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An approximation method for common fixed points of a finite family of asymptotic pointwise nonexpansive mappings

Bancha Nanjaras[1] and Bancha Panyanak[1,2*]

*Correspondence: banchap@chiangmai.ac.th
[1]Department of Mathematics, Faculty of Science, Chaing Mai University, Chiang Mai, 50200, Thailand
[2]Centre of Excellence in Mathematics, CHE, Si Ayutthaya Rd., Bangkok, 10400, Thailand

Abstract
In this article, we consider an iterative scheme to approximate a common fixed point for a finite family of asymptotic pointwise nonexpansive mappings. We obtain weak and strong convergence theorems of the proposed iteration in uniformly convex Banach spaces. The related results for complete CAT(0) spaces are also included.
MSC: 47H09; 47H10

Keywords: common fixed point; asymptotic pointwise nonexpansive mapping; weak convergence; strong convergence; Banach space; CAT(0) space

1 Introduction

It is well known that many of the most important nonlinear problems of applied mathematics reduce to solving a given equation which in turn may be reduced to finding the fixed points of a certain operator. It is important not only to know the fixed points exist, but also to be able to construct that fixed points. Lau is a great mathematician who has published many good papers concerning to the existence and the approximation of fixed points for various types of mappings (see, *e.g.*, [1–11]).

The existence of fixed points for nonexpansive mappings was studied independently by three authors in 1965 (see Browder [12], Göhde [13], and Kirk [14]). Since then the iteration methods for approximating fixed points of nonexpansive mappings has rapidly been developed and many of papers have appeared (see, *e.g.*, [15–21]). One of the popular classes of generalized nonexpansive mappings is the class of asymptotically nonexpansive mappings which was introduced by Goebel and Kirk [22] in 1972. Later on, Kirk and Xu [23] introduced the concept of asymptotic pointwise nonexpansive mappings which generalizes the concept of asymptotically nonexpansive mappings and proved the existence of fixed points for such maps in a uniformly convex Banach space. In 2011, Kozlowski [24] defined an iterative sequence for an asymptotic pointwise nonexpansive mapping T on a convex subset C of a Banach space X by $x_1 \in C$ and

$$\begin{aligned} x_{k+1} &= (1-t_k)x_k + t_k T^{n_k} y_k, \\ y_k &= (1-s_k)x_k + s_k T^{n_k} x_k, \quad k \in \mathbb{N}, \end{aligned} \tag{1}$$

where $\{t_k\}$ and $\{s_k\}$ are sequences in $[0,1]$ and $\{n_k\}$ is an increasing sequence of natural numbers. He proved, under some suitable assumptions, that the sequence $\{x_k\}$ defined by (1) converges weakly to a fixed point of T where X is a uniformly convex Banach space which satisfies the Opial condition and $\{x_k\}$ converges strongly to a fixed point of T provided T^r is a compact mapping for some $r \in \mathbb{N}$. Recently, Pasom and Panyanak [25] extended Kozlowski's results to a finite family of asymptotic pointwise nonexpansive mappings $T_1, \dots, T_m$. Precisely, they proved weak and strong convergence theorems of the iterative process defined by

$$\begin{aligned} x_{k+1} &= (1 - t_{mk})x_k + t_{mk} T_m^{n_k} y_{(m-1)k}, \\ y_{(m-1)k} &= (1 - t_{(m-1)k})x_k + t_{(m-1)k} T_{m-1}^{n_k} y_{(m-2)k}, \\ y_{(m-2)k} &= (1 - t_{(m-2)k})x_k + t_{(m-2)k} T_{m-2}^{n_k} y_{(m-3)k}, \\ &\vdots \\ y_{2k} &= (1 - t_{2k})x_k + t_{2k} T_2^{n_k} y_{1k}, \\ y_{1k} &= (1 - t_{1k})x_k + t_{1k} T_1^{n_k} y_{0k}, \\ y_{0k} &= x_k, \quad k \in \mathbb{N}, \end{aligned} \tag{2}$$

where $\{t_{ik}\}_{k=1}^{\infty}$ are sequences in $[0,1]$ for all $i = 1, 2, \dots, m$, and $\{n_k\}$ be an increasing sequence of natural numbers. On the other hand, Kettapun *et al.* [26] studied the iterative process defined by

$$\begin{aligned} x_{k+1} &= (1 - t_{mk})y_{(m-1)k} + t_{mk} T_m^{n} y_{(m-1)k}, \\ y_{(m-1)k} &= (1 - t_{(m-1)k})y_{(m-2)k} + t_{(m-1)k} T_{m-1}^{n} y_{(m-2)k}, \\ y_{(m-2)k} &= (1 - t_{(m-2)k})y_{(m-3)k} + t_{(m-2)k} T_{m-2}^{n} y_{(m-3)k}, \\ &\vdots \\ y_{2k} &= (1 - t_{2k})y_{1k} + t_{2k} T_2^{n} y_{1k}, \\ y_{1k} &= (1 - t_{1k})y_{0k} + t_{1k} T_1^{n} y_{0k}, \\ y_{0k} &= x_k, \quad k \in \mathbb{N}, \end{aligned} \tag{3}$$

where $T_1, \dots, T_m$ are asymptotically quasi-nonexpansive mappings on C.

