contractive on X then it is $(\mathbb{H}^{\mathcal{L}}_{(1)}, \Upsilon, \Lambda)$-uniformly locally contractive on X.

(**ii**) If (X, T), $T : X \to 2^X$, is $(\mathbb{H}^{\mathcal{L}}_{(2)}, \Lambda)$-contractive on X then it is $(\mathbb{H}^{\mathcal{L}}_{(1)}, \Lambda)$-contractive on X.

(**iii**) Let $i \in \{1, 2\}$. If (X, T), $T : X \to 2^X$, is $\left(\mathbb{H}^{\mathcal{L}}_{(i)}, \Lambda\right)$-contractive on X then it is $\left(\mathbb{H}^{\mathcal{L}}_{(i)}, \Upsilon, \Lambda\right)$-uniformly locally contractive on X.

Statement of results

Definition 10 Let $(X, \mathcal{D})$ be a Hausdorff generalized uniform space and let $x \in X$/We say that a set-valued dynamic system (X, T), $T : X \to 2^X$, is *closed at x* if whenever $(x_m : m \in \mathbb{N})$ is a sequence $\mathcal{D}$-converging to x in X and $(y_m : m \in \mathbb{N})$ is a sequence $\mathcal{D}$-converging to y in X such that $y_m \in T(x_m)$ for all $m \in \mathbb{N}$, then $y \in T(x)$.

The main existence and convergence result of this article we can now state as follows.

Theorem 8 *Assume that $(X, \mathcal{D})$ is a Hausdorff generalized uniform space, $\mathcal{L} \in \mathbb{L}_{(X,\mathcal{D})}$ and one of the following properties holds:*

(**P1**) *$(X, \mathcal{D})$ is $\mathcal{L}$-sequentially complete; or*

(**P2**) *$(X, \mathcal{D})$ is $\mathcal{D}$-sequentially complete.*

Let $i \in \{1, 2\}$, let $\mathbb{H}^{\mathcal{L}}_{(i)} : 2^X \times 2^X \to K^{\mathcal{A}}_{0,+\infty}$ be a $\mathbb{H}^{\mathcal{L}}_{(i)}$ -distance on 2^X and assume that a set-valued dynamic system (X, T), $T : X \to 2^X$, has the property

(**C**) $\forall_{w^0\in X}\forall_{(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)}\forall_{w\in X}$ $\{\lim_{m\to\infty} w^m = w \Rightarrow T$ is closed at $w\}$.

(**I**) *If $\Upsilon = (\varepsilon_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}}$ and $\Lambda = (\lambda_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}}$ satisfy $\forall_{\alpha\in\mathcal{A}}\{\varepsilon_\alpha \in (0, \infty) \wedge \lambda_\alpha \in [0, 1)\}$ and (X, T) is $\left(\mathbb{H}^{\mathcal{L}}_{(i)}, \Upsilon, \Lambda\right)$ -uniformly locally contractive on X then, for each $w^0 \in X$, the following alternative holds: either*

(**A1**) $\forall_{(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)}\forall_{m\in\mathbb{N}}\exists_{\alpha_0\in\mathcal{A}}\{L_{\alpha_0}(w^{m-1}, w^m) \ge \varepsilon_{\alpha_0}\}$*; or*

(**A2**) $\exists_{(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)}\exists_{w\in X}$ $\{w \in Fix(T) \wedge \lim_{m\to\infty} w^m = w \wedge (w^m : m \in \{0\} \cup \mathbb{N})$ *is $\mathcal{L}$-Cauchy*$\}$.

(**II**) *If $\Lambda = (\lambda_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}}$ satisfies $\forall_{\alpha\in\mathcal{A}}\{\lambda_\alpha \in [0, 1)\}$ and (X, T) is $\left(\mathbb{H}^{\mathcal{L}}_{(i)}, \Lambda\right)$-contractive on X then, for each $w^0 \in X$, the following alternative holds: either*

(**B1**) $\forall_{(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)}\forall_{m\in\mathbb{N}}\exists_{\alpha_0\in\mathcal{A}}\{L_{\alpha_0}(w^{m-1}, w^m) = \infty\}$ *; or*

(**B2**) $\exists_{(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)}\exists_{w\in X}$ $\{w \in Fix(T) \wedge \lim_{m\to\infty} w^m = w \wedge (w^m : m \in \{0\} \cup \mathbb{N})$ *is $\mathcal{L}$-Cauchy*$\}$.

Definition 11 Let $(X, \mathcal{D})$ be a Hausdorff generalized uniform space.

(**a**) We say that a nonempty subset Y of X is *closed in X if* $Y = Cl(Y)$ where $Cl(Y)$, the *closure* of Y in X, denotes the set of all $x \in X$ for which there exists a sequence $(x_m : m \in \mathbb{N})$ in Y which is $\mathcal{D}$-convergent to x.

(**b**) The class of all nonempty closed subsets of X is denoted by $C(X)$, i.e. $C(X) = \{Y : Y \in 2^X \wedge Y = Cl(Y)\}$.

Theorem 8 has the following corresponding when $\mathcal{L} = \mathcal{D}$ and when $T : X \to C(X)$.

Theorem 9 *Let* $(X, \mathcal{D})$ *be a Hausdorff* $\mathcal{D}$*-sequentially complete generalized uniform space, let* $i \in \{1, 2\}$ *and assume that* $\mathbb{H}^{\mathcal{D}}_{(i)} : C(X) \times C(X) \to K^{\mathcal{A}}_{0,+\infty}$ *is a* $\mathbb{H}^{\mathcal{D}}_{(i)}$*-distance on* $C(X)$.

(**I**) *If* $\Upsilon = (\varepsilon_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}}$ *and* $\Lambda = (\lambda_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}}$ *satisfy* $\forall_{\alpha\in\mathcal{A}}\{\varepsilon_\alpha \in (0,\infty) \wedge \lambda_\alpha \in [0,1)\}$ *and if a set-valued dynamic system* (X, T) *satisfying* $T : X \to C(X)$ *is* $(\mathbb{H}^{\mathcal{D}}_{(i)}, \Upsilon, \Lambda)$ *-uniformly locally contractive on X then, for each* $w^0 \in X$, *the following alternative holds: either*

(**F1**) $\forall_{(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)}\forall_{m\in\mathbb{N}}\exists_{\alpha_0\in\mathcal{A}}\{L_{\alpha_0}(w^{m-1}, w^m) \geq \varepsilon_{\alpha_0}\}$; *or*

(**F2**) $\exists_{(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)}\exists_{w\in X}\{w \in Fix(T) \wedge \lim_{m\to\infty} w^m = w\}$.

(**II**) *If* $\Lambda = (\lambda_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}}$ *satisfies* $\forall_{\alpha\in\mathcal{A}}\{\lambda_\alpha \in [0,1)\}$ *and a set-valued dynamic system* (X, T) *satisfying* $T : X \to C(X)$ *is* $(\mathbb{H}^{\mathcal{D}}_{(i)}, \Lambda)$*-contractive on X then, for each* $w^0 \in X$, *the following alternative holds: either*

(**G1**) $\forall_{(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)}\forall_{m\in\mathbb{N}}\exists_{\alpha_0\in\mathcal{A}}\{d_{\alpha_0}(w^{m-1}, w^m) = \infty\}$; *or*

(**G2**) $\exists_{(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)}\exists_{w\in X}\{w \in Fix(T) \wedge \lim_{m\to\infty} w^m = w\}$.

Proof of Theorem 8

(**I**) Let $i \in \{1, 2\}$. The proof is divided into three steps.

Step 1. *Assume that* $w^0 \in X$ *and suppose that the assertion* (A1) *does not hold; that is,*

$$\exists_{(v^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,v^0=w^0)}\exists_{m_0\in\mathbb{N}}\forall_{\alpha\in\mathcal{A}}\{L_\alpha(v^{m_0-1}, v^{m_0}) < \varepsilon_\alpha\}. \tag{14}$$

Then there exists $(w^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}(X, T, w^0)$ *which is* $\mathcal{L}$*-Cauchy sequence on X; that is,*

$$\forall_{\alpha\in\mathcal{A}}\{\lim_{n\to\infty}\sup_{m>n} L_\alpha(w^n, w^m) = 0\}. \tag{15}$$

Indeed, since (14) holds, thus, by (12), we get

$$\forall_{\alpha\in\mathcal{A}}\{[\mathbb{H}^{\mathcal{L}}_{(i)}(T(v^{m_0-1}), T(v^{m_0}))]_\alpha \leq \lambda_\alpha L_\alpha(v^{m_0-1}, v^{m_0}) < \lambda_\alpha\varepsilon_\alpha\}. \tag{16}$$

It follows from (16) and Definition 8(c), that there exists $\mathbb{INF}(\mathcal{H}^{\mathcal{L}}_{(i)}(T(v^{m_0-1}), T(v^{m_0})))$ and

$$\forall_{\alpha\in\mathcal{A}}\{[\mathbb{INF}(\mathcal{H}^{(i)}_{\mathcal{L}}(T(v^{m_0-1}), T(v^{m_0})))]_\alpha < \lambda_\alpha\varepsilon_\alpha\}. \tag{17}$$

From this, denoting $\Omega = \{\lambda_\alpha\varepsilon_\alpha : \alpha \in \mathcal{A}\} \in K^{\mathcal{A}}$, we deduce that $\mathbb{INF}(\mathcal{H}^{\mathcal{L}}_{(i)}(T(v^{m_0-1}), T(v^{m_0}))) \prec_{K^{\mathcal{A}}} \Omega$. Consequently, by $(\mathbb{I}2)$, there exists

$$\Theta \in \mathcal{H}^{\mathcal{L}}_{(i)}(T(v^{m_0-1}), T(v^{m_0})) \tag{18}$$

such that $\Theta \prec_{K^{\mathcal{A}}} \Omega$ which implies

$$\forall_{\alpha\in\mathcal{A}}\{[\Theta]_\alpha \le [\Omega]_\alpha = \lambda_\alpha\varepsilon_\alpha\} \text{ and } \Theta \ne \Omega. \tag{19}$$

If $i = 1$, then we note that, by (18), (9), and (10), $T(v^{m_0-1}) \subset U_{\mathcal{L}}(\Theta, T(v^{m_0}))$. Clearly, $v^{m_0} \in T(v^{m_0-1})$. Thus, $v^{m_0} \in U_{\mathcal{L}}(\Theta, T(v^{m_0}))$ and the conclusion

$$\exists_{u^{m_0+1}\in T(v^{m_0})}\forall_{\alpha\in\mathcal{A}}\{L_\alpha(v^{m_0}, u^{m_0+1}) < [\Theta]_\alpha \le \lambda_\alpha\varepsilon_\alpha\}$$

follows directly from (9), (10), (18), and (19).

If $i = 2$, then we also note that, by (18), (9) and (11), $T(v^{m_0-1}) \subset U_{\mathcal{L}}(\Theta, T(v^{m_0}))$ and $T(v^{m_0}) \subset U_{\mathcal{L}}(\Theta, T(v^{m_0-1}))$. Clearly, $v^{m_0} \in T(v^{m_0-1})$. Thus, $v^{m_0} \in U_{\mathcal{L}}(\Theta, T(v^{m_0}))$ and the conclusion

$$\exists_{u^{m_0+1}\in T(v^{m_0})\subset U_{\mathcal{L}}(\Theta, T(v^{m_0-1}))}\forall_{\alpha\in\mathcal{A}}\{L_\alpha(v^{m_0}, u^{m_0+1}) < [\Theta]_\alpha \le \lambda_\alpha\varepsilon_\alpha\}$$

follows directly from (9), (11), (18), and (19).

This proves

$$\exists_{u^{m_0+1}\in T(v^{m_0})}\forall_{\alpha\in\mathcal{A}}\{L_\alpha(v^{m_0}, u^{m_0+1}) < \lambda_\alpha\varepsilon_\alpha\}. \tag{20}$$

Since, by (20), $\forall_{\alpha\in\mathcal{A}}\{L_\alpha(v^{m_0}, u^{m_0+1}) < \varepsilon_\alpha\}$, it follows, using (12) and (20), that

$$\forall_{\alpha\in\mathcal{A}}\{[\mathbb{H}^{\mathcal{L}}_{(i)}(T(v^{m_0}), T(u^{m_0+1}))]_\alpha \le \lambda_\alpha L_\alpha(v^{m_0}, u^{m_0+1}) < (\lambda_\alpha)^2\varepsilon_\alpha\}.$$

That is,

$$\forall_{\alpha\in\mathcal{A}}\{[\mathbb{INF}(\mathcal{H}^{\mathcal{L}}_{(i)}(T(v^{m_0}), T(u^{m_0+1})))]_\alpha < (\lambda_\alpha)^2\varepsilon_\alpha\}. \tag{21}$$

Denoting $\Delta = \{(\lambda_\alpha)^2\varepsilon_\alpha : \alpha \in \mathcal{A}\} \in K^{\mathcal{A}}$, we see that condition (21) implies $\mathbb{INF}(\mathcal{H}^{\mathcal{L}}_{(i)}(T(v^{m_0}), T(u^{m_0+1}))) \prec_{K^{\mathcal{A}}} \Delta$. Hence, by ($\mathbb{I}2$), there exists

$$\Pi \in \mathcal{H}^{\mathcal{L}}_{(i)}(T(v^{m_0}), T(u^{m_0+1})) \tag{22}$$

such that $\Pi \prec_{K^{\mathcal{A}}} \Delta$. This means

$$\forall_{\alpha\in\mathcal{A}}\{[\Pi]_\alpha \le [\Delta]_\alpha = (\lambda_\alpha)^2\varepsilon_\alpha\} \text{ and } \Pi \ne \Delta. \tag{23}$$

Let $i = 1$. Clearly, by (9), (10), and (22), $T(v^{m_0}) \subset U_{\mathcal{L}}(\Pi, T(u^{m_0+1}))$. Moreover, by (20), $u^{m_0+1} \in T(v^{m_0})$. Therefore $u^{m_0+1} \in U_{\mathcal{L}}(\Pi, T(u^{m_0+1}))$. This, by (9), (10) and (21)-(23), implies

$$\exists_{u^{m_0+2}\in T(u^{m_0+1})}\forall_{\alpha\in\mathcal{A}}\{L_\alpha(u^{m_0+1}, u^{m_0+2}) < [\Pi]_\alpha < (\lambda_\alpha)^2\varepsilon_\alpha\}.$$

Let $i = 2$. Clearly, by (9)-(11) and (22), $T(v^{m_0}) \subset U_{\mathcal{L}}(\Pi, T(u^{m_0+1}))$ and $T(u^{m_0+1}) \subset U_{\mathcal{L}}(\Pi, T(v^{m_0}))$. Moreover, $u^{m_0+1} \in T(v^{m_0})$. Therefore $u^{m_0+1} \in U_{\mathcal{L}}(\Pi, T(u^{m_0+1}))$. This, by (9)-(11) and (21)-(23), implies

$$\exists_{u^{m_0+2}\in T(u^{m_0+1})\subset U_{\mathcal{L}}(\Pi, T(v^{m_0}))}\forall_{\alpha\in\mathcal{A}}\{L_\alpha(u^{m_0+1}, u^{m_0+2}) < [\Pi]_\alpha < (\lambda_\alpha)^2\varepsilon_\alpha\}.$$

That is,

$$\exists_{u^{m_0+2}\in T(u^{m_0+1})}\forall_{\alpha\in\mathcal{A}}\{L_\alpha(u^{m_0+1}, u^{m_0+2}) < (\lambda_\alpha)^2\varepsilon_\alpha\}. \tag{24}$$

By (24), we have $\forall_{\alpha\in\mathcal{A}}\{L_\alpha(u^{m_0+1},u^{m_0+2})<\varepsilon_\alpha\}$ and, using (12) and (24), we get

$$\forall_{\alpha\in\mathcal{A}}\{[\mathbb{H}^{\mathcal{L}}_{(i)}(T(u^{m_0+1}),T(u^{m_0+2}))]_\alpha\le\lambda_\alpha L_\alpha(u^{m_0+1},u^{m_0+2})<(\lambda_\alpha)^3\varepsilon_\alpha\}.$$

This means

$$\forall_{\alpha\in\mathcal{A}}\{[\mathbb{INF}(\mathcal{H}^{\mathcal{L}}_{(i)}(T(u^{m_0+1}),T(u^{m_0+2})))]_\alpha<(\lambda_\alpha)^3\varepsilon_\alpha\}. \tag{25}$$

By induction, a similar argument as in the proofs of (17)-(25) shows that

$$\begin{aligned}\exists_{(u^{m_0+n}:n\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,u^{m_0}=v^{m_0})}\forall_{\alpha\in\mathcal{A}}\forall_{n\in\{0\}\cup\mathbb{N}}\{u^{m_0+n+1}\in T(u^{m_0+n})\wedge\\ \wedge L_\alpha(u^{m_0+n},u^{m_0+n+1})<(\lambda_\alpha)^{n+1}\varepsilon_\alpha\wedge\\ \wedge[\mathbb{H}^{\mathcal{L}}_{(i)}(T(u^{m_0+n}),T(v^{m_0+n+1}))]_\alpha\le\lambda_\alpha L_\alpha(u^{m_0+n},u^{m_0+n+1})\}.\end{aligned} \tag{26}$$

It is clear that (26) implies that $(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)$ where $\forall_{m<m_0}\{w^m=v^m\}$, $w^{m_0}=u^{m_0}=v^{m_0}$ and $\forall_{m>m_0}\{w^m=u^m\}$. Additionally, this sequence $(w^m : m\in\{0\}\cap\mathbb{N})$ is a $\mathcal{L}$-Cauchy sequence on X, i.e., (15) holds.

Step 2. *Assume that the condition* (C) *and the property* (P1) *hold. If* $w^0\in X$ *and the assertion* (A1) *does not hold, then* (A2) *holds.*

By Step 1, Definition 8(c) and (P1) (note that then $(X,\mathcal{D})$ is $\mathcal{L}$-sequentially complete), we have that there exists $w\in X$ satisfying

$$\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}L_\alpha(w^m,w)=0\}. \tag{27}$$

Applying (15), (27), and $(\mathcal{L}2)$ (where $(x_m=w^m : m\in\mathbb{N})$ and $(y_m=w : m\in\mathbb{N})$), we find that

$$\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty}d_\alpha(w^m,w)=0\}. \tag{28}$$

Clearly, since $(X,\mathcal{D})$ is Hausdorff, condition (28) implies that such a point w is unique.

We observe that $w\in Fix(T)$. Indeed, we have that a dynamic process $(w^m : m\in\{0\}\cup\mathbb{N})$ satisfies (28). Hence, by (C), T is closed at w and, since $\forall_{m\in\mathbb{N}}\{w^m\in T(w^{m-1})\}$, we get $w\in T(w)$. This proves that the assertion (A2) holds.

This yields the result when (C) and (P1) hold.

Step 3. *Assume that the condition* (C) *and the property* (P2) *hold. If* $w^0\in X$ *and the assertion* (A1) *does not hold, then* (A2) *holds.*

If (A1) does not hold, then, by Step 1, there exists a sequence $(w^m : m\in\{0\}\cap\mathbb{N})$ which satisfies $(w^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,w^0)$ and, additionally, this sequence is a $\mathcal{L}$-Cauchy sequence on X, i.e.

$$\forall_{\alpha\in\mathcal{A}}\{\lim_{n\to\infty}\sup_{m>n}L_\alpha(w^n,w^m)=0\}. \tag{29}$$

We prove that $(w^m:m\in\{0\}\cup\mathbb{N})$ is a $\mathcal{D}$-Cauchy sequence on X, i.e. that

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_0=n_0(\alpha,\varepsilon)\in\mathbb{N}}\forall_{s,l\in\mathbb{N},\ s>l>n_0}\{d_\alpha(w^s,w^l)<\varepsilon\}. \tag{30}$$

Indeed, by (29), we claim that

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_1=n_1(\alpha,\varepsilon)\in\mathbb{N}}\forall_{n>n_1}\{\sup(L_\alpha(w^n,w^m):m>n\}<\varepsilon\}.$$

Hence, in particular,

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_1=n_1(\alpha,\varepsilon)\in\mathbb{N}}\forall_{n>n_1}\forall_{q\in\mathbb{N}}\{L_\alpha(w^n,w^{q+n})<\varepsilon\}. \tag{31}$$

Let now r_0, $j_0 \in \mathbb{N}$, $r_0 > j_0$, be arbitrary and fixed. If we define

$$t^m = w^{r_0+m} \text{ and } z^m = w^{j_0+m} \text{for } m \in \mathbb{N}, \tag{32}$$

then (31) implies that

$$\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} L_\alpha(w^m,t^m) = \lim_{m\to\infty} L_\alpha(w^m,z^m) = 0\}. \tag{33}$$

Therefore, by (29), (33), and ($\mathcal{L}2$), we get

$$\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} d_\alpha(w^m,t^m) = \lim_{m\to\infty} d_\alpha(w^m,z^m) = 0\}. \tag{34}$$

From (32)-(34), we then claim that

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_2=n_2(\alpha,\varepsilon)\in\mathbb{N}}\forall_{m>n_2}\{d_\alpha(w^m,w^{r_0+m})<\varepsilon/2\} \tag{35}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_3=n_3(\alpha,\varepsilon)\in\mathbb{N}}\forall_{m>n_3}\{d_\alpha(w^m,w^{j_0+m})<\varepsilon/2\}. \tag{36}$$

Let now $\alpha_0 \in \mathcal{A}$ and $\varepsilon_0 > 0$ be arbitrary and fixed, let $n_0 = \max\{n_2(\alpha_0, \varepsilon_0), n_3(\alpha_0, \varepsilon_0)\} + 1$ and let s, $l \in \mathbb{N}$ be arbitrary and fixed such that $s > l > n_0$. Then $s = r_0 + n_0$ and $l = j_0 + n_0$ for some r_0, $j_0 \in \mathbb{N}$ such that $r_0 > j_0$ and, using (35) and (36), we get

$$\begin{aligned} d_{\alpha_0}(w^s,w^l) = d_{\alpha_0}(w^{r_0+n_0},w^{j_0+n_0}) &\le d_{\alpha_0}(w^{n_0},w^{r_0+n_0}) + d_{\alpha_0}(w^{n_0},w^{j_0+n_0}) \\ &< \varepsilon_0/2 + \varepsilon_0/2 = \varepsilon_0. \end{aligned}$$

Hence, we conclude that

$$\forall_{\alpha\in\mathcal{A}}\forall_{\varepsilon>0}\exists_{n_0=n_0(\alpha,\varepsilon)\in\mathbb{N}}\forall_{s,l\in\mathbb{N},\ s>l>n_0}\{d_\alpha(w^s,w^l)<\varepsilon\}.$$

The proof of (30) is complete.

Now we see that there exists a unique $w \in X$ such that $\lim_{m\to\infty} w^m = w$. Indeed, since $(X, \mathcal{D})$ is a Hausdorff $\mathcal{D}$-sequentially complete generalized uniform space and the sequence $(w^m : m \in \{0\} \cup \mathbb{N})$ is a $\mathcal{D}$-Cauchy sequence on X, thus there exists a unique $w \in X$ such that $\lim_{m\to\infty} w^m = w$.

Moreover, we observe that $w \in Fix(T)$. Indeed, we have that a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ satisfies $\lim_{m\to\infty} w^m = w$. Hence, by (C), T is closed at w and, since $\forall_{m\in\mathbb{N}} \{w^m \in T(w^{m-1})\}$, we get $w \in T(w)$. We proved that the assertion (A2) holds.

This yields the result when (C) and (P2) hold.

The proof of (I) is complete.

(**II**) Let $i \in \{1, 2\}$. Let $w^0 \in X$, let the condition (C) holds and suppose that the assertion (B1) does not hold, i.e. suppose that

$$\exists_{(v^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,v^0=w^0)}\exists_{m_0\in\mathbb{N}}\forall_{\alpha\in\mathcal{A}}\{L_\alpha(v^{m_0-1},v^{m_0})<\infty\}.$$

This implies that there exists the family $\Upsilon = (\varepsilon_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}}$ such that $\forall_{\alpha\in\mathcal{A}}\{\varepsilon_\alpha \in (0,\infty)\}$ and $\forall_{\alpha\in\mathcal{A}}\{L_\alpha(v^{m_0-1},v^{m_0})<\varepsilon_\alpha<\infty\}$. Consequently,

$$\exists_{(v^m:m\in\{0\}\cup\mathbb{N})\in\mathcal{O}(X,T,v^0=w^0)}\exists_{m_0\in\mathbb{N}}\forall_{\alpha\in\mathcal{A}}\{L_\alpha(v^{m_0-1},v^{m_0})<\varepsilon_\alpha\}.$$

Clearly, (X, T) is $\left(\mathbb{H}^{\mathcal{L}}_{(i)}, \Upsilon, \Lambda\right)$-uniformly locally contractive on X since (X, T) is $\left(\mathbb{H}^{\mathcal{L}}_{(i)}, \Lambda\right)$-contractive on X. From the above and by similar argumentations as in Steps 1-3 of the proof of Theorem 8(I) we conclude that all assumptions of Theorem 8(I) hold and the assertion (A1) of Theorem 8(I) does not hold. Consequently, using Theorem 8(I), we get that the assertion (A2) of Theorem 8(I) holds in the case when the property either (P1) or (P2) holds. Hence, the assertion (B2) of Theorem 8(II) holds.

The proof of Theorem 8 is complete. □

Proof of Theorem 9

(**I**) Let $i \in \{1, 2\}$. Let $w^0 \in X$ be arbitrary and fixed and suppose that the assertion (F1) does not hold. That is

$$\exists_{(v^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}(X,T,v^0=w^0)} \exists_{m_0 \in \mathbb{N}} \forall_{\alpha \in \mathcal{A}} \{d_\alpha(v^{m_0-1}, v^{m_0}) < \varepsilon_\alpha\}. \tag{37}$$

But then, using analogous considerations as in the Step 1 of the proof of Theorem 8 (I), we obtain that

$$\begin{aligned} \exists_{(u^{m_0+n} : n \in \{0\} \cup \mathbb{N}) \in \mathcal{O}(X,T,u^{m_0}=v^{m_0})} \forall_{\alpha \in \mathcal{A}} \forall_{n \in \{0\} \cup \mathbb{N}} \{u^{m_0+n+1} \in T(u^{m_0+n}) \wedge \\ d_\alpha(u^{m_0+n}, u^{m_0+n+1}) < (\lambda_\alpha)^{n+1} \varepsilon_\alpha \wedge \\ \alpha[\mathbb{H}^{\mathcal{D}}_{(i)}(T(u^{m_0+n}), T(u^{m_0+n+1}))] \leq \lambda_\alpha d_\alpha(u^{m_0+n}, u^{m_0+n+1})\}. \end{aligned} \tag{38}$$

Consequently, the sequence $(w^m : m \in \{0\} \cup \mathbb{N})$ such that $\forall_{m<m_0}\{w^m = v^m\}$, $w^{m_0} = u^{m_0} = v^{m_0}$ and $\forall_{m>m_0}\{w^m = u^m\}$ is a dynamic process of T starting at w^0 and, additionally, this sequence is a $\mathcal{D}$-Cauchy sequence on X, i.e.

$$\forall_{\alpha \in \mathcal{A}} \{\lim_{n \to \infty} \sup_{m > n} d_\alpha(w^n, w^m) = 0\}. \tag{39}$$

It is clear that (39) implies

$$\forall_{\alpha \in \mathcal{A}} \{\lim_{m \to \infty} d_\alpha(w^m, w^{m+1}) = 0\} \tag{40}$$

and, since $(X, \mathcal{D})$ is a Hausdorff $\mathcal{D}$-sequentially complete generalized uniform space, there exists a unique $w \in X$ such that

$$\forall_{\alpha \in \mathcal{A}} \{\lim_{m \to \infty} d_\alpha(w^m, w) = 0\}. \tag{41}$$

If, for each $\alpha \in \mathcal{A}$, $x \in X$ and $B \subset Cl(X)$, we denote

$$d_\alpha(x, B) = \inf\{d_\alpha(x, y) : y \in B\} \tag{42}$$

and

$$\omega_\alpha(x) = d_\alpha(x, T(x)), \tag{43}$$

then (42) and (40) implies

$$\begin{aligned} \forall_{\alpha \in \mathcal{A}} \{\lim_{m \to \infty} \omega_\alpha(w^m) &= \lim_{m \to \infty} d_\alpha(w^m, T(w^m)) \\ &\leq \lim_{m \to \infty} d_\alpha(w^m, w^{m+1}) = 0\}. \end{aligned} \tag{44}$$

Let $m \in \mathbb{N}$, $m > m_0$, and $\alpha \in \mathcal{A}$ be arbitrary and fixed and let

$$[\Phi]_\alpha = \varphi_\alpha = [\mathbb{H}^{\mathcal{D}}_{(i)}(T(w^m), T(w))]_\alpha, \Phi \in K^{\mathcal{A}}_{0,+\infty};$$

here m_0 is defined by (37). Then, by (9)-(11) and definition of $\mathbb{H}^{\mathcal{D}}_{(i)}(T(w^m), T(w))$, we get that $\forall_{v \in T(w^m)} \exists_{c_1 \in T(w)} \{d_\alpha(v, c_1) \le \varphi_\alpha\}$ and $\forall_{v \in T(w)} \exists_{c_2 \in T(w^m)} \{d_\alpha(v, c_2) \le \varphi_\alpha\}$. Hence, in particular, if $v \in T(w^m)$ is arbitrary and fixed, then

$$d_\alpha(v, T(w)) = \inf\{d_\alpha(v, z) : z \in T(w)\} \le d_\alpha(v, c_1) \le \varphi_\alpha.$$

This implies

$$\sup_{v \in T(w^m)} d_\alpha(v, T(w)) \le \varphi_\alpha = \alpha[\mathbb{H}^{\mathcal{D}}_{(i)}(T(w^m), T(w))]. \tag{45}$$

Now, by ($\mathcal{D}1$), (remember that $\mathcal{L} = \mathcal{D}$), for each $u \in T(w)$ and $v \in T(w^m)$, we have

$$d_\alpha(w, u) \le d_\alpha(w, w^m) + d_\alpha(w^m, v) + d_\alpha(v, u).$$

Hence, by (42) and ($\mathcal{D}1$), for each $v \in T(w^m)$, it follows

$$d_\alpha(w, T(w)) = \omega_\alpha(w) \le d_\alpha(w, w^m) + d_\alpha(w^m, v) + d_\alpha(v, T(w)).$$

Further, by (38), (43), (44), and (11), we get

$$\begin{aligned} d_\alpha(w, T(w)) = \omega_\alpha(w) &\le d_\alpha(w, w^m) + \inf_{v \in T(w^m)} \{d_\alpha(w^m, v) + d_\alpha(v, T(w))\} \\ &\le d_\alpha(w, w^m) + \inf_{v \in T(w^m)} d_\alpha(w^m, v) + \sup_{v \in T(w^m)} d_\alpha(v, T(w)) \\ &\le d_\alpha(w, w^m) + \omega_\alpha(w^m) + [\mathbb{H}^{\mathcal{D}}_{(i)}(T(w^m), T(w))]_\alpha \\ &\le d_\alpha(w, w^m) + \omega_\alpha(w^m) + \lambda_\alpha d_\alpha(w^m, w). \end{aligned}$$

Hence, by (41) and (44), $\forall_{\alpha \in \mathcal{A}} \{\omega_\alpha(w) = d_\alpha(w, T(w)) = 0\}$. However, this property of w, i.e.

$$d_\alpha(w, T(w)) = \inf\{d_\alpha(w, y) : y \in T(w)\} = 0,$$

and fact that $T(w)$ is closed, gives $w \in T(w)$. This and (41) yield that (F2) holds.

(**II**) Let $i \in \{1, 2\}$. Let $w^0 \in X$ and suppose that the assertion (G1) does not hold, i.e. suppose that

$$\exists_{(v^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}(X, T, v^0 = w^0)} \exists_{m_0 \in \mathbb{N}} \forall_{\alpha \in \mathcal{A}} \{d_\alpha(v^{m_0 - 1}, v^{m_0}) < \infty\}.$$

This implies that there exists the family $\Upsilon = (\varepsilon_\alpha : \alpha \in \mathcal{A}) \in K^{\mathcal{A}}$ such that $\forall_{\alpha \in \mathcal{A}} \{\varepsilon_\alpha \in (0, \infty)\}$ and $\forall_{\alpha \in \mathcal{A}} \{d_\alpha(v^{m_0 - 1}, v^{m_0}) < \varepsilon_\alpha < \infty\}$. Consequently,

$$\exists_{(v^m : m \in \{0\} \cup \mathbb{N}) \in \mathcal{O}(X, T, v^0 = w^0)} \exists_{m_0 \in \mathbb{N}} \forall_{\alpha \in \mathcal{A}} \{d_\alpha(v^{m_0 - 1}, v^{m_0}) < \varepsilon_\alpha\}.$$

Clearly, (X, T) is $(\mathbb{H}^{\mathcal{D}}_{(i)}, \Upsilon, \Lambda)$- uniformly locally contractive on X since (X, T) is $\mathbb{H}^{\mathcal{D}}_{(i)}$-contractive on X. Using now similar argumentation as in the proof of Theorem 8 (II), we obtain that (G2) holds.

The proof of Theorem 9 is complete. □

Generalized locally convex spaces $(X, \mathcal{P})$

We want to show an immediate consequence of the Section "Generalized uniform spaces $(X, \mathcal{D})$ and the class $\mathbb{L}_{(X,\mathcal{D})}$ of $\mathcal{L}$-families of generalized pseu-dodistances on $(X, \mathcal{D})$".

Definition 12 Let X be a vector space over $\mathbb{R}$.

(i) The family

$$\mathcal{P} = \{p_\alpha : X \to [0, +\infty], \alpha \in \mathcal{A}\}$$

is said to be a $\mathcal{P}$ *-family of generalized seminorms on* X ($\mathcal{P}$*-family*, for short) if the following three conditions hold:

($\mathcal{P}$1) $\forall_{a\in\mathcal{A}}\forall_{x\in X}\left\{0 \le p_\alpha(x) \wedge x = 0 \Rightarrow p_\alpha(x) = 0\right\}$;

($\mathcal{P}$2) $\forall_{\alpha\in\mathcal{A}}\forall_{\lambda\in\mathbb{R}}\forall_{x\in X}\left\{p_\alpha(\lambda x) = |\lambda|\, p_\alpha(x)\right\}$; and

($\mathcal{P}$3) If $\alpha \in \mathcal{A}$ and $x, y \in X$ and if $p_\alpha(x)$ and $p_\alpha(y)$ are finite, then $p_\alpha(x + y)$ is finite and $p_\alpha(x + y) \le p_\alpha(x) + p_\alpha(y)$.