In this article, motivated by the results mentioned above, we obtain weak and strong convergence theorems of the iterative process defined by

$$\begin{aligned} x_{k+1} &= (1 - t_{mk})y_{(m-1)k} + t_{mk} T_m^{n_k} y_{(m-1)k}, \\ y_{(m-1)k} &= (1 - t_{(m-1)k})y_{(m-2)k} + t_{(m-1)k} T_{m-1}^{n_k} y_{(m-2)k}, \\ y_{(m-2)k} &= (1 - t_{(m-2)k})y_{(m-3)k} + t_{(m-2)k} T_{m-2}^{n_k} y_{(m-3)k}, \\ &\vdots \\ y_{2k} &= (1 - t_{2k})y_{1k} + t_{2k} T_2^{n_k} y_{1k}, \\ y_{1k} &= (1 - t_{1k})y_{0k} + t_{1k} T_1^{n_k} y_{0k}, \\ y_{0k} &= x_k, \quad k \in \mathbb{N}, \end{aligned} \tag{4}$$

where $T_1, \dots, T_m$ are asymptotic pointwise nonexpansive mappings on C, $\{t_{ik}\}_{k=1}^{\infty}$ are sequences in $[0,1]$ for all $i = 1, 2, \dots, m$, and $\{n_k\}$ be an increasing sequence of natural numbers.

2 Preliminaries and lemmas

Let C be a nonempty subset of a metric space (X, d) and T be a mapping on C. A point x in C is called a *fixed point* of T if $x = Tx$. We shall denote by $F(T)$ the set of fixed points of T. The mapping $T : C \to C$ is said to be

(i) *nonexpansive* if $d(Tx, Ty) \leq (x, y)$ for all $x, y \in C$,

(ii) *asymptotically nonexpansive* if there is a sequence $\{k_n\}$ of positive numbers with the property $\lim_{n\to\infty} k_n = 1$ and such that

$$d\big(T^n x, T^n y\big) \leq k_n d(x, y), \quad \text{for all } x, y \in C \text{ and } n \geq 1,$$

(iii) *asymptotically quasi-nonexpansive* if there is a sequence $\{k_n\}$ of positive numbers with the property $\lim_{n\to\infty} k_n = 1$ and such that

$$d\big(T^n x, p\big) \leq k_n d(x, p), \quad \text{for all } x \in C, p \in F(T) \text{ and } n \geq 1,$$

(iv) *asymptotic pointwise nonexpansive* if there exists a sequence of functions $\alpha_n : C \to [0, \infty)$ such that $\limsup_{n\to\infty} \alpha_n(x) \leq 1$ and

$$d\big(T^n x, T^n y\big) \leq \alpha_n(x) d(x, y), \quad \text{for all } x, y \in C \text{ and } n \geq 1.$$

The following implications hold.

$$\begin{array}{c} T \text{ is nonexpansive} \Rightarrow T \text{ is asymptotically nonexpansive} \Rightarrow T \text{ is asymptotically quasi-nonexpansive} \\ \Downarrow \\ T \text{ is asymptotic pointwise nonexpansive} \end{array}$$

The existence of fixed points for asymptotic pointwise nonexpansive mappings in uniformly convex Banach spaces was proved by Kirk and Xu [23] as the following result.

Theorem 2.1 *Let C be a nonempty bounded closed and convex subset of a uniformly convex Banach space X. Then every asymptotic pointwise nonexpansive mapping $T : C \to C$ has a fixed point. Moreover, $F(T)$ is closed and convex.*

For common fixed points of a family of commuting mappings, Pasom and Panyanak [27] obtained the following result.

Theorem 2.2 *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Then every commuting family $\mathcal{S}$ of asymptotic pointwise nonexpansive mappings on C has a nonempty closed convex common fixed point set.*

Let C be a nonempty subset of a metric space (X, d). We shall denote by $\mathcal{T}(C)$ the class of all asymptotic pointwise nonexpansive mappings from C into C. Let $T_1, \ldots, T_m \in \mathcal{T}(C)$, without loss of generality, we can assume that there exists a sequence of mappings $\alpha_n : C \to [0, \infty)$ such that for all $x, y \in C$, $i = 1, \ldots, m$, and $n \in \mathbb{N}$,

$$d\big(T_i^n x, T_i^n y\big) \leq \alpha_n(x) d(x, y) \quad \text{and} \quad \limsup_{n\to\infty} \alpha_n(x) \leq 1. \tag{5}$$

Let $a_n(x) = \max\{\alpha_n(x), 1\}$. Again, without loss of generality, we can assume that

$$d\big(T_i^n x, T_i^n y\big) \leq a_n(x) d(x, y), \qquad \lim_{n\to\infty} a_n(x) = 1 \quad \text{and} \quad a_n(x) \geq 1, \tag{6}$$

for all $x, y \in C$, $i = 1, \ldots, m$, and $n \in \mathbb{N}$. We define $b_n(x) = a_n(x) - 1$, then for each $x \in C$ we have $\lim_{n\to\infty} b_n(x) = 0$.