(ii) If $\mathcal{P}$ is $\mathcal{P}$-family, then the pair $(X, \mathcal{P})$ is called a *generalized locally convex space.*

(iii) A $\mathcal{P}$-family $\mathcal{P}$ is said to be *separating* if

($\mathcal{P}$4) $\forall_{x\in X}\left\{x \ne 0 \Rightarrow \exists_{\alpha\in\mathcal{A}}\left\{0 < p_\alpha(x)\right\}\right\}$.

(iv) If a $\mathcal{P}$-family $\mathcal{P}$ is separating, then the pair $(X, \mathcal{P})$ is called a *Hausdorff generalized locally convex space.*

Remark 6 It is clear that each generalized locally convex space is an generalized uniform space. Indeed, if X is a vector space over $\mathbb{R}$ and $(X, \mathcal{P})$ is a generalized locally convex space, then $\mathcal{D} = \left\{d_\alpha : X \times X \to [0, +\infty], \alpha \in \mathcal{A}\right\}$ where $d_\alpha(x,y) = p_\alpha(x - y)$, $(x,y) \in X \times X$, $\alpha \in \mathcal{A}$, is $\mathcal{D}$-family and $(X, \mathcal{D})$ is a generalized uniform space.

Examples of the decompositions of the generalized uniform spaces

Example 3 For each $n \in \mathbb{N}$, let $Z_n = [2n - 2, 2n - 1]$ and let $q_n : Z_n \times Z_n \to [0, +\infty)$ where $q_n(x,y) = |x - y|$ for $x,y \in Z_n$. Let $Z = \bigcup_{n=1}^{\infty} Z_n$ and define $q : Z \times Z \to [0, +\infty]$ by the formula

$$q(x, y) = \begin{cases} q_n(x, y) & \text{if } x, y \in Z_n, n \in \mathbb{N} \\ +\infty & \text{if } x \in Z_n, y \in Z_m, n \ne m, \; n, m \in \mathbb{N} \end{cases}. \tag{46}$$

Then (Z, q) is a complete generalized metric space.

Example 4 Let $Y = \mathbb{R}^{\mathbb{N}} = \mathbb{R} \times \mathbb{R} \times \ldots$ be a non-normable real Hausdorff and sequentially complete locally convex space with the family $\mathcal{C} = \{c_n, n \in \mathbb{N}\}$ of calibrations $c_n, n \in \mathbb{N}$, defined as follows:

$$c_n(\mathbf{x}) = \left|[\mathbf{x}]_n\right| = |x_n|\,, \mathbf{x} = (x_1, x_2, x_3, \ldots) \in Y, \; n \in \mathbb{N}.$$

For each $s \in \mathbb{N}$, let $P_s = [2s - 2, 2s - 1]^{\mathbb{N}}$ be a Hausdorff sequentially complete uniform space with uniformity defined by the saturated family $\{p_{s,n} : n \in \mathbb{N}\}$ of pseudometrics $p_{s,n} : P_s \times P_s \to [0, +\infty)$, $n \in \mathbb{N}$, defined as follows:

$$p_{s,n}(\mathbf{x}, \mathbf{y}) = c_n(\mathbf{x} - \mathbf{y}), \; \mathbf{x}, \mathbf{y} \in P_s, \; n \in \mathbb{N}.$$

Let $P = \bigcup_{s=1}^{\infty} P_s$ and define $p_n : P \times P \to [0, +\infty]$, $n \in \mathbb{N}$, as follows

$$p_n(\mathbf{x}, \mathbf{y}) = \begin{cases} p_{s,n}(\mathbf{x}, \mathbf{y}) \text{ if } \mathbf{x}, \mathbf{y} \in P_s \\ +\infty \quad \begin{array}{l} \text{if } \mathbf{x} \in P_{s_1}, \mathbf{y} \in P_{s_2}, \\ s_1 \neq s_2, s_1, s_2 \in \mathbb{N} \end{array} \end{cases}, \mathbf{x}, \mathbf{y} \in P, n \in \mathbb{N}. \tag{47}$$

Then $(P, \{p_n : P \times P \to [0, +\infty], n \in \mathbb{N}\})$ is a Hausdorff $\{p_n : P \times P \to [0, +\infty], n \in \mathbb{N}\}$-sequentially complete generalized uniform space.

Examples of elements of the class $\mathbb{L}_{(X,\mathcal{D})}$

In this section we describe some elements of the class $\mathbb{L}_{(X,\mathcal{D})}$.

Example 5 Let $(X, \mathcal{D})$ be a Hausdorff generalized uniform space where , $\mathcal{A}$-index set, is a $\mathcal{D}$-family. Let the set $E \subset X$, containing at least two different points, be arbitrary and fixed and, for each $\alpha \in \mathcal{A}$, let $L_a : X \times X \to [0, +\infty]$ be defined by the formula:

$$L_\alpha(x, y) = \begin{cases} d_\alpha(x, y) \text{ if } E \cap \{x, y\} = \{x, y\} \\ +\infty \quad \text{ if } E \cap \{x, y\} \neq \{x, y\} \end{cases}, x, y \in X. \tag{48}$$

We show that the family $\mathcal{L} = \{L_\alpha : \alpha \in \mathcal{A}\}$ is $\mathcal{L}$-family on $(X, \mathcal{D})$.

First, we observe that the condition $(\mathcal{L}1)$ holds. Indeed, let $\alpha \in \mathcal{A}$ and $x, y, z \in X$ be arbitrary and fixed and such that $L_\alpha(x, z) < +\infty$ and $L_\alpha(z, y) < +\infty$. By (48), this implies that: $x, y, z \in E$; $d_\alpha(x, z) = L_\alpha(x, z) < +\infty$; and $d_\alpha(z, y) = L_\alpha(z,y) < +\infty$. Then, by $(\mathcal{D}3)$, we get that $d_\alpha(x,y) < +\infty$ and $d_\alpha(x, y) \leq d_\alpha(x,z) + d_a(z,y)$. Consequently, since $x,y,z \in E$, this mean that $L_\alpha(x,y) = d_a(x,y) < +\infty$ and $L_a(x,y) \leq L_a(x,z) + L_a(z,y)$. Therefore, the condition $(\mathcal{L}1)$ holds.

To prove that $(\mathcal{L}2)$ holds, we assume that the sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X satisfy (6) and (7). Then, in particular, (7) is of the form

$$\forall_{\alpha\in\mathcal{A}} \forall_{0<\varepsilon_\alpha<+\infty} \exists_{m_0=m_0(\varepsilon_\alpha,\alpha)\in\mathbb{N}} \forall_{m\geq m_0} \{L_\alpha(x_m, y_m) < \varepsilon_\alpha\}.$$

By definition of $\mathcal{L}$, this implies that

$$\forall_{\alpha\in\mathcal{A}} \forall_{0<\varepsilon_\alpha<+\infty} \exists_{m_0=m_0(\varepsilon_\alpha,\alpha)\in\mathbb{N}} \forall_{m\geq m_0} \{E \cap \{x_m, y_m\} = \{x_m, y_m\} \\ \wedge d_\alpha(x_m, y_m) < \varepsilon_\alpha < +\infty\}.$$

Therefore, we obtain that

$$\forall_{\alpha\in\mathcal{A}} \forall_{0<\varepsilon_\alpha<+\infty} \exists_{m_0=m_0(\varepsilon_\alpha,\alpha)\in\mathbb{N}} \forall_{m\geq m_0} \{d_\alpha(x_m, y_m) < \varepsilon_\alpha\}.$$

This means that the sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ satisfy (8). Hence we conclude that the condition $\mathcal{L}2$ is satisfied.

Example 6 Let $(X, \mathcal{D})$ be a generalized metric space where $\mathcal{D} = \{d : X \times X \to [0, +\infty]\}$ is a $\mathcal{D}$-family. Let the set $E \subset X$, containing at least two different points, be arbitrary and fixed and let $L : X \times X \to [0, +\infty]$ be defined by the formula (see (48)):

$$L(x, y) = \begin{cases} d(x, y) \text{ if } E \cap \{x, y\} = \{x, y\} \\ +\infty \quad \text{ if } E \cap \{x, y\} \neq \{x, y\} \end{cases}, x, y \in X. \tag{49}$$

By Example 5, the family $\mathcal{L} = \{L\}$ is $\mathcal{L}$-family on X.

Example 7 Let $(X, \mathcal{D})$ be a Hausdorff generalized uniform space where , $\mathcal{A}$-index set, is a $\mathcal{D}$-family. Let the sets E and F satisfying $E \subset F \subset X$ be arbitrary and fixed and such that E contains at least two different points and F contains at least three different points. Let $0 < a_\alpha < b_\alpha < c_\alpha < +\infty$, $\alpha \in \mathcal{A}$, and let, for each $\alpha \in \mathcal{A}$, $L_\alpha : X \times X \to [0, +\infty]$ be defined by the formula:

$$L_\alpha(x,y) = \begin{cases} d_\alpha(x,y) + a_\alpha & \text{if } \{x,y\} \cap E = \{x,y\} \\ d_\alpha(x,y) & \text{if } x \in E \wedge y \in F \backslash E \\ d_\alpha(x,y) + c_\alpha & \text{if } x \in F \backslash E \wedge y \in E \\ d_\alpha(x,y) + b_\alpha & \text{if } \{x,y\} \cap F \backslash E = \{x,y\} \\ +\infty & \text{if } \{x,y\} \cap F \neq \{x,y\} \end{cases}, x, y \in X. \tag{50}$$

We show that the family $\mathcal{L} = \{L_\alpha : \alpha \in \mathcal{A}\}$ is $\mathcal{L}$-family on X.

First, we observe that the condition $(\mathcal{L}1$ holds. Indeed, let $\alpha \in \mathcal{A}$ and $x, y, z \in X$ satisfying $L_\alpha(x, z) < +\infty$ and $L_\alpha(z, y) < +\infty$ be arbitrary and fixed. Clearly, by definition of L_α, this implies that $x, y, z \in F$. We consider the following cases:

Case 1. If $L_\alpha(x, y) = d_\alpha(x, y) + b_\alpha$, then by (50) we conclude that, $\{x, y\} \cap F \backslash E = \{x, y\}$. Now, if $z \in E$, then $L_\alpha(x, z) = d_\alpha(x, z) + c_a$; $L_\alpha(z, y) = d_\alpha(z, y)$; and consequently, since $b_\alpha < c_\alpha$, by $(\mathcal{D}3)$, we get

$$\begin{aligned} L_\alpha(x,y) &= d_\alpha(x,y) + b_\alpha \leq d_\alpha(x,z) + c_\alpha + d_\alpha(z,y) \\ &= L_\alpha(x,z) + L_\alpha(x,y). \end{aligned}$$

If $z \in F \setminus E$, then $L_\alpha(x, z) = d_\alpha(x, z) + b_\alpha$; $L_\alpha(z, y) = d_\alpha(z, y) + b_\alpha$; and consequently, by $(\mathcal{D}3)$, we get

$$\begin{aligned} L_\alpha(x,y) &= d_\alpha(x,y) + b_\alpha \leq d_\alpha(x,z) + b_\alpha + d_\alpha(z,y) + b_\alpha \\ &= L_\alpha(x,z) + L_\alpha(x,y). \end{aligned}$$

Case 2. If $L_\alpha(x, y) = d_\alpha(x, y) + c_\alpha$, then by (50) we conclude that, $x \in F \setminus E \wedge y \in E$. Now, if $z \in E$ then $L_\alpha(x, z) = d_\alpha(x, z) + c_\alpha$; $L_\alpha(z, y) = d_\alpha(z, y) + \alpha_\alpha$; and consequently, by $(\mathcal{D}3)$, we get

$$\begin{aligned} L_\alpha(x,y) &= d_\alpha(x,y) + c_\alpha \leq d_\alpha(x,z) + c_\alpha + d_\alpha(z,y) + a_\alpha \\ &= L_\alpha(x,z) + L_\alpha(z,y). \end{aligned}$$

If $z \in F \setminus E$, then $L_\alpha(x, z) = d_\alpha(x, z) + b_\alpha$; $L_\alpha(z, y) = d_\alpha(z, y) + c_\alpha$; and consequently, by $(\mathcal{D}3)$, we get

$$\begin{aligned} L_\alpha(x,y) &= d_\alpha(x,y) + c_\alpha \leq d_\alpha(x,z) + b_\alpha + d_\alpha(z,y) + c_\alpha \\ &= L_\alpha(x,z) + L_\alpha(z,y). \end{aligned}$$

Case 3. If $L_\alpha(x, y) = d_\alpha(x, y)$, then by (50) we conclude that, $x \in E \wedge y \in F \backslash E$. Now, if $z \in E$ then $L_\alpha(x, z) = d_\alpha(x, z) + a_\alpha$; $L_\alpha(z, y) = d_\alpha(z, y)$; and consequently, by $(\mathcal{D}3)$, we get

$$\begin{aligned} L_\alpha(x,y) &= d_\alpha(x,y) \leq d_\alpha(x,z) + a_\alpha + d_\alpha(z,y) \\ &= L_\alpha(x,z) + L_\alpha(z,y). \end{aligned}$$

If $z \in F \backslash E$, then $L_\alpha(x, z) = d_\alpha(x, z)$; $L_\alpha(z, y) = d_\alpha(z, y) + b_\alpha$; and consequently, by $(\mathcal{D}3)$, we get

$$L_\alpha(x,y) = d_\alpha(x,y) \le d_\alpha(x,z) + d_\alpha(z,y) + b_\alpha$$
$$= L_\alpha(x,z) + L_\alpha(z,y).$$

Case 4. If $L_\alpha(x, y) = d_\alpha(x, y) + a_\alpha$, then by (50) we conclude that, $x \in E \wedge y \in E$. Now, if $z \in E$ *then* $L_\alpha(x, z) = d_\alpha(x, z) + a_\alpha$; $L_\alpha(z, y) = d_\alpha(z, y) + a_\alpha$; and consequently, by $(\mathcal{D}3)$, we get

$$L_\alpha(x,y) = d_\alpha(x,y) + a_\alpha \le d_\alpha(x,z) + a_\alpha + d_\alpha(z,y) + a_\alpha$$
$$= L_\alpha(x,z) + L_\alpha(z,y).$$

If $z \in F \backslash E$, then $L_\alpha(x, z) = d_\alpha(x, z)$; $L_\alpha(z, y) = d_\alpha(z, y) + c_\alpha$; and consequently, since $a_\alpha < c_\alpha$, by $(\mathcal{D}3)$, we get

$$L_\alpha(x,y) = d_\alpha(x,y) + a_\alpha \le d_\alpha(x,z) + d_\alpha(z,y) + c_\alpha$$
$$= L_\alpha(x,z) + L_\alpha(z,y).$$

Consequently, the condition $\mathcal{L}1$ holds.

To prove that $\mathcal{L}2$ holds, we assume that the sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X satisfy (6) and (7). Then, in particular, (7) is of the form

$$\forall_{\alpha\in\mathcal{A}} \forall_{0<\varepsilon_\alpha<a_\alpha} \exists_{m_0=m_0(\varepsilon_\alpha,\alpha)\in\mathbb{N}} \forall_{m\ge m_0} \{L_\alpha(x_m, y_m) < \varepsilon_\alpha\}.$$

By definition of $\mathcal{L}$, this implies that

$$\forall_{\alpha\in\mathcal{A}} \forall_{0<\varepsilon_\alpha<a_\alpha} \exists_{m_0=m_0(\varepsilon_\alpha,\alpha)\in\mathbb{N}} \forall_{m\ge m_0} \{x_m \in E \wedge y_m \in F\backslash E$$
$$\wedge d_\alpha(x_m, y_m) < \varepsilon_\alpha < a_\alpha\}.$$

As a consequence of this, we get

$$\forall_{\alpha\in\mathcal{A}} \forall_{0<\varepsilon_\alpha<a_\alpha} \exists_{m_0=m_0(\varepsilon_\alpha,\alpha)\in\mathbb{N}} \forall_{m\ge m_0} \{d_\alpha(x_m, y_m)$$
$$= L_\alpha(x_m, y_m) < \varepsilon_\alpha\}.$$

This means that the sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ satisfy (8). Therefore, the property ($\mathcal{L}2$ holds.

It is worth noticing that, there exists $x, y \in X$ such that, for each $\alpha \in \mathcal{A}$, $L_\alpha(x, y) = L_\alpha(y, x)$ does not hold. Indeed, if $x \in E$ and $y \in F \setminus E$, then

$$\forall_{\alpha\in\mathcal{A}} \{d_\alpha(x,y) = L_\alpha(x,y) \ne L_\alpha(y,x) = d_\alpha(y,x) + c_\alpha\}.$$

Example 8 Let X, $\mathcal{D}$ be a generalized metric space where $\mathcal{D} = \{d : X \times X \to [0, +\infty]\}$ is a $\mathcal{D}$-family. Let the sets E and F satifying $E \subset F \subset X$ be arbitrary and fixed and such that E contains at least two different points and F contains at least three different points. Let $L : X \times X \to [0, +\infty]$ be defined by the formula:

$$L(x,y) = \begin{cases} d_\alpha(x,y) + 1 & \text{if } \{x,y\} \cap E = \{x,y\} \\ d_\alpha(x,y) & \text{if } x \in E \wedge y \in F\backslash E \\ d_\alpha(x,y) + 4 & \text{if } x \in F\backslash E \wedge y \in E \\ d_\alpha(x,y) + 3 & \text{if } \{x,y\} \cap F\backslash E = \{x,y\} \\ +\infty & \text{if } \{x,y\} \cap F \ne \{x,y\} \end{cases}, x, y \in X. \quad (51)$$

By Example 7, the family $\mathcal{L} = \{L\}$ is $\mathcal{L}$-family on X.