Definition 2.3 [24] Define $\mathcal{T}_r(C)$ as a class of all $T \in \mathcal{T}(C)$ such that

$$\sum_{n=1}^{\infty} b_n(x) < \infty \quad \text{for every } x \in C, \quad \text{and} \tag{7}$$

$$a_n \text{ is a bounded function for every } n \in \mathbb{N}. \tag{8}$$

Let C be a nonempty subset of a Banach space X and $T_1, \ldots, T_m \in \mathcal{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset (0,1)$ be bounded away from 0 and 1 for all $i = 1, 2, \ldots, m$ and $\{n_k\}$ be an increasing sequence of natural numbers. Let $x_1 \in C$ and define a sequence $\{x_k\}$ in C as

$$\begin{aligned} x_{k+1} &= (1 - t_{mk})y_{(m-1)k} + t_{mk}T_m^{n_k}y_{(m-1)k}, \\ y_{(m-1)k} &= (1 - t_{(m-1)k})y_{(m-2)k} + t_{(m-1)k}T_{m-1}^{n_k}y_{(m-2)k}, \\ y_{(m-2)k} &= (1 - t_{(m-2)k})y_{(m-3)k} + t_{(m-2)k}T_{m-2}^{n_k}y_{(m-3)k}, \\ &\vdots \\ y_{2k} &= (1 - t_{2k})y_{1k} + t_{2k}T_2^{n_k}y_{1k}, \\ y_{1k} &= (1 - t_{1k})y_{0k} + t_{1k}T_1^{n_k}y_{0k}, \\ y_{0k} &= x_k, \quad k \in \mathbb{N}. \end{aligned} \tag{9}$$

We say that the sequence $\{x_k\}$ in (9) is well defined if $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$. As in [24], we observe that $\lim_{k\to\infty} a_k(x) = 1$ for every $x \in C$. Hence, we can always choose a subsequence $\{a_{n_k}\}$ which makes $\{x_k\}$ well defined.

Definition 2.4 A strictly increasing sequence $\{n_k\} \subset \mathbb{N}$ is called *quasi-periodic* if the sequence $\{n_{k+1} - n_k\}$ is bounded, or equivalently if there exists a number $p \in \mathbb{N}$ such that any block of p consecutive natural numbers must contain a term of the sequence $\{n_k\}$. The smallest of such numbers p will be called a quasi-period of $\{n_k\}$.

Recall that a mapping $T: C \to C$ is called *semi-compact* if for any sequence $\{x_n\}$ in C such that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $q \in C$ such that $\lim_{j\to\infty} x_{n_j} = q$. A family of mapping $\{T_i : i = 1, 2, \ldots, m\}$ on C is said to satisfy *Condition* (A'') if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, T_jx) \geq f(\operatorname{dist}(x, F))$, for some $j = 1, \ldots, m$ for all $x \in C$, where $\operatorname{dist}(x, F) = \inf\{d(x,p) : p \in F = \bigcap_{i=1}^m F(T_i)\}$.

Lemma 2.5 [28, Lemma 2.2] *Let $\{s_n\}$ and $\{u_n\}$ be sequences of nonnegative real numbers satisfy:*

$$s_{n+1} \leq (1 + u_n)s_n, \quad \textit{for all } n \in \mathbb{N}, \quad \textit{and} \quad \sum_{n=1}^{\infty} u_n < \infty.$$

Then (i) $\lim_n s_n$ *exists (ii) if* $\liminf_n s_n = 0$, *then* $\lim_n s_n = 0$.

Lemma 2.6 [29, Lemma 1] *Suppose $\{r_k\}$ is a bounded sequence of real numbers and $\{d_{k,l}\}$ is a doubly index sequence of real numbers which satisfy*

$$\limsup_{k\to\infty} \limsup_{l\to\infty} d_{k,l} \leq 0, \quad \textit{and} \quad r_{k+l} \leq r_k + d_{k,l}$$

for each $k, l \in \mathbb{N}$. Then $\lim_{k\to\infty} r_k = a$ for some $a \in \mathbb{R}$.

Lemma 2.7 [30, 31] *Let X be a uniformly convex Banach space and let $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0,1)$. Suppose that $\{u_n\}$ and $\{v_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} \|u_n\| \leq r, \qquad \limsup_{n\to\infty} \|v_n\| \leq r, \quad \textit{and} \quad \lim_{n\to\infty} \|t_nu_n + (1 - t_n)v_n\| = r,$$

for some $r \geq 0$. Then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

Lemma 2.8 [24, Lemma 3.1] *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $T \in \mathcal{T}_r(C)$. If $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ then for any $m \in \mathbb{N}$, $\lim_{n\to\infty}\|x_n - T^m x_n\| = 0$.*

Lemma 2.9 [24, Theorem 3.1] *Let X be a uniformly convex Banach space with the Opial property and let C be a nonempty closed convex subset of X. Let $T \in \mathcal{T}_r(C)$ and let $\omega \in X$, $\{x_n\} \subset X$ be such that weak-$\lim_{n\to\infty} x_n = \omega$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Then $\omega \in F(T)$.*

3 Results in Banach spaces

3.1 Results for bounded domains

Recall that a subset C of a metric space (X,d) is said to be *bounded* if

$$\operatorname{diam}(C) := \sup\{d(x,y) : x,y \in C\} < \infty.$$

Lemma 3.1 *Let C be a nonempty closed convex subset of a Banach space X and $T_1,\ldots,T_m \in \mathcal{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [0,1]$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (9) is well defined. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Then for each $p \in F$, there are sequences of nonnegative real numbers $\{\gamma_k\}$ and $\{\delta_k\}$ (depending on p) such that $\sum_{k=1}^{\infty}\gamma_k < \infty$, $\sum_{k=1}^{\infty}\delta_k < \infty$ and the following statements hold:*

- *(i) $\|T_i^{n_k} y_{(i-1)k} - p\| \le (1+\gamma_k)\|y_{(i-1)k} - p\|$, for all $i = 1,\ldots,m$;*
- *(ii) $\|y_{ik} - p\| \le (1+\gamma_k)^i\|x_k - p\|$, for all $i = 1,\ldots,m-1$;*
- *(iii) $\|x_{k+1} - p\| \le (1+\delta_k)\|x_k - p\|$;*
- *(iv) if C is bounded, then $\lim_{k\to\infty}\|x_k - p\|$ exists.*

Proof Let $p \in F$ and $\gamma_k = b_{n_k}(p)$ for all $k \in \mathbb{N}$. Then $\sum_{k=1}^{\infty}\gamma_k < \infty$.

(i) For $i = 1,2,\ldots,m$, we have

$$\|T_i^{n_k} y_{(i-1)k} - p\| \le (1+\gamma_k)\|y_{(i-1)k} - p\|.$$

(ii) By (9), we obtain

$$\begin{aligned}
\|y_{1k} - p\| &= \|(1-t_{1k})(x_k - p) + t_{1k}(T_1^{n_k}x_k - p)\| \\
&\le (1-t_{1k})\|x_k - p\| + t_{1k}\|T_1^{n_k}x_k - p\| \\
&\le (1-t_{1k})\|x_k - p\| + t_{1k}(1+\gamma_k)\|x_k - p\| \\
&\le (1+\gamma_k)\|x_k - p\|.
\end{aligned}$$

We assume that $\|y_{jk} - p\| \le (1+\gamma_k)^j\|x_k - p\|$ holds for some $j = 1,\ldots,m-2$. From part (i), we have

$$\begin{aligned}
\|y_{(j+1)k} - p\| &= \|(1-t_{(j+1)k})(y_{jk} - p) + t_{(j+1)k}(T_{j+1}^{n_k}y_{jk} - p)\| \\
&\le (1-t_{(j+1)k})\|y_{jk} - p\| + t_{(j+1)k}\|T_{j+1}^{n_k}y_{jk} - p\| \\
&\le (1-t_{(j+1)k})\|y_{jk} - p\| + t_{(j+1)k}(1+\gamma_k)\|y_{jk} - p\| \\
&\le (1+\gamma_k)\|y_{jk} - p\| \\
&\le (1+\gamma_k)(1+\gamma_k)^j\|x_k - p\| \\
&= (1+\gamma_k)^{j+1}\|x_k - p\|.
\end{aligned}$$

By mathematical induction, we obtain

$$\|y_{ik} - p\| \le (1+\gamma_k)^i\|x_k - p\|, \quad \text{for all } i = 1,\ldots,m-1.$$

(iii) By part (ii), we get

$$\begin{aligned}
\|x_{k+1} - p\| &= \left\|(1 - t_{mk})(y_{(m-1)k} - p) + t_{mk}\left(T_m^{n_k} y_{(m-1)k} - p\right)\right\| \\
&\le (1 - t_{mk})\|y_{(m-1)k} - p\| + t_{mk}\left\|T_m^{n_k} y_{(m-1)k} - p\right\| \\
&\le (1 - t_{mk})\|y_{(m-1)k} - p\| + t_{mk}(1 + \gamma_k)\|y_{(m-1)k} - p\| \\
&\le (1 + \gamma_k)\|y_{(m-1)k} - p\| \\
&\le (1 + \gamma_k)(1 + \gamma_k)^{m-1}\|x_k - p\| \\
&\le (1 + \gamma_k)^m\|x_k - p\| \\
&\le (1 + \delta_k)\|x_k - p\|,
\end{aligned}$$

where $\delta_k = \binom{m}{1}\gamma_k + \binom{m}{2}\gamma_k^2 + \cdots + \binom{m}{m}\gamma_k^m$. Since $\sum_{k=1}^{\infty}\gamma_k < \infty$, then $\sum_{k=1}^{\infty}\delta_k < \infty$.