Example 9 Let $(X, \mathcal{D})$ be a Hausdorff generalized uniform space where , $\mathcal{A}$-index set, is a $\mathcal{D}$-family. Let the sets E and F satisfying $E \subset F \subset X$ be arbitrary and fixed and such that E contains at least two different points and F contains at least three different points. Let $0 < b_\alpha < c_\alpha < +\infty$, $\alpha \in \mathcal{A}$, and let, for each $\alpha \in \mathcal{A}$, $L_\alpha : X \times X \to [0, +\infty]$ be defined by the formula:

$$L_\alpha(x, y) = \begin{cases} d_\alpha(x, y) & \begin{array}{l}\text{if } \{x, y\} \cap E = \{x, y\} \\ \text{or } x \in E \wedge y \in F \backslash E\end{array} \\ d_\alpha(x, y) + c_\alpha & \text{if } x \in F\backslash E \wedge y \in E \\ d_\alpha(x, y) + b_\alpha & \text{if } \{x, y\} \cap F\backslash E = \{x, y\} \\ +\infty & \text{if } \{x, y\} \cap F \neq \{x, y\} \end{cases}, x, y \in X. \quad (52)$$

We show that the family $\mathcal{L} = \{L_\alpha : \alpha \in \mathcal{A}\}$ is $\mathcal{L}$-family on X.

First, we observe that the condition ($\mathcal{L}1$ holds. Indeed, let $\alpha \in \mathcal{A}$ and $x, y, z \in X$ satisfying $L_\alpha(x, z) < +\infty$ and $L_\alpha(z, y) < +\infty$ be arbitrary and fixed. Clearly, by definition of L_α, this implies that $x, y, z \in F$. We consider the following cases:

Case 1. If $L_\alpha(x, y) = d_\alpha(x, y) + b_\alpha$, then by (52) we conclude that, $\{x, y\} \cap F\backslash E = \{x, y\}$. Now, if $z \in E$, then

$$L_\alpha(x, z) = d_\alpha(x, z) + c_\alpha; L_\alpha(z, y) = d_\alpha(z, y);$$

and consequently, since $b_\alpha < c_\alpha$, by ($\mathcal{D}3$), we get

$$\begin{aligned} L_\alpha(x, y) &= d_\alpha(x, y) + b_\alpha \leq d_\alpha(x, z) + c_\alpha + d_\alpha(z, y) \\ &= L_\alpha(x, z) + L_\alpha(z, y). \end{aligned}$$

If $z \in F \setminus E$, then

$$L_\alpha(x, z) = d_\alpha(x, z) + b_\alpha; L_\alpha(z, y) = d_\alpha(z, y) + b_\alpha;$$

and consequently, by ($\mathcal{D}3$), we get

$$\begin{aligned} L_\alpha(x, y) &= d_\alpha(x, y) + b_\alpha \leq d_\alpha(x, z) + b_\alpha + d_\alpha(z, y) + b_\alpha \\ &= L_\alpha(x, z) + L_\alpha(z, y). \end{aligned}$$

Case 2. If $L_\alpha(x, y) = d_\alpha(x, y) + c_\alpha$, then by (52) we conclude that, $x \in F\backslash E \wedge y \in E$. Now, if $z \in E$ then

$$L_\alpha(x, z) = d_\alpha(x, z) + c_\alpha; L_\alpha(z, y) = d_\alpha(z, y);$$

and consequently, by ($\mathcal{D}3$), we get

$$\begin{aligned} L_\alpha(x, y) &= d_\alpha(x, y) + c_\alpha \leq d_\alpha(x, z) + c_\alpha + d_\alpha(z, y) \\ &= L_\alpha(x, z) + L_\alpha(z, y). \end{aligned}$$

If $z \in F \setminus E$, then

$$L_\alpha(x, z) = d_\alpha(x, z) + b_\alpha; L_\alpha(z, y) = d_\alpha(z, y) + c_\alpha;$$

and consequently, by ($\mathcal{D}3$), we get

$$\begin{aligned} L_\alpha(x, y) &= d_\alpha(x, y) + c_\alpha \leq d_\alpha(x, z) + b_\alpha + d_\alpha(z, y) + c_\alpha \\ &= L_\alpha(x, z) + L_\alpha(z, y). \end{aligned}$$

Case 3. If $L_\alpha(x, y) = d_\alpha(x, y)$, then by (52) we conclude that, $x \in E \wedge y \in E$ or $x \in E \wedge y \in F\backslash E$. First, assume that $x \in E \wedge y \in E$. Now, if $z \in E$ then

$$L_\alpha(x, z) = d_\alpha(x, z); L_\alpha(z, y) = d_\alpha(z, y);$$

and consequently, by $(\mathcal{D}3)$, we get

$$\begin{aligned} L_\alpha(x,y) &= d_\alpha(x,y) \le d_\alpha(x,z) + d_\alpha(z,y) \\ &= L_\alpha(x,z) + L_\alpha(z,y). \end{aligned}$$

If $z \in F \setminus E$, then

$$L_\alpha(x,z) = d_\alpha(x,z); L_\alpha(z,y) = d_\alpha(z,y) + c_\alpha;$$

and consequently, by $(\mathcal{D}3)$, we get

$$\begin{aligned} L_\alpha(x,y) &= d_\alpha(x,y) \le d_\alpha(x,z) + d_\alpha(z,y) + c_\alpha \\ &= L_\alpha(x,z) + L_\alpha(z,y). \end{aligned}$$

Next, we assume that $x \in E \wedge y \in F \backslash E$. Now, if $z \in E$ then $L_\alpha(x, z) = d_\alpha(x, z)$; $L_\alpha(z, y) = d_\alpha(z, y)$; and consequently, by $(\mathcal{D}3)$, we get

$$\begin{aligned} L_\alpha(x,y) &= d_\alpha(x,y) \le d_\alpha(x,z) + d_\alpha(z,y) \\ &= L_\alpha(x,z) + L_\alpha(z,y). \end{aligned}$$

If $z \in F \setminus E$, then

$$L_\alpha(x,z) = d_\alpha(x,z); L_\alpha(z,y) = d_\alpha(z,y) + b_\alpha;$$

and consequently, by $(\mathcal{D}3)$, we get

$$\begin{aligned} L_\alpha(x,y) &= d_\alpha(x,y) \le d_\alpha(x,z) + d_\alpha(z,y) + b_\alpha \\ &= L_\alpha(x,z) + L_\alpha(z,y). \end{aligned}$$

Consequently, the condition ($\mathcal{L}1$ holds.

To prove that ($\mathcal{L}2$ holds, we assume that the sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X satisfy (6) and (7). Then, in particular, (7) is of the form

$$\forall_{\alpha\in\mathcal{A}} \forall_{0<\varepsilon_\alpha<a_\alpha} \exists_{m_0=m_0(\varepsilon_\alpha,\alpha)\in\mathbb{N}} \forall_{m\ge m_0} \{L_\alpha(x_m, y_m) < \varepsilon_\alpha\}.$$

By definition of $\mathcal{L}$, this implies that

$$\begin{aligned} \forall_{\alpha\in\mathcal{A}} \forall_{0<\varepsilon_\alpha<a_\alpha} \exists_{m_0=m_0(\varepsilon_\alpha,\alpha)\in\mathbb{N}} \forall_{m\ge m_0} \{[(x_m \in E \wedge y_m \in F\backslash E) \\ \vee (x_m, y_m \in E)] \wedge d_\alpha(x_m, y_m) < \varepsilon_\alpha < a_\alpha\}. \end{aligned}$$

As a consequence of this, we get

$$\begin{aligned} \forall_{\alpha\in\mathcal{A}} \forall_{0<\varepsilon_\alpha<a_\alpha} \exists_{m_0=m_0(\varepsilon_\alpha,\alpha)\in\mathbb{N}} \forall_{m\ge m_0} \{d_\alpha(x_m, y_m) \\ = L_\alpha(x_m, y_m) < \varepsilon_\alpha\}. \end{aligned}$$

This means that the sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ satisfy (8). Therefore, the property ($\mathcal{L}2$ holds.

It is worth noticing that, there exists $x, y \in X$ such that, for each $\alpha \in \mathcal{A}$, $L_\alpha(x, y) = L_\alpha(y, x)$ does not hold. Indeed, if $x \in E$ and $y \in F \setminus E$, then

$$\forall_{\alpha\in\mathcal{A}} \{d_\alpha(x,y) = L_\alpha(x,y) \ne L_\alpha(y,x) = d_\alpha(y,x) + c_\alpha\}.$$

Example 10 Let $(X, \mathcal{D})$ be a generalized metric space where $\mathcal{D} = \{d : X \times X \to [0, +\infty]\}$ is a $\mathcal{D}$-family. Let the sets E and F satisfying $E \subset F \subset X$ be arbitrary and fixed, and such that E contains at least two different points and F contains at least three different points. Let $L : X \times X \to [0, +\infty]$ be defined by the formula:

$$L(x,y)=\begin{cases} d(x,y) & \text{if } \{x,y\}\cap E=\{x,y\} \\ & \text{or } x\in E\wedge y\in F\backslash E \\ d(x,y)+4 & \text{if } x\in F\backslash E\wedge y\in E \\ d(x,y)+3 & \text{if } \{x,y\}\cap F\backslash E=\{x,y\} \\ +\infty & \text{if } \{x,y\}\cap F\neq\{x,y\}\end{cases},\ x,y\in X. \tag{53}$$

By Example 9, the family $\mathcal{L}=\{L\}$ is $\mathcal{L}$-family on X.

Examples which illustrate our theorems

The following example illustrates the Theorem 8(I) in the case when $(X, \mathcal{D})$ is $\mathcal{D}$-sequentially complete and (X, T) is $(\mathbb{H}^{\mathcal{L}}_{(2)}, \overline{\mathbf{1/2}}, \overline{\mathbf{1/7}})$-uniformly locally contractive on X where $\mathcal{L}\neq\mathcal{D}$ and $\mathcal{L}\in\mathbb{L}_{(X,\mathcal{D})}$.

Example 11 Let P and $\{p_n : P\times P\to[0,+\infty],\ n\in\mathbb{N}\}$ be as in Example 4. Let $X = P\cap[0,9]^{\mathbb{N}}$ and let $\mathcal{D}=\{d_n : n\in\mathbb{N}\}$, $d_n : X\times X\to[0,+\infty]$, $n\in\mathbb{N}$, where, for each $n\in\mathbb{N}$, we define $d_n = p_n|_{[0,9]^{\mathbb{N}}}$. Then $(X, \mathcal{D})$ is a Hausdorff $\mathcal{D}$-sequentially complete generalized uniform space. This gives that the property (P2) of Theorem 8 holds.

The elements of $\mathbb{R}^{\mathbb{N}}$ we denote by $\mathbf{x} = (x_1,x_2,\ldots)$. In particular, the element $(x,x,\ldots)\in\mathbb{R}^{\mathbb{N}}$ we denote by $\bar{\mathbf{x}}$.

Let $F=\{\bar{\mathbf{1}},\bar{\mathbf{7}}\}\subset X$ and let a set-valued dynamic system (X, T) be given by the formula

$$T(x)=\begin{cases}\{\bar{\mathbf{1}},\bar{\mathbf{2}}\} & \text{if } \mathrm{x}\in X\backslash F\\ \{\bar{\mathbf{4}},\bar{\mathbf{5}}\} & \text{if } \mathrm{x}\in F.\end{cases} \tag{54}$$

Let $E=\{\bar{\mathbf{0}},\bar{\mathbf{1}},\bar{\mathbf{2}}\}\cup[\mathbf{4},\mathbf{5}]^{\mathbb{N}}\cup\{\bar{\mathbf{6}},\bar{\mathbf{8}}\}$ and let $\mathcal{L}$ be a family of the maps given by the formula:

$$L_n(\mathbf{x},\mathbf{y})=\begin{cases} d_n(\mathbf{x},\mathbf{y}) & \text{if}\{\mathbf{x},\mathbf{y}\}\cap E=\{\mathbf{x},\mathbf{y}\}\\ +\infty & \text{if}\{\mathbf{x},\mathbf{y}\}\cap E\neq\{\mathbf{x},\mathbf{y}\}\end{cases},\ \mathbf{x},\mathbf{y}\in X,\ n\in\mathbb{N}. \tag{55}$$

By Example 4, the family $\mathcal{L}=\{L_n\}$ is $\mathcal{L}$-family on X.

Now, we show that, for $\varepsilon=\overline{\mathbf{1/2}}$ and $\lambda=\overline{\mathbf{1/7}}$, (X, T) is $(\mathbb{H}^{\mathcal{L}}_{(2)},\varepsilon,\lambda)$-uniformly locally contractive on X, i.e. that

$$\forall_{n\in\mathbb{N}}\forall_{\mathbf{x},\mathbf{y}\in X}\{(L_n(\mathbf{x},\mathbf{y})<1/2)\Rightarrow[\mathbb{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}),T(\mathbf{y}))]_n\leq(1/7)L_n(\mathbf{x},\mathbf{y})\}, \tag{56}$$

where

$$\begin{aligned}&\mathbb{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}),T(\mathbf{y}))=\begin{cases}\mathbb{I} & \text{if } \mathbb{I} \text{exists and } \forall_{n\in\mathbb{N}}\{[\mathbb{I}]_n<+\infty\}\\ \Theta_{+\infty} & \text{otherwise}\end{cases},\\ &\mathbb{I}=\mathbb{INF}(\mathcal{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}),T(\mathbf{y}))),\end{aligned} \tag{57}$$

$$\begin{aligned}\mathcal{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}),T(\mathbf{y}))=\{\Theta\in K^{\mathbb{N}}_{+\infty} : {}& T(\mathbf{x})\subset U_{\mathcal{L}}(\Theta,T(\mathbf{y}))\\ &\wedge T(\mathbf{y})\subset U_{\mathcal{L}}(\Theta,T(\mathbf{x}))\},\end{aligned} \tag{58}$$

$$\begin{aligned}U_{\mathcal{L}}(\Theta,T(\mathbf{y}))&=\{\mathbf{u}\in X : \exists_{\mathbf{z}\in T(\mathbf{y})}\forall_{n\in\mathbb{N}}\{L_n(\mathbf{u},\mathbf{z})<\eta_n\}\},\\ U_{\mathcal{L}}(\Theta,T(\mathbf{x}))&=\{\mathbf{u}\in X : \exists_{\mathbf{z}\in T(\mathbf{x})}\forall_{n\in\mathbb{N}}\{L_n(\mathbf{u},\mathbf{z})<\eta_n\}\}.\end{aligned} \tag{59}$$

Indeed, let $\mathbf{x},\mathbf{y}\in X$ be arbitrary and fixed. Since, by (55), this family $\mathcal{L}$ is symmetric on X, we may consider only the following four cases:

Case 1. *Let* $\mathbf{x} \in F$ *and let* $\mathbf{y} \in X \backslash F$.

If $\mathbf{x} = \bar{\mathbf{1}}$, then, since $\bar{\mathbf{1}} \in E$, by (55), for each $n \in \mathbb{N}$, we have

$$L_n(\mathbf{x}, \mathbf{y}) = \begin{cases} d_n(\bar{\mathbf{1}}, \mathbf{y}) & \text{if } \mathbf{y} \in E \\ +\infty & \text{if } \mathbf{y} \notin E. \end{cases}$$

By (47), from this, for each $n \in \mathbb{N}$, we get

$$L_n(\mathbf{x}, \mathbf{y}) = \begin{cases} d_{1,n}(\bar{\mathbf{1}}, \bar{\mathbf{0}}) = c_n(\bar{\mathbf{1}} - \bar{\mathbf{0}}) = |1 - 0| = 1 & \text{if } \mathbf{y} \in E \text{ and } \mathbf{y} = \bar{\mathbf{0}} \\ +\infty & \text{if } \mathbf{y} \in E \text{ and } \mathbf{y} \neq \bar{\mathbf{0}} \\ +\infty & \text{if } \mathbf{y} \notin E \end{cases}.$$

If $\mathbf{x} = \bar{\mathbf{7}}$, then, since $\bar{\mathbf{7}} \notin E$, by (55), we obtain that $\forall_{n\in\mathbb{N}}\{L_n(\mathbf{x}, \mathbf{y}) = L_n(\bar{\mathbf{7}}, \mathbf{y}) = +\infty\}$ for each $\mathbf{y} \in X \setminus F$. Consequently, for each $n \in \mathbb{N}$, $\mathbf{x} \in F$ and $\mathbf{y} \in X \backslash F$, inequality $L_n(\mathbf{x}, \mathbf{y}) < 1/2$ in (56) does not hold and this case we do not have to consider this case.