(iv) By part (iii), we have $\|x_{k+1} - p\| \le \|x_k - p\| + \operatorname{diam}(C)\delta_k$ for all $k \in \mathbb{N}$. Thus, for each $l \in \mathbb{N}$,

$$\|x_{k+l} - p\| \le \|x_k - p\| + \operatorname{diam}(C)\sum_{i=k}^{k+l-1}\delta_i.$$

Since $\sum_{i=1}^{\infty}\delta_i < \infty$, $\limsup_{k\to\infty}\limsup_{l\to\infty}\sum_{i=k}^{k+l-1}\delta_i = 0$. The conclusion follows from Lemma 2.6 by letting $r_k = \|x_k - p\|$ and $d_{k,l} = \operatorname{diam}(C)\sum_{i=k}^{k+l-1}\delta_i$. □

Lemma 3.2 *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and $T_1, \dots, T_m \in \mathcal{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [a,b] \subset (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (9) is well defined. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Then*

(i) $\lim_{k\to\infty}\|y_{(i-1)k} - T_i^{n_k} y_{(i-1)k}\| = 0$, for all $i = 1, 2, \dots, m$;

(ii) $\lim_{k\to\infty}\|x_k - T_i^{n_k} y_{(i-1)k}\| = 0$, for all $i = 1, 2, \dots, m$;

(iii) If the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then $\lim_{k\to\infty}\|x_k - T_i x_k\| = 0$, for all $i = 1, 2, \dots, m$.

Proof (i) Let $p \in F$, then by Lemma 3.1(iv) we have $\lim_{k\to\infty}\|x_k - p\|$ exists. Let

$$\lim_{k\to\infty}\|x_k - p\| = c. \tag{10}$$

By (10) and Lemma 3.1(ii), we get that

$$\limsup_{k\to\infty}\|y_{ik} - p\| \le c, \quad \text{for all } i = 1, \dots, m-1. \tag{11}$$

Note that

$$\begin{aligned}
\|x_{k+1} - p\| &\le (1 - t_{mk})\|y_{(m-1)k} - p\| + t_{mk}\left\|T_m^{n_k} y_{(m-1)k} - p\right\| \\
&\le (1 - t_{mk})\|y_{(m-1)k} - p\| + t_{mk}(1 + \gamma_k)\|y_{(m-1)k} - p\| \\
&\le (1 + \gamma_k)\|y_{(m-1)k} - p\| \\
&= (1 + \gamma_k)\left\|(1 - t_{(m-1)k})(y_{(m-2)k} - p) + t_{(m-1)k}\left(T_{m-1}^{n_k} y_{(m-2)k} - p\right)\right\| \\
&\le (1 + \gamma_k)\left((1 - t_{(m-1)k})\|y_{(m-2)k} - p\| + t_{(m-1)k}(1 + r_k)\|y_{(m-2)k} - p\|\right) \\
&\le (1 + \gamma_k)^2\|y_{(m-2)k} - p\| \\
&\;\;\vdots \\
&\le (1 + \gamma_k)^{m-i}\|y_{ik} - p\|,
\end{aligned}$$

for all $i = 1, \dots, m-1$. So that

$$c \le \liminf_{k\to\infty}\|y_{ik} - p\|, \quad \text{for all } i = 1, \dots, m-1. \tag{12}$$

From (11) and (12), we have

$$\lim_{k\to\infty} \|y_{ik} - p\| = c, \quad \text{for all } i = 1, 2, \ldots, m-1. \tag{13}$$

That is,

$$\lim_{k\to\infty} \left\|(1 - t_{ik})(y_{(i-1)k} - p) + t_{ik}\left(T_i^{n_k} y_{(i-1)k} - p\right)\right\| = c, \quad \text{for all } i = 1, 2, \ldots, m-1. \tag{14}$$

By Lemma 3.1(i) and (13), we get that

$$\limsup_{k\to\infty} \left\|T_i^{n_k} y_{(i-1)k} - p\right\| \le c, \quad \text{for all } j = 1, 2, \ldots, m-1. \tag{15}$$

By (11), (14), (15), and Lemma 2.7, we obtain

$$\lim_{k\to\infty} \left\|y_{(i-1)k} - T_i^{n_k} y_{(i-1)k}\right\| = 0, \quad \text{for all } i = 1, 2, \ldots, m-1. \tag{16}$$

For the case $i = m$, by Lemma 3.1(i), we have

$$\left\|T_m^{n_k} y_{(m-1)k} - p\right\| \le (1 + \gamma_k)\|y_{(m-1)k} - p\|.$$

This implies by (13) that

$$\limsup_{k\to\infty} \left\|T_m^{n_k} y_{(m-1)k} - p\right\| \le c. \tag{17}$$

Moreover,

$$\lim_{k\to\infty} \left\|(1 - t_{mk})(y_{(m-1)k} - p) + t_{mk}\left(T_m^{n_k} y_{(m-1)k} - p\right)\right\| = \lim_{k\to\infty} \|x_{k+1} - p\| = c.$$

Again, by Lemma 2.7, we get that

$$\lim_{k\to\infty} \left\|y_{(m-1)k} - T_m^{n_k} y_{(m-1)k}\right\| = 0. \tag{18}$$

Thus, (16) and (18) imply that

$$\lim_{k\to\infty} \left\|y_{(i-1)k} - T_i^{n_k} y_{(i-1)k}\right\| = 0, \quad \text{for all } i = 1, \ldots, m. \tag{19}$$

(ii) From (9), we have

$$\|y_{ik} - y_{(i-1)k}\| = t_{ik}\left\|T_i^{n_k} y_{(i-1)k} - y_{(i-1)k}\right\|, \quad \text{for all } i = 1, \ldots, m-1.$$

By (19), we obtain

$$\lim_{k\to\infty} \|y_{ik} - y_{(i-1)k}\| = 0, \quad \text{for } i = 1, \ldots, m-1. \tag{20}$$