Case 2. *Let* $\mathbf{x}, \mathbf{y} \in F$ *be such that* $\mathbf{x} \neq \mathbf{y}$ *or* $\mathbf{x} = \mathbf{y} = \bar{\mathbf{7}}$. Then, by definition of F, $\mathbf{x} = \bar{\mathbf{7}}$ or $\mathbf{y} = \bar{\mathbf{7}}$. But, $\bar{\mathbf{7}} \notin E$, therefore, by (55), we get $\forall_{n\in\mathbb{N}}\{L_n(\mathbf{x},\mathbf{y}) = +\infty\}$. Therefore, by (56), this case we can also be omitted.

Case 3. *Let* $\mathbf{x}, \mathbf{y} \in F$ *be such that* $\mathbf{x} = \mathbf{y} = \bar{\mathbf{1}}$. Then, since $\bar{\mathbf{1}} \in E$, by (55) and (47), we get

$$\forall_{n\in\mathbb{N}}\{L_n(\mathbf{x}, \mathbf{y}) = d_n(\bar{\mathbf{1}}, \bar{\mathbf{1}}) = 0\} \tag{60}$$

and, consequently, for each $n \in \mathbb{N}$, the inequality $L_n(\mathbf{x}, \mathbf{y}) < 1/2$ holds. In virtue ofthis, we show that the inequalities $\forall_{n\in\mathbb{N}}\{[\mathbb{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}), T(\mathbf{y}))]_n \leq (1/7)L_n(\mathbf{x}, \mathbf{y})\}$ in (56) hold. With this aim, we see that:

($\mathbf{3_i}$) By (54), we have $T(\mathbf{x}) = T(\mathbf{y}) = T(\bar{\mathbf{1}}) = \{\bar{\mathbf{4}}, \bar{\mathbf{5}}\} \subset E$;

($\mathbf{3_{ii}}$) Next, if $\Theta = (\eta_n : n \in \mathbb{N}) \in K^{\mathbb{N}}_{+\infty}$, then, by ($3_i$),

$$\begin{aligned} &U_{\mathcal{L}}(\Theta, T(\mathbf{x})) \\ &\quad = \{\mathbf{u} \in X : \exists_{\mathbf{z}\in T(\mathbf{x})=T(\mathbf{y})}\forall_{n\in\mathbb{N}}\{L_n(\mathbf{u}, \mathbf{z}) < \eta_n\}\} \\ &\quad = \{\mathbf{u} \in X : \exists_{\mathbf{z}\in\{\bar{4},\,\bar{5}\}}\forall_{n\in\mathbb{N}}\{L_n(\mathbf{u}, \mathbf{z}) < \eta_n\}\} \\ &\quad = \{\mathbf{u} \in X : \{\forall_{n\in\mathbb{N}}\{L_n(\mathbf{u},\ \bar{\mathbf{4}}) < \eta_n\} \vee \forall_{n\in\mathbb{N}}\{L_n(\mathbf{u},\ \bar{\mathbf{5}}) < \eta_n\}\}\}; \end{aligned}$$

($\mathbf{3_{iii}}$) Now, by ($3i$), ($3ii$), (58), and (59), we get

$$\begin{aligned} &\mathcal{H}^{(2)}_{\mathcal{L}}(T(\mathbf{x}), T(\mathbf{y})) \\ &\quad = \{\Theta \in K^{\mathbb{N}}_{+\infty} : T(\mathbf{x}) \subset U_{\mathcal{L}}(\Theta, T(\mathbf{y})) \wedge T(\mathbf{y}) \subset U_{\mathcal{L}}(\Theta, T(\mathbf{x}))\} \\ &\quad = \{\Theta \in K^{\mathbb{N}}_{+\infty} : \{\bar{\mathbf{4}}, \bar{\mathbf{5}}\} \subset U_{\mathcal{L}}(\Theta, \{\bar{\mathbf{4}}, \bar{\mathbf{5}}\}) \wedge \{\bar{\mathbf{4}}, \bar{\mathbf{5}}\} \subset U_{\mathcal{L}}(\Theta, \{\bar{\mathbf{4}}, \bar{\mathbf{5}}\})\} \\ &\quad = \{\Theta \in K^{\mathbb{N}}_{+\infty} : [\bar{\mathbf{4}} \in U_{\mathcal{L}}(\Theta, \{\bar{\mathbf{4}}, \bar{\mathbf{5}}\}) \wedge \bar{\mathbf{5}} \in U_{\mathcal{L}}(\Theta, \{\bar{\mathbf{4}}, \bar{\mathbf{5}}\})]\} \\ &\quad = \{\Theta \in K^{\mathbb{N}}_{+\infty} : [\forall_{n\in\mathbb{N}}\{L_n(\bar{\mathbf{4}}, \bar{\mathbf{4}}) = 0 < \eta_n\} \vee \forall_{n\in\mathbb{N}}\{L_n(\bar{\mathbf{4}}, \bar{\mathbf{5}}) < \eta_n\}] \\ &\qquad\qquad \wedge [\forall_{n\in\mathbb{N}}\{L_n(\bar{\mathbf{5}}, \bar{\mathbf{4}}) < \eta_n\} \vee \forall_{n\in\mathbb{N}}\{L_n(\bar{\mathbf{5}}, \bar{\mathbf{5}}) = 0 < \eta_n]\}; \end{aligned}$$

($\mathbf{3_{iv}}$) Therefore, by (3_{iii}), we have $\mathbb{INF}(\mathcal{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}), T(\mathbf{y}))) = \Theta_0$;

($\mathbf{3_v}$) The consequence of (57) and (3_{iv}) is

$$\mathbb{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}), T(\mathbf{y})) = \mathbb{INF}(\mathcal{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}), T(\mathbf{y}))) = \Theta_0.$$

Hence, by (60), we conclude that

$$\forall_{n\in\mathbb{N}}\{[\mathbb{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}),T(\mathbf{y}))]_n=0\leq(1/7)L_n(x,y)\}$$

holds.

Case 4. *Let* $\mathbf{x}$, $\mathbf{y}\notin F$. Then we see that:

($4_{\mathbf{i}}$) By (54), we have $T(\mathbf{x})=T(\mathbf{y})=\{\bar{\mathbf{1}},\bar{\mathbf{2}}\}\subset E$;

($4_{\mathbf{ii}}$) Next, if $\Theta=(\eta_n:n\in\mathbb{N})\in K^{\mathbb{N}}_{+\infty}$, then

$$\begin{aligned}U_{\mathcal{L}}(\Theta,T(\mathbf{x}))&=\{\mathbf{u}\in X:\exists_{\mathbf{z}\in T(\mathbf{x})=T(\mathbf{y})}\forall_{n\in\mathbb{N}}\{L_n(\mathbf{u},\mathbf{z})<\eta_n\}\}\\&=\{\mathbf{u}\in X:\exists_{\mathbf{z}\in\{\bar{\mathbf{1}},\bar{\mathbf{2}}\}}\forall_{n\in\mathbb{N}}\{L_n(\mathbf{u},\mathbf{z})<\eta_n\}\}\}\\&=\{\mathbf{u}\in X:\forall_{n\in\mathbb{N}}\{L_n(\mathbf{u},\bar{\mathbf{1}})<\eta_n\}\vee\forall_{n\in\mathbb{N}}\{L_n(\mathbf{u},\bar{\mathbf{2}})<\eta_n\}\};\end{aligned}$$

($4_{\mathbf{iii}}$) Now, by (4_i) and (4_{ii}), we get

$$\begin{aligned}&\mathcal{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}),T(\mathbf{y}))\\&=\{\Theta\in K^{\mathbb{N}}_{+\infty}:T(\mathbf{x})\subset U_{\mathcal{L}}(\Theta,T(\mathbf{y}))\wedge T(\mathbf{y})\subset U_{\mathcal{L}}(\Theta,T(\mathbf{x}))\}\\&=\{\Theta\in K^{\mathbb{N}}_{+\infty}:\{\bar{\mathbf{1}},\bar{\mathbf{2}}\}\subset U_{\mathcal{L}}(\Theta,\{\bar{\mathbf{1}},\bar{\mathbf{2}}\})\wedge\{\bar{\mathbf{1}},\bar{\mathbf{2}}\}\subset U_{\mathcal{L}}(\Theta,\{\bar{\mathbf{1}},\bar{\mathbf{2}}\})\}\\&=\{\Theta\in K^{\mathbb{N}}_{+\infty}:[\bar{\mathbf{1}}\in U_{\mathcal{L}}(\Theta,\{\bar{\mathbf{1}},\bar{\mathbf{2}}\})\wedge\bar{\mathbf{2}}\in U_{\mathcal{L}}(\Theta,\{\bar{\mathbf{1}},\bar{\mathbf{2}}\})]\}\\&=\{\Theta\in K^{\mathbb{N}}_{+\infty}:[\forall_{n\in\mathbb{N}}\{L_n(\bar{\mathbf{1}},\bar{\mathbf{1}})=0<\eta_n\}\vee\forall_{n\in\mathbb{N}}\{L_n(\bar{\mathbf{1}},\bar{\mathbf{2}})<\eta_n]\\&\qquad\wedge[\forall_{n\in\mathbb{N}}\{L_n(\bar{\mathbf{2}},\bar{\mathbf{1}})<\eta_n\}\vee\forall_{n\in\mathbb{N}}\{L_n(\bar{\mathbf{2}},\bar{\mathbf{2}})=0<\eta_n\}]\};\end{aligned}$$

($4_{\mathbf{iv}}$) Therefore, by (4_{iii}), $\mathbb{INF}(\mathcal{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}),T(\mathbf{y})))=\Theta_0$;

($4_{\mathbf{v}}$) According to (57) and (4_{iv}), we have

$$\mathbb{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}),T(\mathbf{y}))=\mathbb{INF}(\mathcal{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}),T(\mathbf{y})))=\Theta_0.$$

Consequently, by (60),

$$\forall_{n\in\mathbb{N}}\{[\mathbb{H}^{\mathcal{L}}_{(2)}(T(\mathbf{x}),T(\mathbf{y}))]_n=0\leq(1/7)L_n(\mathbf{x},\mathbf{y})\}.$$

We proved that (X, T) is ($\mathbb{H}^{\mathcal{L}}_{(2)}$, $\overline{\mathbf{1/2}}$, $\overline{\mathbf{1/7}}$)-uniformly locally contractive on X. We see also that (C) holds.

Finally, we see that $\forall_{m\geq3}\{T^{[m]}(X)\subset\{\bar{\mathbf{1}},\bar{\mathbf{2}}\}\}$. Hence, for each $w^0\in X$, there exists a dynamic process $(w^m:m\in\{0\}\cup\mathbb{N})$ such that: (i) $\forall_{m\geq3}\{w^m=\bar{\mathbf{2}}\}$; (ii) $\lim_{m\to\infty}w^m=\bar{\mathbf{2}}$; and (iii) $\bar{\mathbf{2}}\in Fix(T)$.

The following example illustrates the Theorem 8(I) in the case when (X, $\mathcal{D}$) is $\mathcal{L}$-sequentially complete for some $\mathcal{L}\in\mathbb{L}_{(X,\mathcal{D})}$, $\mathcal{L}\neq\mathcal{D}$, but not $\mathcal{D}$-sequentially complete and (X, T) is ($\mathbb{H}^{\mathcal{L}}_{(2)}$, $\overline{\mathbf{1/2}}$, $\overline{\mathbf{1/7}}$)-uniformly locally contractive on X.

Example 12 Let X and $\{p_n:P\times P\to[0,+\infty],\ n\in\mathbb{N}\}$ be as in Example 4. Let $X=(P\cap[0,9]^{\mathbb{N}})\backslash\{\bar{\mathbf{3}},\bar{\mathbf{8}}\}$ and let $\mathcal{D}=\{d_k:k\in\mathbb{N}\}$, $d_k:X\times X\to[0,\infty]$, $k\in\mathbb{N}$, where, for each $k\in\mathbb{N}$, we define $d_k=p_k|_{[0,9]^{\mathbb{N}}}$. Then ($X$, $\mathcal{D}$) is a Hausdorff generalized uniform space.

We observe that (X, $\mathcal{D}$) is not a $\mathcal{D}$-sequentially complete space. Indeed, we consider the sequence $(x_m:m\in\mathbb{N})$ defined as follows: $x_m=\bar{\mathbf{8}}+\overline{\mathbf{1/m}}=(8+1/m,8+1/m,\ldots)$, $m\in\mathbb{N}$, $m\in\mathbb{N}$. Of course, the sequence $(x_m:m\in\mathbb{N})$ is $\mathcal{D}$-Cauchy sequence on X. Indeed, we have $\forall_{m\in\mathbb{N}}\left\{x_m\in(8,9]^{\mathbb{N}}\subset P_5\right\}$ which implies that

$$
\begin{aligned}
&\forall_{n\in\mathbb{N}}\forall_{m,n\in\mathbb{N}}\{d_k(x_m,x_n)\\
&= p_k(x_m,x_n) = p_{5,k}(x_m,x_n)\\
&= c_k(x_m-x_n) = \left|[x_m-x_n]_k\right|\\
&= \left|[(8+1/m,8+1/m,\ldots)-(8+1/n,8+1/n,\ldots)]_k\right|\\
&= [((8+1/m)-(8+1/n),(8+1/m)-(8+1/n),\ldots)]_k\\
&= \left|1/m-1/n\right|\}.
\end{aligned}
$$

Consequently,

$$\forall_{k\in\mathbb{N}}\{\lim_{n\to\infty}\sup_{m>n} d_k(x_m,x_n) = \lim_{n\to\infty}\sup_{m>n}\left|1/m-1/n\right| = 0\}.$$

However, there does not exist $x\in X$ such that $\lim_{m\to\infty} x_m = x$. Therefore, X is not $\mathcal{D}$-sequentially complete.

Let $E = \{\bar{\mathbf{0}},\bar{\mathbf{1}},\bar{\mathbf{2}}\}\cup[4,5]^{\mathbb{N}}\cup\{\bar{\mathbf{6}}\}$ and let $\mathcal{L} = \left\{L_k : X\times X\to[0,+\infty], k\in\mathbb{N}\right\}$ be a family of the maps given by the formula:

$$L_k(\mathbf{x},\mathbf{y}) = \begin{cases} d_k(\mathbf{x},\mathbf{y}) & \text{if}\{\mathbf{x},\mathbf{y}\}\cap E = \{\mathbf{x},\mathbf{y}\}\\ +\infty & \text{if}\{\mathbf{x},\mathbf{y}\}\cap E \neq \{\mathbf{x},\mathbf{y}\}\end{cases}, \mathbf{x},\mathbf{y}\in X, k\in\mathbb{N}\ ,$$

By (47), this gives

$$L_k(\mathbf{x},\mathbf{y}) = \begin{cases} d_{s,k}(\mathbf{x},\mathbf{y}) & \text{if}\{\mathbf{x},\mathbf{y}\}\cap E\cap P_s = \{\mathbf{x},\mathbf{y}\}, s\in N\\ +\infty & \begin{array}{l}\text{if } \mathbf{x}\in E\cap P_{s_1}, \mathbf{y}\in E\cap P_{s_2}\\ \text{and } s_1\neq s_2, s_1,s_2\in N\end{array}\\ +\infty & \text{if}\{\ \mathbf{x},\mathbf{y}\}\ \cap E\neq\{\mathbf{x},\mathbf{y}\}\end{cases}, \tag{61}$$

where $N = \{0,1, 2, 3, 4, 5\}$, $\mathbf{x}$, $\mathbf{y}\in X$ and $k\in\mathbb{N}$.

By Example 4, the familly $\mathcal{L} = \{L_k : k\in\mathbb{N}\}$ is $\mathcal{L}$-family on X.

We show that X is $\mathcal{L}$-sequentially complete space. Indeed, let $(x_m : m\in\mathbb{N})$ be arbitrary and fixed $\mathcal{L}$-Cauchy sequence in X, i.e.

$$\forall_{k\in\mathbb{N}}\{\lim_{n\to\infty}\sup_{m>n} L_k(x_n,x_m) = 0\}.$$

This implies that

$$\forall_{k\in\mathbb{N}}\forall_{\varepsilon>0}\exists_{n_0(k,\varepsilon)}\forall_{m>n>n_0}\{L_k(x_n,x_m) < \varepsilon\}. \tag{62}$$

Hence, in particular, we conclude that

$$\forall_{k\in\mathbb{N}}\exists_{n_0(k)}\forall_{m>n>n_0}\{L_k(x_n,x_m) < 1\}. \tag{63}$$

Now, (63) and (61) gives that

$$\forall_{k\in\mathbb{N}}\exists_{n_0(k)}\exists_{s_0\in N}\forall_{m>n_0}\{x_m\in E\cap P_{s_0}\}.$$

Of course, since $(x_m : m\in\mathbb{N})$ is arbitrary nad fixed, then there exists a unique $s_0\in N$ for all $k\in\mathbb{N}$. Now, putting $l_0 = \min_{k\in\mathbb{N}}\{n_0(k)\}$ we obtain that

$$\forall_{m>l_0}\{x_m\in E\cap P_{s_0}\}. \tag{64}$$

The property (61) and (64) gives that

$$\exists_{l_0\in\mathbb{N}}\forall_{k\in\mathbb{N}}\forall_{m>n>l_0}\{L_k(x_n,x_m) = d_k(x_n,x_m) = p_{s_0,k}(x_n,x_m) < 1\}. \tag{65}$$

Using (65), (64) and definition of E, we may consider only the following two cases:

Case 1. If $\forall_{m>l_0}\{x_m = \bar{0}\}$ or $\forall_{m>l_0}\{x_m = \bar{1}\}$ or $\forall_{m>l_0}\{x_m = \bar{2}\}$ or $\forall_{m>l_0}\{x_m = \bar{6}\}$, then in each of these situations the sequence, as a constant sequence, is, by (61), $\mathcal{L}$-convergent to $\bar{0}, \bar{1}, \bar{2}, \bar{6}$, respectively.

Case 2. If $\forall_{m>l_0}\left\{x_m \in [4,5]^{\mathbb{N}} = P_3\right\}$, then

$$\forall_{k\in\mathbb{N}}\forall_{m>n>l_0}\{L_k(x_n, x_m) = p_{3,k}(x_n, x_m)\},$$

so by (65) and (62), we obtain

$$\forall_{k\in\mathbb{N}}\forall_{\varepsilon>0}\exists_{n_1=\max\{n_0(k,\varepsilon),l_0\}}\forall_{m>n>n_1}\{p_{3,k}(x_n, x_m) = L_k(x_n, x_m) < \varepsilon\}.$$

This gives that $(x_m : m \in \mathbb{N})$ is a $\mathcal{D}$-Cauchy sequence in X, so also the sequence $(y_n = x_{l_0+(n-1)} : n \in \mathbb{N})$ is a $\mathcal{D}$-Cauchy sequence in $[4,5]^{\mathbb{N}}$. Since $[4,5]^{\mathbb{N}}$ is a $\mathcal{D}$-complete uniform space, so there exists $x \in X$ such that

$$\forall_{k\in\mathbb{N}} \lim_{m\to\infty} L_k(x_m, x) = \lim_{m\to\infty} p_{3,k}(x_m, x) = 0\},$$

i.e $(x_m : m \in \mathbb{N})$ is $\mathcal{L}$-convergent. In consequence, X is $\mathcal{L}$-sequentially complete generalized uniform space.

Now, let $F = \{\bar{1}, \bar{7}\} \subset X$ and let (X, T) be given by the formula

$$T(x) = \begin{cases} \{\bar{1}, \bar{2}\} \text{ if } \mathbf{x} \in X\backslash F \\ \{\bar{4}, \bar{5}\} \text{ if } \mathbf{x} \in F \end{cases}.$$

By the same reasoning as in Example 11, we obtain that, for $\varepsilon = \overline{1/2}$ and $\lambda = \overline{1/7}$, (X, T) is $(\mathbb{H}^{\mathcal{L}}_{(2)}, \varepsilon, \lambda)$-uniformly locally contractive on X, for each $w^0 \in X$ there exists a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ such that $\lim_{m\to\infty} w^m = \bar{2}$ and $\bar{2} \in Fix(T)$.

Now, in Example 13, for given $(X, \mathcal{D})$ and (X, T), we study the assertions of Theorem 8(I) with respect to changing of the family of $\mathcal{L}$ and of the point $w^0 \in X$.

Example 13 Let $(X, \mathcal{D})$ be a complete metric space where $X = [0,1]$ and let $\mathcal{D} = \{d\}, d : X \times X \to [0,\infty), d(x,y) = |x-y|, x, y \in X$, d:$X \times X \to [0,\infty)$, $d(x, y) = |x-y|$, $x, y \in X$. Let a dynamic system (X, T) be given by the formula:

$$T(x) = \begin{cases} [7/8, 1] & \text{if } x \in [0, 1/4) \\ [3/4, 7/8] & \text{if } x \in [1/4, 1/2) \\ \{x/2 + 1/2\} & \text{if } x \in [1/2, 1] \end{cases}. \tag{66}$$

Question 2 *For these* (X, D) *and* (X, T) *and for* $\varepsilon = 1/2$ *and* $\lambda = 1/2$, *what are the assertions of our theorems with respect to changing of the family L and of the point* $w^0 \in X$?

Answer 1 *We show that there exists* $\mathcal{L}$*-family on X such that:* (**a**) (X,T) *is not* $(\mathbb{H}^{\mathcal{L}}_{(2)}, 1/2, 1/2)$*-uniformly locally contractive on X; and* (**b**) (X, T) *is* $(\mathbb{H}^{\mathcal{L}}_{(1)}, 1/2, 1/2)$*-uniformly locally contractive on X and for each* $w^0 \in X$ *the assertion* (A1) *holds.*

(**a**) Let $E = (1/2,1)$ and $F = (1/2,1] \subset X$ (we see that $E \subset F \subset X$)and let $L : X \times X \to [0,+\infty]$ be defined by (51). It follows from Example 8 that the family $\mathcal{L} = \{L\}$ is $\mathcal{L}$-family on X.

We see that (X, T) is not $(\mathbb{H}^{\mathcal{L}}_{(2)}, 1/2, 1/2)$-uniformly locally contractive on X. Otherwise, $\forall_{x,y\in X}\{(L(x,y) < 1/2) \Rightarrow \mathbb{H}^{\mathcal{L}}_{(2)}(T(x), T(y)) \leq (1/2)L(x,y)\}$, where

$$\mathbb{H}^{\mathcal{L}}_{(2)}(T(x),T(y)) = \begin{cases} \mathbb{I} & \text{if } \mathbb{I} \text{ is finite} \\ +\infty & \text{otherwise} \end{cases},$$

$$\begin{aligned}
&\mathbb{I} = \mathbb{INF}(\mathcal{H}^{\mathcal{L}}_{(2)}(T(x),T(y))),\\
&\mathcal{H}^{\mathcal{L}}_{(2)}(T(x),T(y)) = \{\eta > 0 : T(x) \subset U_{\mathcal{L}}(\eta, T(y)) \wedge T(y) \subset U_{\mathcal{L}}(\eta, T(x))\},\\
&U_{\mathcal{L}}(\eta, T(y)) = \{u \in X : \exists_{z \in T(y)}\{L(u,z) < \eta\}\},\\
&U_{\mathcal{L}}(\eta, T(x)) = \{u \in X : \exists_{z \in T(x)}\{L(u,z) < \eta\}\}.
\end{aligned}$$

We note, by (51), (66) and definitions of E and F, that the condition

$$L(x,y) < 1/2 \tag{67}$$

implies, in particular,

$$x \in (1/2, 1), y = 1, T(x) = \{x/2 + 1/2\}, T(y) = \{1\}, \tag{68}$$

$$L(x,y) = d(x,y) \tag{69}$$

and, for $\eta > 0$, then the following hold

$$\begin{aligned}
&U_{\mathcal{L}}(\eta, T(y)) = \{u \in X : L(u,1) < \eta\},\\
&U_{\mathcal{L}}(\eta, T(x)) = \{u \in X : L(u, x/2 + 1/2) < \eta\}.
\end{aligned} \tag{70}$$

Indeed, if x, $y \in X$ satisfying (67) are arbitrary and fixed, then from (51) we conclude that (67) holds only if $x \in E$ and $y \in F \setminus E$. Hence, we get that $x \in (1/2,1)$, $y = 1$ and $d(x,y) < 1/2$, which, by (66), gives (68). Of course, by (51), the equality (69) holds. Now, if $\eta > 0$, then, by (68),

$$\begin{aligned}
U_{\mathcal{L}}(\eta, T(y)) &= \{u \in X : \exists_{z \in T(y) = \{1\}}\{L(u,z) < \eta\}\}\\
&= \{u \in X : L(u,1) < \eta\}
\end{aligned}$$

and

$$\begin{aligned}
U_{\mathcal{L}}(\eta, T(x)) &= \{u \in X : \exists_{z \in T(x) = \{x/2+1/2\}}\{L(u,z) < \eta\}\}\\
&= \{u \in X : L(u, x/2 + 1/2) < \eta\}.
\end{aligned}$$

Thus, (70) holds.

Now, by (67)-(70), we see that

$$\begin{aligned}
&\mathcal{H}^{\mathcal{L}}_{(2)}(T(x),T(y))\\
&= \{\eta > 0 : T(x) \subset U_{\mathcal{L}}(\eta, T(y)) \wedge T(y) \subset U_{\mathcal{L}}(\eta, T(x))\}\\
&= \{\eta > 0 : [x/2 + 1/2 \in U_{\mathcal{L}}(\eta, \{1\})] \wedge [1 \in U_{\mathcal{L}}(\eta, \{x/2 + 1/2\})]\}\\
&= \{\eta > 0 : L(x/2 + 1/2, 1) = d(x/2 + 1/2, 1) < \eta \wedge L(1, x/2 + 1/2)\\
&= d(x/2 + 1/2, 1) + 4 < \eta\}\\
&= \{\eta > 0 : 1/2 - x/2 < \eta \wedge 9/2 - x/2 < \eta\}\\
&= \{\eta > 0 : 9/2 - x/2 < \eta\};
\end{aligned}$$

that is, for $x \in (1/2,1)$ and $y = 1$, we have $\forall_{\gamma \in \mathcal{H}^{\mathcal{L}}_{(2)}(T(x),T(y))}\{9/2 - x/2 < \gamma\}$.

Therefore,

$$\begin{aligned}
\mathbb{H}^{\mathcal{L}}_{(2)}(T(x),T(y)) &= \inf \mathcal{H}^{\mathcal{L}}_{(2)}(T(x),T(y))\\
&= 9/2 - x/2 = (1/2)(9 - x) > (1/2)d(x,1)\\
&= (1/2)L(x,1) = (1/2)L(x,y).
\end{aligned}$$

Consequently, we proved that (X, T) is not $(\mathbb{H}^{\mathcal{L}}_{(2)}, 1/2, 1/2)$-uniformly locally contractive on X.

This gives that the assumptions of Theorem 8(I) for $i = 2$ and for $\mathcal{L}$ defined by (51) where $X = [0,1]$, $E = (1/2,1)$ and $F = (1/2,1]$ does not hold.

(b) However, by (67)-(70), we get

$$\begin{aligned}\mathcal{H}^{\mathcal{L}}_{(1)}(T(x), T(y)) &= \{\eta > 0 : T(x) \subset U_{\mathcal{L}}(\eta, T(y))\}\\ &= \{\eta > 0 : \{x/2 + 1/2\} \subset U_{\mathcal{L}}(\eta, \{1\})\}\\ &= \{\eta > 0 : \{x/2 + 1/2\} \in U_{\mathcal{L}}(\eta, \{1\})\}\\ &= \{\eta > 0 : L(x/2 + 1/2, 1) = d(x/2 + 1/2, 1) < \eta\}\\ &= \{\eta > 0 : 1/2 - x/2 < \eta\};\end{aligned}$$

that is, for $x \in (1/2,1)$ and $y = 1$, we have $\forall_{\gamma\in\mathcal{H}^{\mathcal{L}}_{(1)}(T(x),T(y))}\{1/2 - x/2 < \gamma\}$. Therefore,

$$\begin{aligned}\mathbb{H}^{\mathcal{L}}_{(1)}(T(x), T(y)) &= \inf \mathcal{H}^{\mathcal{L}}_{(1)}(T(x), T(y))\\ &= (1/2)(1 - x) = (1/2)d(x, 1)\\ &= (1/2)L(x, 1) = (1/2)L(x, y).\end{aligned}$$

Consequently, we proved that (X, T) is $(\mathbb{H}^{\mathcal{L}}_{(1)}, 1/2, 1/2)$-uniformly locally contractive on X.

This gives that the assumptions of Theorem 8(I) for $i = 1$ and for $\mathcal{L}$ defined by (51) where $X = [0,1]$, $E = (1/2, 1)$ and $F = (1/2, 1]$ hold.

Now, we see that, for each $w^0 \in X$, the assertion (A1) holds. Indeed, we have:

Case 1. *Let* $w^0 \in [0,1/4)$. Then, for each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X, T) starting at w^0, by (66), we have: (i) if $w^1 \neq 1$, then $\forall_{m\in\mathbb{N}}\{w^m \in E\}$ and, by (51), $L(w^0, w^1) = +\infty > 1/2$ and

$$\forall_{m>1}\{L(w^{m-1}, w^m) = d(w^{m-1}, w^m) + 1 > 1/2\};$$

or (ii) if $w^1 = 1$, then $\forall_{m\in\mathbb{N}}\{w^m = 1 \in F\setminus E\}$ and, by (51), $L(w^0, w^1) = +\infty > 1/2$ and

$$\forall_{m\geq 2}\{L(w^{m-1}, w^m) = d(w^{m-1}, w^m) + 3 > 1/2\}.$$

Consequently, for each $w^0 \in [0,1/4)$, each a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X, T) starting at w^0 satisfies $\forall_{m\in\mathbb{N}}\{L(w^{m-1}, w^m) > 1/2\}$, i.e. for each $w^0 \in [0,1/4)$, the assertion (A1) holds.

Case 2. *Let* $w^0 \in [1/4,1)$. Then, for each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X, T) starting at w^0, by (66), we have that $\forall_{m\in\mathbb{N}}\{w^m \in E\}$ and, by (51),

$$L(w^0, w^1) = \begin{cases} +\infty > 1/2 & \text{if } w^0 \in [1/4, 1/2)\\ d(w^0, w^1) + 1 > 1/2 & \text{if } w^0 \in [1/2, 1)\end{cases}$$

and

$$\forall_{m\geq 2}\{L(w^{m-1}, w^m) = d(w^{m-1}, w^m) + 1 > 1/2\}.$$

Consequently, for each $w^0 \in [1/4,1)$, each a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X,T) starting at w^0 satisfies $\forall_{m\in\mathbb{N}}\{L(w^{m-1}, w^m) > 1/2\}$, i.e. for each $w^0 \in [1/4,1)$, the assertion (A1) holds.

Case 3. *Let* $w^0 = 1$. Then, for a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X, T) starting at w^0, by (66), we have that $\forall_{m\in\mathbb{N}}\{w^m = 1 \in F\setminus E\}$ and, by (51),

$$\forall_{m\in\mathbb{N}}\{L(w^{m-1},w^m)=d(w^{m-1},w^m)+3>1/2\}.$$

Consequently, if $w^0 = 1$, a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X, T) starting at w^0 satisfies $\forall_{m\in\mathbb{N}}\{L(w^{m-1}, w^m) > 1/2\}$, i.e. for $w^0 = 1$, the assertion (A1) holds.

Remark 7 Let us observe that, for each $w^0 \in X$, there exists a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ starting at w^0 such that $\lim_{m\to\infty} w^m = 1$, $\lim_{m\to\infty} L(w^m, 1) = \lim_{m\to\infty} d(w^m, 1) = 0$ and $1 \in Fix(T)$. However, assertion (A2) does not hold since from Cases 1-3 it follows that, for each $w^0 \in X$, each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ starting at w^0 is not $\mathcal{L}$-Cauchy.

Answer 2 *We show that there exists* $\mathcal{L}$ *-family on X such that* (X, T) *is* $(\mathbb{H}^{\mathcal{L}}_{(2)}, 1/2, 1/2)$*-uniformly locally contractive on X and, for each* $w^0 \in X$*, the assertion* (A2) *holds.*

Let $E = [1/2,1] \subset X$ and let $L : X \times X \to [0, +\infty]$ be defined by the formula:

$$L(x,y)=\begin{cases} d(x,y) & \text{if}\{x,y\}\cap E=\{x,y\} \\ +\infty & \text{if}\{x,y\}\cap E\neq\{x,y\}\end{cases}. \tag{71}$$

It follows, from Example 6, that the family $\mathcal{L} = \{L\}$ is $\mathcal{L}$-family on X.

We see that (X, T) is $(\mathbb{H}^{\mathcal{L}}_{(2)}, 1/2, 1/2)$-uniformly locally contractive on X, i.e.

$$\forall_{x,y\in X}\{L(x,y)<1/2\Rightarrow \mathbb{H}^{\mathcal{L}}_{(2)}(T(x),T(y))\leq(1/2)L(x,y)\},$$

where

$$\mathbb{H}^{\mathcal{L}}_{(2)}(T(x),T(y))=\begin{cases}\mathbb{I} & \text{if } \mathbb{I} \text{ is finite} \\ +\infty & \text{otherwise}\end{cases},$$

$$\begin{aligned}&\mathbb{I}=\mathbb{INF}(\mathcal{H}^{\mathcal{L}}_{(2)}(T(x),T(y))),\\ &\mathcal{H}^{\mathcal{L}}_{(2)}(T(x),T(y))=\{\eta>0:T(x)\subset U_{\mathcal{L}}(\eta,T(y))\wedge T(y)\subset U_{\mathcal{L}}(\eta,T(x))\},\\ &U_{\mathcal{L}}(\eta,T(y))=\{u\in X:\exists_{z\in T(y)}\{L(u,z)<\eta\}\},U_{\mathcal{L}}(\eta,T(x))\\ &\qquad=\{u\in X:\exists_{z\in T(x)}\{L(u,z)<\eta\}\}.\end{aligned}$$

Indeed, first, we see that, by (66) and (71),

$$L(x,y)<1/2 \tag{72}$$

implies

$$x,y\in[1/2,1],T(x)=\{x/2+1/2\},T(y)=\{y/2+1/2\}, \tag{73}$$

$$L(x,y)=d(x,y) \tag{74}$$

and, for $\eta > 0$,

$$\begin{aligned}&U_{\mathcal{L}}(\eta,T(y))=\{u\in X:L(u,y/2+1/2)<\eta\},\\ &U_{\mathcal{L}}(\eta,T(x))=\{u\in X:L(u,x/2+1/2)<\eta\}.\end{aligned} \tag{75}$$

Indeed, if $x, y \in X$ satisfying (72) are arbitrary and fixed, then from (71) we conclude that (72) holds only if $x, y \in E$. Hence, we get that $x, y \in [1/2,1]$ and $d(x, y) < 1/2$, which, by (66), gives (73). Of course, by (49), (74) holds. Now, if $\eta > 0$, then, by (73),

$$U_{\mathcal{L}}(\eta, T(y)) = \{u \in X : \exists_{z \in T(y)=\{y/2+1/2\}}\{L(u,z) < \eta\}\}$$
$$= \{u \in X : L(u, y/2 + 1/2) < \eta\}$$

and

$$U_{\mathcal{L}}(\eta, T(x)) = \{u \in X : \exists_{z \in T(x)=\{x/2+1/2\}}\{L(u,z) < \eta\}\}$$
$$= \{u \in X : L(u, x/2 + 1/2) < \eta\}.$$

Thus, (75) holds.