From

$$\|x_k - y_{ik}\| \le \|x_k - y_{1k}\| + \|y_{1k} - y_{2k}\| + \cdots + \|y_{(i-1)k} - y_{ik}\|, \quad \text{for all } i = 1, \ldots, m-1,$$

it follows by (20) that

$$\lim_{k\to\infty} \|x_k - y_{ik}\| = 0, \quad \text{for all } i = 1, \ldots, m-1. \tag{21}$$

From

$$\left\|x_k - T_i^{n_k} y_{(i-1)k}\right\| \le \|x_k - y_{(i-1)k}\| + \left\|y_{(i-1)k} - T_i^{n_k} y_{(i-1)k}\right\|,$$

it implies by (19) and (21) that

$$\lim_{k\to\infty} \left\|x_k - T_i^{n_k} y_{(i-1)k}\right\| = 0, \quad \text{for all } i = 1, 2, \ldots, m. \tag{22}$$

(iii) For $i = 1$, from (ii) we have

$$\lim_{k\to\infty} \left\|T_1^{n_k} x_k - x_k\right\| = 0. \tag{23}$$

If $i = 2, 3, \ldots, m$, then

$$\begin{aligned}\left\| T_i^{n_k} x_k - x_k \right\| &\le \left\| T_i^{n_k} x_k - T_i^{n_k} y_{(i-1)k} \right\| + \left\| T_i^{n_k} y_{(i-1)k} - x_k \right\| \\ &\le a_{n_k}(x_k) \| x_k - y_{(i-1)k} \| + \left\| T_i^{n_k} y_{(i-1)k} - x_k \right\|.\end{aligned}$$

By (21), (22), and $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$, we get

$$\limsup_{k\to\infty} \left\| T_i^{n_k} x_k - x_k \right\| = 0 \quad \text{for all } i = 2, 3, \ldots, m. \tag{24}$$

By (23) and (24), we have

$$\lim_{k\to\infty} \left\| T_i^{n_k} x_k - x_k \right\| = 0 \quad \text{for all } i = 1, 2, \ldots, m. \tag{25}$$

From (9), we have

$$\begin{aligned}\| x_{k+1} - x_k \| &\le (1 - t_{mk}) \| y_{(m-1)k} - x_k \| + t_{mk} \left\| T_m^{n_k} y_{(m-1)k} - x_k \right\| \\ &\le (1 - t_{mk}) \| y_{(m-1)k} - x_k \| + t_{mk} \left(\left\| T_m^{n_k} y_{(m-1)k} - y_{(m-1)k} \right\| + \| y_{(m-1)k} - x_k \| \right) \\ &= \| y_{(m-1)k} - x_k \| + t_{mk} \left\| T_m^{n_k} y_{(m-1)k} - y_{(m-1)k} \right\|.\end{aligned}$$

From (19) and (21),

$$\lim_{k\to\infty} \| x_{k+1} - x_k \| = 0. \tag{26}$$

The proof of the remaining part is identical to the proof of [25, Lemma 4.8(iii)] upon replacing $d(\cdot,\cdot)$ with $\|\cdot\|$. □

By using Lemma 3.1 and the argument in the proof of [26, Theorem 3.2], we can obtain the following result.

Lemma 3.3 *Let C be a nonempty bounded closed convex subset of a Banach space X and $T_1, \ldots, T_m \in \mathcal{T}_r(C)$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [0,1]$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (9) is well defined. Assume that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Then $\{x_k\}$ converges strongly to a point in F if and only if $\liminf_{k\to\infty} \operatorname{dist}(x_k, F) = 0$.*

Theorem 3.4 *Let X be a uniformly convex Banach space with the Opial property and C be a nonempty bounded closed convex subset of X. Let $T_1, \ldots, T_m \in \mathcal{T}_r(C)$ be such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. $\{t_{ik}\}_{k=1}^{\infty} \subset [a,b] \subset (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (9) is well defined. If the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then the sequence $\{x_k\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, \ldots, m\}$.*

Proof We have by Lemma 3.1 that $\lim_{n\to\infty} \| x_k - p \|$ exists for every $p \in F$. We shall prove that $\{x_k\}$ has a unique weak subsequential limit in F. For this, we suppose that there are subsequences $\{x_{k_l}\}$ and $\{x_{k_j}\}$ of $\{x_k\}$ which converge weakly to u and v, respectively. By Lemma 3.2(iii), $\lim_{k\to\infty} \| T_i x_k - x_k \| = 0$ for all $i = 1, \ldots, m$. It follows from Lemma 2.9 that $u, v \in F(T_i)$ for all $i = 1, \ldots, m$. That is $u, v \in F$. Finally, we prove that $u = v$. Suppose not, then by the Opial property we get that

$$\begin{aligned}\lim_{k\to\infty} \| x_k - u \| &= \lim_{l\to\infty} \| x_{k_l} - u \| \\ &< \lim_{l\to\infty} \| x_{k_l} - v \| \\ &= \lim_{k\to\infty} \| x_k - v \| \\ &= \lim_{j\to\infty} \| x_{k_j} - v \|\end{aligned}$$

$$
\begin{aligned}
&< \lim_{j\to\infty} \|x_{k_j} - u\| \\
&= \lim_{k\to\infty} \|x_k - u\|.
\end{aligned}
$$