Now, by (72)-(75), we see that

$$\begin{aligned}
&\mathcal{H}^{\mathcal{L}}_{(2)}(T(x), T(y)) \\
&= \{\eta > 0 : T(x) \subset U_{\mathcal{L}}(\eta, T(y)) \wedge T(y) \subset U_{\mathcal{L}}(\eta, T(x))\} \\
&= \{\eta > 0 : \{x/2 + 1/2\} \subset U_{\mathcal{L}}(\eta, \{y/2 + 1/2\}) \wedge \{y/2 + 1/2\} \subset U_{\mathcal{L}}(\eta, \{x/2 + 1/2\})\} \\
&= \{\eta > 0 : [x/2 + 1/2 \in U_{\mathcal{L}}(\eta, \{y/2 + 1/2\})] \wedge [y/2 + 1/2 \in U_{\mathcal{L}}(\eta, \{x/2 + 1/2\})]\} \\
&= \{\eta > 0 : L(x/2 + 1/2, y/2 + 1/2) = d(x/2 + 1/2, y/2 + 1/2) \\
&= (1/2)\left|x - y\right| < \eta \wedge L(y/2 + 1/2, x/2 + 1/2) \\
&= d(x/2 + 1/2, y/2 + 1/2) = \left|x - y\right|/2 < \eta\} = \{\eta > 0 : \left|x - y\right|/2 < \eta\};
\end{aligned}$$

that is, for x, $y \in [1/2,1]$, we have $\forall_{\gamma \in \mathcal{H}^{\mathcal{L}}_{(2)}(T(x),T(y))}\{\left|x - y\right|/2 < \gamma\}$. Therefore,

$$\mathbb{H}^{\mathcal{L}}_{(2)}(T(x), T(y)) = \inf \mathcal{H}^{\mathcal{L}}_{(2)}(T(x), T(y))$$
$$= \left|x - y\right|/2 \leq \left|x - y\right|/2 = (1/2)d(x, y) = (1/2)L(x, y).$$

Consequently, we proved that (X, T) is $(\mathbb{H}^{\mathcal{L}}_{(2)}, 1/2, 1/2)$-uniformly locally contractive on X.

This gives that the assumptions of Theorem 8(I) for $\mathcal{L}$ defined by (71) and for $i = 2$ hold.

We see that, for each $w^0 \in X$, the assertion (A2) holds. Indeed, we have that: $1 \in Fix(T)$; for each $w^0 \in X$ and for each dynamic processes $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X,T) starting at w^0, by (66), we have that $\forall_{m \geq 2} \{w^m \in E\}$, so $\lim_{m\to\infty} L(w^m, 1) = \lim_{m\to\infty} d(w^m, 1) = 0$ and $\lim_{n\to\infty} \sup_{m>n} L(w^n,w^m) = \lim_{n\to\infty} \sup_{m>n} d(w^n, w^m) = 0$. Therefore, the sequence $(w^m : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{L}$-Cauchy.

Remark 8 We see that $L(1, 1) = \mathbb{H}^{\mathcal{L}}_{(2)}(T(1), T(1)) = 0$. Indeed, by (71), $L(1,1) = d(1,1) = 0$ and

$$\mathbb{H}^{\mathcal{L}}_{(2)}(T(1), T(1)) = \mathbb{H}^{\mathcal{L}}_{(2)}(1, 1) = \inf \mathbb{H}^{\mathcal{L}}_{(2)}(\{1\}, \{1\})$$
$$= \inf\{\eta > 0 : \{1\} \subset U_{\mathcal{L}}(\eta, \{1\})\}$$
$$= \inf\{\eta > 0 : L(1, 1) = 0 < \eta\} = 0.$$

Answer 3 *We show that there exists $\mathcal{L}$-family on X such that:* **(i)** (X, T) *is* $(\mathbb{H}^{\mathcal{L}}_{(1)}, 1/2, 1/2)$*-uniformly locally contractive on X*; **(ii)** *There exists $w \in X$ such that $End(T) = \{w\}$*; **(iii)** *For each $w^0 \in X \backslash End(T)$ the assertion* (A2) *holds; and* **(iv)** *For $w^0 = w$ the assertion* (A1) *holds (since $L(w, w) = 3$ where $\mathcal{L} = \{L\}$).*

Define $E = (1/2,1)$ and $F = (1/2,1] \subset X$ (we see that $E \subset F \subset X$) and let $L:X \times X \to [0, +\infty]$ be defined by (53). It follows from Example 10 that the family $\mathcal{L} = \{L\}$ is $\mathcal{L}$-family on X.

First, we show that (X,T) is not $(\mathbb{H}^{\mathcal{L}}_{(2)}, 1/2, 1/2)$-uniformly locally contractive on X. Otherwise,

$$\forall_{x,y\in X}\{L(x,y) < 1/2 \Rightarrow \mathbb{H}^{\mathcal{L}}_{(2)}(T(x),T(y)) \le (1/2)L(x,y)\},$$

where

$$\mathbb{H}^{\mathcal{L}}_{(2)}(T(x),T(y)) = \begin{cases} \mathbb{I} & \text{if } \mathbb{I} \text{ is finite} \\ +\infty & \text{otherwise} \end{cases},$$

$$\begin{aligned} &\mathbb{I} = \mathbb{INF}(\mathcal{H}^{\mathcal{L}}_{(2)}(T(x),T(y))), \\ &\mathcal{H}^{\mathcal{L}}_{(2)}(T(x),T(y)) = \{\eta > 0 : T(x) \subset U_{\mathcal{L}}(\eta,T(y)) \wedge T(y) \subset U_{\mathcal{L}}(\eta,T(x))\}, \\ &U_{\mathcal{L}}(\eta,T(y)) = \{u \in X : \exists_{z\in T(y)}\{L(u,z) < \eta\}\}, \\ &U_{\mathcal{L}}(\eta,T(x)) = \{u \in X : \exists_{z\in T(x)}\{L(u,z) < \eta\}\}. \end{aligned}$$

Let us notice that, by (53) and (66),

$$L(x,y) < 1/2 \tag{76}$$

implies

$$x \in (1/2,1), y = 1, T(x) = \{x/2 + 1/2\}, T(y) = \{1\}, \tag{77}$$

$$L(x,y) = d(x,y) \tag{78}$$

and, for $\eta > 0$,

$$\begin{aligned} &U_{\mathcal{L}}(\eta,T(y)) = \{u \in X : L(u,1) < \eta\}, \\ &U_{\mathcal{L}}(\eta,T(x)) = \{u \in X : L(u,x/2+1/2) < \eta\}. \end{aligned} \tag{79}$$

Indeed, if x, $y \in X$ satisfying (76) are arbitrary and fixed, then from (53) we conclude that (76) holds only in two following cases: (i) $(x, y) \in E \times (F \backslash E)$ or (ii) $(x,y) \in E \times E$.

Now we see that, in particular, if $x \in E$ and $y \in F \setminus E$, then we get that $x \in (1/2,1)$, $y = 1$ and $d(x, y) < 1/2$, which, by (66), gives (77). Of course, by (53), (78) holds. Now, if $\eta > 0$, then, by (77),

$$\begin{aligned} U_{\mathcal{L}}(\eta,T(y)) &= \{u \in X : \exists_{z\in T(y)=\{1\}}\{L(u,z) < \eta\}\} \\ &= \{u \in X : L(u,1) < \eta\} \end{aligned}$$

and

$$\begin{aligned} U_{\mathcal{L}}(\eta,T(x)) &= \{u \in X : \exists_{z\in T(x)=\{x/2+1/2\}}\{L(u,z) < \eta\}\} \\ &= \{u \in X : L(u,x/2+1/2) < \eta\}. \end{aligned}$$

Thus, (79) holds. Further, by (76)-(79), we see that

$$\begin{aligned} &\mathcal{H}^{\mathcal{L}}_{(2)}(T(x),T(y)) \\ &= \{\eta > 0 : T(x) \subset U_{\mathcal{L}}(\eta,T(y)) \wedge T(y) \subset U_{\mathcal{L}}(\eta,T(x))\} \\ &= \{\eta > 0 : \{x/2+1/2\} \subset U_{\mathcal{L}}(\eta,\{1\}) \wedge \{1\} \subset U_{\mathcal{L}}(\eta,\{x/2+1/2\})\} \\ &= \{\eta > 0 : [x/2+1/2 \in U_{\mathcal{L}}(\eta,\{1\})] \wedge [1 \in U_{\mathcal{L}}(\eta,\{x/2+1/2\})]\} \\ &= \{\eta > 0 : L(x/2+1/2,1) = d(x/2+1/2,1) < \eta \wedge L(1,x/2+1/2) \\ &= d(x/2+1/2,1) + 4 < \eta\} \\ &= \{\eta > 0 : 1/2 - x/2 < \eta \wedge 9/2 - x/2 < \eta\} = \{\eta > 0 : 9/2 - x/2 < \eta\}; \end{aligned}$$

that is, for $x \in (1/2,1)$ and $y = 1$, we have $\forall_{\gamma\in\mathcal{H}^{\mathcal{L}}_{(1)}(T(x),T(y))}\{(1/2)\left|x-y\right| < \gamma\}$.

Therefore,

$$\begin{aligned}\mathbb{H}^{\mathcal{L}}_{(2)}(T(x),T(y)) &= \inf \mathcal{H}^{\mathcal{L}}_{(2)}(T(x),T(y)) = 9/2 - x/2\\ &= (1/2)(9-x) > (1/2)d(x,1)\\ &= (1/2)L(x,1) = (1/2)L(x,y).\end{aligned}$$

Consequently, we proved that (X,T) is not $(\mathbb{H}^{\mathcal{L}}_{(2)}, 1/2, 1/2)$-uniformly locally contractive on X. This gives that the assumptions of Theorem 8(I) for such $\mathcal{L}$ and for $i = 2$ do not hold.

Next, to prove that (X, T) is $\forall_{x,y\in X}\{(L(x,y) < 1/2) \Rightarrow \mathbb{H}^{\mathcal{L}}_{(2)}(T(x),T(y)) \leq (1/2)L(x,y)\}$-uniformly locally contractive on X, we assume that $x, y \in X$ satisfying (76) are arbitrary and fixed. Then, by (53), we conclude that (76) holds only in the following two cases:

Case 1. *Let* $x \in E$ *and let* $y \in F\backslash E$. By (76)-(79), we get

$$\begin{aligned}\mathcal{H}^{\mathcal{L}}_{(1)}(T(x),T(y)) &= \{\eta > 0 : T(x) \subset U_{\mathcal{L}}(\eta, T(y))\}\\ &= \{\eta > 0 : \{x/2 + 1/2\} \subset U_{\mathcal{L}}(\eta, \{1\})\}\\ &= \{\eta > 0 : x/2 + 1/2 \in U_{\mathcal{L}}(\eta, \{1\})\}\\ &= \{\eta > 0 : L(x/2 + 1/2, 1) = d(x/2 + 1/2, 1) < \eta\}\\ &= \{\eta > 0 : 1/2 - x/2 < \eta\};\end{aligned}$$

that is, for $x \in (1/2,1)$ and $y = 1$, we have $\forall_{\gamma\in\mathcal{H}^{\mathcal{L}}_{(1)}(T(x),T(y))}\{1/2 - x/2 < \gamma\}$.

Therefore,

$$\begin{aligned}\mathbb{H}^{\mathcal{L}}_{(1)}(T(x),T(y)) &= \inf \mathcal{H}^{\mathcal{L}}_{(1)}(T(x),T(y)) = (1/2)(1-x)\\ &= (1/2)d(x,1) = (1/2)L(x,1) = (1/2)L(x,y).\end{aligned}$$

Case 2. *Let* $x, y \in E$. By (66), $T(x) = \{x/2 + 1/2\}$, $T(y) = \{y/2 + 1/2\}$, and, consequently, we get

$$\begin{aligned}&\mathcal{H}^{\mathcal{L}}_{(2)}(T(x),T(y))\\ &= \{\eta > 0 : T(x) \subset U_{\mathcal{L}}(\eta, T(y))\}\\ &= \{\eta > 0 : \{x/2 + 1/2\} \subset U_{\mathcal{L}}(\eta, \{y/2 + 1/2\})\}\\ &= \{\eta > 0 : x/2 + 1/2 \in U_{\mathcal{L}}(\eta, \{y/2 + 1/2\})\}\\ &= \{\eta > 0 : L(x/2 + 1/2, y/2 + 1/2) = d(x/2 + 1/2, y/2 + 1/2) < \eta\}\\ &= \{\eta > 0 : (1/2)\left|x - y\right| < \eta\};\end{aligned}$$

that is, for $x, y \in (1/2,1)$, we have $\forall_{\gamma\in\mathcal{H}^{\mathcal{L}}_{(1)}(T(x),T(y))}\{(1/2)\left|x-y\right| < \gamma\}$. Therefore,

$$\begin{aligned}\mathbb{H}^{\mathcal{L}}_{(1)}(T(x),T(y)) &= \inf \mathcal{H}^{\mathcal{L}}_{(1)}(T(x),T(y))\\ &= (1/2)\left|x-y\right| = (1/2)d(x,y) = (1/2)L(x,y).\end{aligned}$$

From Cases 1 and 2 it follows that (X, T) is $\forall_{x,y\in X}\{(L(x,y) < 1/2) \Rightarrow \mathbb{H}^{\mathcal{L}}_{(2)}(T(x),T(y)) \leq (1/2)L(x,y)\}$-uniformly locally contractive on X.

It is clear that the assumptions of Theorem 8(I) for such $\mathcal{L}$ and for $i = 1$ hold.

Now we prove that if $w^0 \in [0,1)$, then the assertion (A2) holds and if $w^0 = 1$ then the assertion (A1) holds. Indeed, we have the following three cases:

Case 1. *Let* $w^0 \in [0,1/4)$. Then, by (66), there exists a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X, T) starting at w^0 of the form: $w^1 \neq 1$ and $\forall_{m\in\mathbb{N}}\{w^m \in [1/2, 1) = E\}$. Then, by (53), $L(w^0, w^1) = +\infty$ and $\forall_{m\geq 2}\left\{L\left(w^{m-1}, w^m\right) = d\left(w^{m-1}, w^m\right)\right\}$. Consequently, a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{L}$-Cauchy on X, $\lim_{m\to\infty} w^m = 1$ and $1 \in Fix(T)$, i.e. for each $w^0 \in [0,1/4)$, the assertion (A2) holds.

Case 2. *Let* $w^0 \in [1/4,1)$. Then, for each a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X, T) starting at w^0, by (66), we have that $\forall_{m\in\mathbb{N}}\{w^m \in E\}$ and, by (53),

$$L(w^0, w^1) = \begin{cases} +\infty & \text{if } w^0 \in [1/4, 1/2] \\ d(w^0, w^1) & \text{if } w^0 \in [1/2, 1] \end{cases}$$

and $\forall_{m\geq 2}\left\{L\left(w^{m-1}, w^m\right) = d\left(w^{m-1}, w^m\right)\right\}$. Consequently, for each $w^0 \in [1/4,1)$, each a dynamic process $(w^m : m \in \{0\}\cup \mathbb{N})$ of (X, T) starting at w^0 is $\mathcal{L}$-Cauchy on X, $\lim_{m\to\infty} w^m = 1$ and $1 \in Fix(T)$, i.e., for each $w^0 \in [1/4,1)$, the assertion (A2) holds.

Case 3. *Let* $w^0 = 1$. Then, for a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X, T) starting at w^0, by (66), we have that $\forall_{m\in\mathbb{N}}\{w^m = 1 \in F\backslash E\}$ and, by (53), $\forall_{m\in\mathbb{N}}\{L(w^{m-1}, w^m) = d(w^{m-1}, w^m) + 3 > 1/2\}$. Consequently, if $w^0 = 1$, a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X, T) starting at w^0 satisfies $\forall_{m\in\mathbb{N}}\{L(w^{m-1}, w^m) > 1/2\}$, i.e. for $w^0 = 1$, the assertion (A1) holds.

Finally, we see that, for each $w^0 \in X$, there exists a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ such that $\lim_{m\to\infty} w^m = 1$, $\lim_{m\to\infty} L(w^m, 1) = \lim_{m\to\infty} d(w^m, 1) = 0$ and $1 \in Fix(T)$.