This is a contradiction. Therefore, the proof is complete. □

Theorem 3.5 *Let X be a uniformly convex Banach space and C be a nonempty bounded closed convex subset of X. Let $T_1, \ldots, T_m \in \mathcal{T}_r(C)$ be such that T_i^l is semi-compact for some $i \in \{1, \ldots, m\}$ and $l \in \mathbb{N}$. $\{t_{ik}\}_{k=1}^{\infty} \subset [a,b] \subset (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (9) is well defined. Suppose that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic. Then $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$.*

Proof By Lemma 3.2, we have

$$\lim_{k\to\infty} \|x_k - T_i x_k\| = 0, \quad \text{for all } i = 1, \ldots, m. \tag{27}$$

Let $i \in \{1, \ldots, m\}$ be such that T_i^l is semi-compact. Thus, by Lemma 2.8,

$$\lim_{k\to\infty} \left\| x_k - T_i^l x_k \right\| = 0.$$

We can also find a subsequence $\{x_{n_j}\}$ of $\{x_k\}$ such that $\lim_{j\to\infty} x_{k_j} = q \in C$. Hence, from (27), we have

$$\|q - T_i q\| = \lim_{j\to\infty} \|x_{k_j} - T_i x_{k_j}\| = 0, \quad \text{for all } i = 1, \ldots, m.$$

Thus $q \in F$. Therefore, $\{x_{k_j}\}$ converges strongly to $q \in F$. But since $\lim_{k\to\infty} \|x_k - q\|$ exists, $\{x_k\}$ must itself converges to q. This completes the proof. □

Theorem 3.6 *Let X be a uniformly convex Banach space and C be a nonempty bounded closed convex subset of X. Let $\{T_1, \ldots, T_m\} \subset \mathcal{T}_r(C)$ be satisfy Condition (A''). Let $\{t_{ik}\}_{k=1}^{\infty} \subset [a,b] \subset (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (9) is well defined. Suppose that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic. Then $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$.*

Proof By Lemma 3.2, $\lim_{k\to\infty} \|x_k - T_i x_k\| = 0$, for all $i = 1, 2, \ldots, m$. By using Condition (A''), there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$ such that

$$\lim_{k\to\infty} f\big(\operatorname{dist}(x_k, F)\big) \leq \lim_{k\to\infty} \|x_k - T_j x_k\| = 0 \quad \text{for some } j = 1, \ldots, m.$$

This implies that $\lim_{k\to\infty} \operatorname{dist}(x_k, F) = 0$. The conclusion follows from Lemma 3.3. □

3.2 Results for unbounded domains

To relax the boundedness of the domains we have to add some condition on the sequence $\{b_{n_k}\}$.

Lemma 3.7 *Let C be a nonempty closed convex subset of a Banach space X and $T_1, \ldots, T_m \in \mathcal{T}_r(C)$ be such that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [0,1]$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (9) is well defined. Assume that $\sum_{k=1}^{\infty} \sup_{x\in C} b_{n_k}(x) < \infty$. Then for $p \in F$, we have $\lim_{k\to\infty} \|x_k - p\|$ exists.*

Proof Similar to the proof of Lemma 3.1, we can show that $\|x_{k+1} - p\| \leq (1 + \eta_k)\|x_k - p\|$ for all $k \in \mathbb{N}$, where $\eta_k = \binom{m}{1}s_k + \binom{m}{2}s_k^2 + \cdots + \binom{m}{m}s_k^m$ and $s_k = \sup_{x\in C} b_{n_k}(x)$. By assumption, we have $\sum_{k=1}^{\infty} s_k^i < \infty$ for all $i = 1, \ldots, m$. It follows that $\sum_{k=1}^{\infty} \eta_k < \infty$. By Lemma 2.5, we get that $\lim_{k\to\infty} \|x_k - p\|$ exists. □

By using Lemma 3.7 and the argument in Section 3.1 we can obtain the following results.

Lemma 3.8 *Let C be a nonempty closed convex subset of a Banach space X and $T_1, \ldots, T_m \in \mathcal{T}_r(C)$ be such that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [0,1]$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (9) is well defined. Assume that $\sum_{k=1}^{\infty} \sup_{x\in C} b_{n_k}(x) < \infty$. Then*

(i) $\lim_{k\to\infty} \|y_{(i-1)k} - T_i^{n_k} y_{(i-1)k}\| = 0$, for all $i = 1, 2, \ldots, m$;

(ii) $\lim_{k\to\infty} \|x_k - T_i^{n_k} y_{(i-1)k}\| = 0$, for all $i = 1, 2, \ldots, m$;

(iii) If the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then $\lim_{k\to\infty} \|x_k - T_i x_k\| = 0$, for all $i = 1, 2, \ldots, m$.

Lemma 3.9 *Let C be a nonempty closed convex subset of a Banach space X and $T_1, \ldots, T_m \in \mathcal{T}_r(C)$ be such that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [0,1]$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (9) is well defined. Assume that $\sum_{k=1}^{\infty} \sup_{x\in C} b_{n_k}(x) < \infty$. Then $\{x_k\}$ converges strongly to a point in F if and only if $\liminf_{k\to\infty} \operatorname{dist}(x_k, F) = 0$.*

Theorem 3.10 *Let X be a uniformly convex Banach space with the Opial property and C be a nonempty closed convex subset of X. Let $T_1, \ldots, T_m \in \mathcal{T}_r(C)$ be such that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$. $\{t_{ik}\}_{k=1}^{\infty} \subset [a,b] \subset (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (9) is well defined. Assume that $\sum_{k=1}^{\infty} \sup_{x\in C} b_{n_k}(x) < \infty$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic. Then the sequence $\{x_k\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, \ldots, m\}$.*

Theorem 3.11 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T_1, \ldots, T_m \in \mathcal{T}_r(C)$ be such that T_i^l is semi-compact for some $i \in \{1, \ldots, m\}$ and $l \in \mathbb{N}$, $\{t_{ik}\}_{k=1}^{\infty} \subset [a,b] \subset (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (9) is well defined. Suppose that $\sum_{k=1}^{\infty} \sup_{x\in C} b_{n_k}(x) < \infty$, $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic. Then $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$.*

Theorem 3.12 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $\{T_1, \ldots, T_m\} \subset \mathcal{T}_r(C)$ be satisfy Condition (A''). Let $\{t_{ik}\}_{k=1}^{\infty} \subset [a,b] \subset (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (9) is well defined. Suppose that $\sum_{k=1}^{\infty} \sup_{x\in C} b_{n_k}(x) < \infty$, $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic. Then $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$.*

4 Results in CAT(0) spaces

A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [32]), $\mathbb{R}$-trees (see [33]), Euclidean buildings (see [34]), the complex Hilbert ball with a hyperbolic metric (see [35]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [32].

Let $x, y \in X$, by Lemma 2.1(iv) of [36] for each $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that

$$d(x,z) = td(x,y) \quad \text{and} \quad d(y,z) = (1-t)d(x,y). \tag{28}$$

From now on, we will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (28).

Let $\{x_n\}$ be a bounded sequence in a metric space (X,d). For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [37] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point. We now give the definition of Δ-convergence.

Definition 4.1 [38, 39] A sequence $\{x_n\}$ in a metric space X is said to Δ-*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and call x the Δ-*limit* of $\{x_n\}$.

Let C be a nonempty closed convex subset of a CAT(0) space X and fix $x_1 \in C$. Define a sequence $\{x_k\}$ in C as

$$\begin{aligned}
x_{k+1} &= (1-t_{mk})y_{(m-1)k} \oplus t_{mk} T_m^{n_k} y_{(m-1)k}, \\
y_{(m-1)k} &= (1-t_{(m-1)k})y_{(m-2)k} \oplus t_{(m-1)k} T_{m-1}^{n_k} y_{(m-2)k}, \\
y_{(m-2)k} &= (1-t_{(m-2)k})y_{(m-3)k} \oplus t_{(m-2)k} T_{m-2}^{n_k} y_{(m-3)k}, \\
&\vdots \\
y_{2k} &= (1-t_{2k})y_{1k} \oplus t_{2k} T_2^{n_k} y_{1k}, \\
y_{1k} &= (1-t_{1k})y_{0k} \oplus t_{1k} T_1^{n_k} y_{0k}, \\
y_{0k} &= x_k, \quad k \in \mathbb{N},
\end{aligned} \tag{29}$$

where $T_1, \dots, T_m \in \mathcal{T}(C)$, $\{t_{ik}\}_{k=1}^{\infty}$ are sequences in $[0,1]$ for all $i = 1, 2, \dots, m$, and $\{n_k\}$ be an increasing sequence of natural numbers.

By using the argument in Section 3 together with the results in [25, 36, 40, 41], we can also obtain the analogous results for CAT(0) spaces.

Theorem 4.2 *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T_1, \dots, T_m \in \mathcal{T}_r(C)$ be such that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$, $\{t_{ik}\}_{k=1}^{\infty} \subset [a,b] \subset (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (29) is well defined. Suppose that either C is bounded or $\sum_{k=1}^{\infty} \sup_{x\in C} b_{n_k}(x) < \infty$. If the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then the sequence $\{x_k\}$ Δ-converges to a common fixed point of the family $\{T_i : i = 1, \dots, m\}$.*

Theorem 4.3 *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T_1, \dots, T_m \in \mathcal{T}_r(C)$ be such that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and T_i^l is semi-compact for some*

$i \in \{1,\dots,m\}$ and $l \in \mathbb{N}$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [a,b] \subset (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (29) is well defined. Suppose that either C is bounded or $\sum_{k=1}^{\infty} \sup_{x\in C} b_{n_k}(x) < \infty$. If the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1,2,\dots,m\}$.

Theorem 4.4 *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{T_1,\dots,T_m\} \subset \mathcal{T}_r(C)$ be satisfy Condition (A″) and $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{t_{ik}\}_{k=1}^{\infty} \subset [a,b] \subset (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (29) is well defined. Suppose that either C is bounded or $\sum_{k=1}^{\infty} \sup_{x\in C} b_{n_k}(x) < \infty$. If the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1,2,\dots,m\}$.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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