Remark 9 Let us point out that $L(1, 1) = \mathbb{H}^{\mathcal{L}}_{(1)}(T(1), T(1)) = 3 > 1/2$. Indeed, by (53), $L(1, 1) = d(1, 1) + 3 = 3$ and

$$\begin{aligned} &\mathbb{H}^{\mathcal{L}}_{(1)}(T(1), T(1)) \\ &= \inf \mathcal{H}^{\mathcal{L}}_{(1)}(\{1\}, \{1\}) = \inf\{\eta > 0 : \{1\} \subset U_{\mathcal{L}}(\eta, \{1\})\} \\ &= \inf\{\eta > 0 : L(1, 1) < \eta\} = \inf\{\eta > 0 : d(1, 1) + 3 < \eta\} = 3. \end{aligned}$$

Examples and comparisons of our results with Banach's, Nadler's, Covitz-Nadler's and Suzuki's results

It is worth noticing that our results in metric spaces and in generalized metric spaces include Banach's [3], Nadler's [[4], Th. 5], Covitz-Nadler's [[5], Theorem 1] and Suzuki's [[10], Theorem 3.7] results.

Clearly, it is not otherwise. More precisely: (a) In Example 14 we construct $\mathcal{D}$-complete generalized metric space $(X, \mathcal{D})$, a $\mathcal{L}$-family on X satisfying $\mathcal{L} \neq \mathcal{D}$ and a set-valued dynamic system (X, T) which is $(\mathbb{H}^{\mathcal{L}}_{(2)}, 1/2, 1/7)$-uniformly locally contractive on X and next we show that the assertion (A2) holds; (b) In Example 15 we show that, for each $\varepsilon \in (0, \infty)$, $\lambda \in [0, 1)$ and $i \in \{1, 2\}$, the set-valued dynamic system (X, T) defined in Example 14 is not $(\mathbb{H}^{\mathcal{D}}_{(i)}, \varepsilon, \lambda)$-uniformly locally contractive on X and thus we cannot use Theorems 1, 2 and 4-7; (c) In Example 16 we construct a complete metric space $(X, \mathcal{D})$, $\mathcal{L} = \{L : X \times X \to [0, +\infty]\}$ which is $\mathcal{L}$-family on X and $\forall_{x,y\in X}\{(L(x, y) < 1/2) \Rightarrow \mathbb{H}^{\mathcal{L}}_{(2)}(T(x), T(y)) \leq (1/2)L(x, y)\}$-uniformly locally contractive set-valued dynamic system (X, T) such that, for each $w^0 \in X$, the assertion (A2) holds and, additionally, $L(w, w) > 0$ for $w \in Fix(T)$ which gives that our theorems are different from Theorem 7.

Example 14 Let Z and q be as in Example 3. Let $X = Z \cap [0,9]$ and let $\mathcal{D} = \{d\}$ where $d = q|_{[0,9]}$. Then $(X, \mathcal{D})$ is a $\mathcal{D}$-complete generalized metric space. Let $F = \{1, 7\}$ and let (X, T) be given by the formula

$$T(x) = \begin{cases} \{1,2\} \text{ if } x \in X \backslash F \\ \{4,5\} \text{ if } x \in F \end{cases};$$

we see that $T : X \to C(X)$. Let $E = \{0, 1, 2\} \cup [4,5] \cup \{6, 8\}$ and let L be of the form

$$L(x,y) = \begin{cases} d(x,y) \text{ if}\{x,y\} \cap E = \{x,y\} \\ +\infty \quad \text{ if}\{x,y\} \cap E \neq \{x,y\} \end{cases}.$$

By Example 6, the family $\mathcal{L} = \{L\}$ is $\mathcal{L}$-family on X. By the similar reasoning as in Example 11, we show that (X, T) is $(\mathbb{H}^{\mathcal{L}}_{(2)}, 1/2, 1/7)$-uniformly locally contractive on X. We see that for each $w^0 \in X$ there exists a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ such that $\lim_{m\to\infty} w^m = 2$ and $2 \in Fix(T)$.

Remark 10 We notice that $L(2,2) = \mathbb{H}^{\mathcal{L}}_{(2)}(T(2), T(2)) = 0$.

Example 15 Let X, $\mathcal{D} = \{d\}$ and T be such as in Example 14. We show that, for any $\varepsilon \in (0, \infty)$, $\lambda \in [0, 1)$ and $i \in \{1, 2\}$, T is not $(\mathbb{H}^{\mathcal{D}}_{(i)}, \varepsilon, \lambda)$-uniformly locally contractive on X.

Otherwise, there exist $\varepsilon_0 \in (0, \infty)$, $\lambda_0 \in [0, 1)$ and $i \in \{1, 2\}$ such that

$$\forall_{x,y \in X}\{\{d(x,y) < \varepsilon_0\} \Rightarrow \{\mathbb{H}^{\mathcal{D}}_{(i)}(T(x), T(y)) \leq \lambda_0 d(x,y)\}\}. \tag{80}$$

We consider the following three cases:

Case 1. If $\varepsilon_0 = 1$, then, in particular, for $x_0 = 1$ and $y_0 = 1/2$, since $x_0, y_0 \in [0,1]$, by formula (46), we get

$$d(x_0, y_0) = d(1, 1/2) = q_1(1, 1/2) = \left|1 - 1/2\right| = 1/2 < \varepsilon_0.$$

However, $T(x_0) = \{4, 5\}$, $T(y_0) = \{1, 2\}$, and, by (46),

$$\begin{aligned} &d(5,1) = d(1,5) = q(1,5) = +\infty, \\ &d(1,4) = d(4,1) = +\infty, \\ &d(2,5) = d(5,2) = d(2,4) = d(4,2) = +\infty. \end{aligned}$$

Hence

$$\begin{aligned} \inf\{\eta > 0 : &[d(1,5) < \eta \vee d(1,4) < \eta] \wedge [d(2,5) < \eta \vee d(2,4) < \eta] \\ &\wedge [d(4,1) < \eta \vee d(4,2) < \eta] \wedge [d(5,1) < \eta \vee d(5,2) < \eta]\} = +\infty. \end{aligned}$$

Consequently,

$$\mathbb{H}^{\mathcal{D}}_{(i)}(T(x_0), T(y_0)) = \mathbb{H}^{\mathcal{D}}_{(i)}(T(1), T(1/2)) = +\infty$$

and (80) gives

$$\begin{aligned} \mathbb{H}^{\mathcal{D}}_{(i)}(T(x_0), T(y_0)) &= +\infty \leq \lambda_0(1/2) = \lambda_0 \left|1 - 1/2\right| \\ &= \lambda_0 q_1(1, 1/2) = \lambda_0 d(x_0, y_0). \end{aligned}$$

This leads to a contradiction.

Case 2. If $\varepsilon_0 \in (1, \infty)$, then by a similar reasoning as in Case 1 we prove that (80) does not hold.

Case 3. If $\varepsilon_0 \in (0, 1)$, then, in particular, for $x_0 = 1$ and $y_0 = ((1 - \varepsilon_0)/2)$, we obtain that $x_0, y_0 \in [0,1]$ and by a similar reasoning as in Case 1 we prove that (80) does not hold.

Example 16 Let $X = [0,1]$ and $\mathcal{D} = \{d\}$ where $d : X \times X \to [0, \infty)$ is defined by the formula $d(x, y) = |x - y|$, $x, y \in X$. Then $(X, \mathcal{D})$ is a complete metric space. Let $E = [1/2, 1)$ and $F = [1/2, 1] \subset X$ (we see that $E \subset F \subset X$) and let $L : X \times X \to [0, +\infty]$ be defined by (53). It follows from Example 10 that the family $\mathcal{L} = \{L\}$ is $\mathcal{L}$-family on X. Let (X, T) be given by the formula:

$$T(x) = \begin{cases} [7/8, 1] & \text{if } x \in [0, 1/4) \\ [3/4, 7/8] & \text{if } x \in [1/4, 1/2) \\ \{x/2 + 1/2\} & \text{if } x \in [1/2, 1) \\ \{(1/2), 1\} & \text{if } x = 1 \end{cases}. \tag{81}$$

First, we show that (X, T) is $\forall_{x,y\in X}\{(L(x,y) < 1/2) \Rightarrow \mathbb{H}^{\mathcal{L}}_{(2)}(T(x), T(y)) \leq (1/2)L(x,y)\}$-uniformly locally contractive on X. Assume that $x, y \in X$ satisfying $L(x, y) < 1/2$ are arbitrary and fixed. Then from (53) we conclude that $L(x, y) < 1/2$ implies $(x, y) \in E \times (F \setminus E)$ or $(x, y) \in E \times E$. Consequently, the following two cases hold:

Case 1. *Let* $x \in E$ *and* $y \in F \setminus E$. Then, by (81) we get: $T(x) = \{x/2+1/2\}$;

$$\begin{aligned} &T(y) \\ &= \{1/2, 1\}; \mathcal{H}^{\mathcal{L}}_{(1)}(T(x), T(y)) = \{\eta > 0 : T(x) \subset U_{\mathcal{L}}(\eta, T(y))\} \\ &= \{\eta > 0 : \{x/2 + 1/2\} \subset U_{\mathcal{L}}(\eta, \{1/2, 1\})\} \\ &= \{\eta > 0 : x/2 + 1/2 \in U_{\mathcal{L}}(\eta, \{1/2, 1\})\} \\ &= \{\eta > 0 : L(x/2 + 1/2, 1/2) = d(x/2 + 1/2, 1/2) < \eta \vee L(x/2 + 1/2, 1) \\ &= d(x/2 + 1/2, 1) < \eta\} = \{\eta > 0 : x/2 < \eta \vee 1/2 - x/2 < \eta\}; \end{aligned}$$

that is, for $x \in [1/2, 1)$ and $y = 1$, we have $1/2 - x/2 \leq 1/4 \leq x/2$ and $\forall_{\eta\in\mathcal{H}^{\mathcal{L}}_{(1)}(T(x),T(y))}\{1/2 - x/2 < \eta\}$. Therefore,

$$\begin{aligned} \mathbb{H}^{\mathcal{L}}_{(1)}(T(x), T(y)) &= \inf \mathcal{H}^{\mathcal{L}}_{(1)}(T(x), T(y)) = (1/2)(1 - x) \\ &= (1/2)d(x, 1) = (1/2)L(x, 1) = (1/2)L(x, y). \end{aligned}$$

Case 2. *Let* $x, y \in E$. Then, by (81), $T(x) = \{x/2 + 1/2\}$, $T(y) = \{y/2 + 1/2\}$ and, consequently, we get

$$\begin{aligned} &\mathcal{H}^{\mathcal{L}}_{(1)}(T(x), T(y)) \\ &= \{\eta > 0 : T(x) \subset U_{\mathcal{L}}(\eta, T(y))\} \\ &= \{\eta > 0 : \{x/2 + 1/2\} \subset U_{\mathcal{L}}(\eta, \{y/2 + 1/2\})\} \\ &= \{\eta > 0 : x/2 + 1/2 \in U_{\mathcal{L}}(\eta, \{y/2 + 1/2\})\} \\ &= \{\eta > 0 : L(x/2 + 1/2, y/2 + 1/2) \\ &= d(x/2 + 1/2, y/2 + 1/2) < \eta\} = \{\eta > 0 : (1/2)\left|x - y\right| < \eta\}; \end{aligned}$$

that is, for $x, y \in (1/2, 1)$, we have $\forall_{\eta\in\mathcal{H}^{\mathcal{L}}_{(1)}(T(x),T(y))}\{(1/2)\left|x - y\right| < \eta\}$. Therefore,

$$\begin{aligned} \mathbb{H}^{\mathcal{L}}_{(1)}(T(x), T(y)) &= \inf \mathcal{H}^{\mathcal{L}}_{(1)}(T(x), T(y)) \\ &= (1/2)\left|x - y\right| = (1/2)d(x, y) = (1/2)L(x, y). \end{aligned}$$

Consequently, we proved that (X, T) is $\forall_{x,y\in X}\{(L(x,y) < 1/2) \Rightarrow \mathbb{H}^{\mathcal{L}}_{(2)}(T(x),T(y)) \le (1/2)L(x,y)\}$ -uniformly locally contractive on X. We also see that all assumptions of Theorem 8(I) for this $\mathcal{L}$ and for $i = 1$ hold.

Now, we show that, for each $w^0 \in X$, the assertion (A2) holds. Indeed, we have the following three cases:

Case 1. *Let* $w^0 \in [0, 1/4)$. Then, there exists a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X, T) starting at w^0 of the form: $w^1 \neq 1$, and $\forall_{m\in\mathbb{N}}\{w^m \in [1/2, 1) = E\}$. Then, by (53), $L(w^0, w^1) = +\infty$ and $\forall_{m\ge 2}\{L(w^{m-1}, w^m) = d(w^{m-1}, w^m)\}$. Consequently, a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{L}$-Cauchy on X, $\lim_{m\to\infty} w^m = 1$ and $1 \in Fix(T)$, i.e. for each $w^0 \in [0, 1/4)$, the assertion (A2) holds.

Case 2. *Let* $w^0 \in [1/4, 1)$. Then, for each a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X, T) starting at w^0, by (81), we have that $\forall_{m\in\mathbb{N}}\{w^m \in E\}$ and, by (53),

$$L(w^0, w^1) = \begin{cases} +\infty & \text{if } w^0 \in [1/4, 1/2) \\ d(w^0, w^1) & \text{if } w^0 \in [1/2, 1) \end{cases}$$

and $\forall_{m\ge 2}\{L(w^{m-1}, w^m) = d(w^{m-1}, w^m)\}$. Consequently, for each $w^0 \in [1/4, 1)$, each a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X, T) starting at w^0 is $\mathcal{L}$-Cauchy on X, $\lim_{m\to\infty} w^m = 1$ and $1 \in Fix(T)$, i.e. for each $w^0 \in [1/4, 1)$, the assertion (A2) holds.

Case 3. *Let* $w^0 = 1$. Then, there exists a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X, T) starting at w^0, of the form: $w^0 = 1$, $w^1 = 1/2$, $\forall_{m\ge 2}\{w^m \in E\}$, and, by (53), $L(w^0, w^1) = d(w^0, w^1)+4$ and $\forall_{m\ge 2}\{L(w^{m-1}, w^m) = d(w^{m-1}, w^m)\}$. Consequently, this dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{L}$-Cauchy on X, $\lim_{m\to\infty} w^m = 1$ and $1 \in Fix(T)$, i.e. for $w^0 = 1$, the assertion (A2) holds.

Remark 11 One can also notice that $L(1, 1) = 3 > 0$ and $\mathbb{H}^{\mathcal{L}}_{(1)}(T(1),T(1)) = 1/2 > 0$. Indeed, we have $L(1, 1) = d(1, 1) + 3 = 3$ and

$$\begin{aligned}
&\mathbb{H}^{\mathcal{L}}_{(1)}(T(1),T(1)) \\
&= \inf \mathcal{H}^{\mathcal{L}}_{(1)}(\{1/2,1\},\{1/2,1\}) \\
&= \inf\{\eta > 0 : \{1/2,1/2\} \subset U_{\mathcal{L}}(\eta,\{1/2,1\})\} \\
&= \inf\{\eta > 0 : L(1/2,1/2) < \eta \vee L(1/2,1) < \eta \vee L(1,1/2) < \eta \vee L(1,1) < \eta\} \\
&= \inf\{\eta > 0 : d(1/2,1/2) < \eta \vee d(1/2,1) < \eta \vee d(1,1/2) + 4 < \eta \vee d(1,1) + 3 < \eta\} \\
&= \inf\{\eta > 0 : 1/2 < \eta \vee 1/2 + 4 < \eta \vee 3 < \eta\} = 1/2.
\end{aligned}$$

Concluding remarks

The Caristi [41] and Ekeland [42] results can be read, respectively, as follows.

Theorem 10 [41]*Let (X, d) be a complete metric space. Let $T : X \to X$ be a single-valued map. Let $\phi : X \to (-\infty, +\infty]$ be a map which is proper lower semicontinuous and bounded from below; we say that a map $\phi : X \to (-\infty, +\infty]$ is proper if its effective domain, $dom(\phi) = \{x : \phi(x) < +\infty\}$, is nonempty. Assume $\forall_{x\in X}\{d(x, T(x)) \le \phi(x) - \phi(T(x))\}$. Then T has a fixed point w in X, i.e. $w = T(w)$.*

Theorem 11 [42]*Let (X, d) be a complete metric space. Let $\phi : X \to (-\infty, +\infty]$ be a proper lower semicontinuous and bounded from below. Then, for every $\varepsilon > 0$ and for every $x_0 \in dom(\phi)$, there exists $w \in X$ such that: (i) $\phi(w)+\varepsilon d(x_0, w) \le \phi(x_0)$; and (ii) $\forall_{x\in X\setminus\{w\}}\{\phi(w) < \phi(x) + \varepsilon d(x, w)\}$.*

The Banach [3], Nadler [[4], Th. 5], Caristi [41], and Ekeland [42] results have extensive applications in many fields of mathematics and applied mathematics, they have been extended in many different directions and a number of authors have found their simpler proofs. Caristi's and Nadler's results yield Banach's result and Caristi's and Ekeland's results are equivalent. Jachymski [[14], Theorem 5], using a similar idea as in Takahashi [13], proved that Caristi's result yields Nadler's result.

Regarding this, we raise a question:

Question 3 *Is it possible to find some analogons of Caristi's and Ekeland's theorems in generalized uniform spaces (or in generalized locally convex spaces or in generalized metric spaces) with generalized pseudodistances, and without lower semicontinuity assumptions as in* [30]*?*

It is also natural to ask the following question:

Question 4 *What additional assumptions in Theorems 8 and 9 (and thus also in Theorems 2 and 4-7) guarantee the uniqueness of fixed points?*

Authors' Contributions

The authors have equitably contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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