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# A new iterative algorithm for solving common solutions of generalized mixed equilibrium problems, fixed point problems and variational inclusion problems with minimization problems

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## Abstract

In this article, we introduce a new general iterative method for solving a common element of the set of solutions of fixed point for nonexpansive mappings, the set of solutions of generalized mixed equilibrium problems and the set of solutions of the variational inclusion for a  $\beta$ -inverse-strongly monotone mapping in a real Hilbert space. We prove that the sequence converges strongly to a common element of the above three sets under some mild conditions. Our results improve and extend the corresponding results of Marino and Xu (J. Math. Anal. Appl. 318:43-52, 2006), Su *et al.* (Nonlinear Anal. 69:2709-2719, 2008), Tan and Chang (Fixed Point Theory Appl. 2011:915629, 2011) and some authors.

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## 1 Introduction

Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$  with the inner product  $\langle \cdot, \cdot \rangle$  and the norm  $\| \cdot \|$ , respectively. A mapping  $S : C \rightarrow C$  is said to be *nonexpansive* if  $\|Sx - Sy\| \leq \|x - y\|$ ,  $\forall x, y \in C$ . If  $C$  is bounded closed convex and  $S$  is a nonexpansive mapping of  $C$  into itself, then  $F(S) := \{x \in C : Sx = x\}$  is nonempty [1]. A mapping  $S : C \rightarrow C$  is said to be a *k*-strictly pseudo-contraction [2] if there exists  $0 \leq k < 1$  such that  $\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2$ ,  $\forall x, y \in C$ , where  $I$  denotes the identity operator on  $C$ . We denote weak convergence and strong convergence by notations  $\rightharpoonup$  and  $\rightarrow$ , respectively. A mapping  $A$  of  $C$  into  $H$  is called *monotone* if  $\langle Ax - Ay, x - y \rangle \geq 0$ ,  $\forall x, y \in C$ . A mapping  $A$  is called  *$\alpha$ -inverse-strongly monotone* if there exists a positive real number  $\alpha$  such that  $\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2$ ,  $\forall x, y \in C$ . A mapping  $A$  is called  *$\alpha$ -strongly monotone* if there exists a positive real number  $\alpha$  such that  $\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2$ ,  $\forall x, y \in C$ . It is obvious that any  $\alpha$ -inverse-strongly monotone mappings  $A$  is a monotone and  $\frac{1}{\alpha}$ -Lipschitz continuous mapping. A linear bounded operator  $A$  is called *strongly positive* if there exists a constant  $\bar{\gamma} > 0$  with the property  $\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2$ ,  $\forall x \in H$ . A self

mapping  $f : C \rightarrow C$  is called *contraction* on  $C$  if there exists a constant  $\alpha \in (0, 1)$  such that  $\|f(x) - f(y)\| \leq \alpha \|x - y\|, \forall x, y \in C$ .

Let  $B : H \rightarrow H$  be a single-valued nonlinear mapping and  $M : H \rightarrow 2^H$  be a set-valued mapping. The *variational inclusion problem* is to find  $x \in H$  such that

$$\theta \in B(x) + M(x), \tag{1.1}$$

where  $\theta$  is the zero vector in  $H$ . The set of solutions of (1.1) is denoted by  $I(B, M)$ . The variational inclusion has been extensively studied in the literature. See, e.g. [3–10] and the reference therein.

A set-valued mapping  $M : H \rightarrow 2^H$  is called *monotone* if  $\forall x, y \in H, f \in M(x)$  and  $g \in M(y)$  imply  $\langle x - y, f - g \rangle \geq 0$ . A monotone mapping  $M$  is *maximal* if its graph  $G(M) := \{(f, x) \in H \times H : f \in M(x)\}$  of  $M$  is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping  $M$  is maximal if and only if for  $(x, f) \in H \times H, \langle x - y, f - g \rangle \geq 0$  for all  $(y, g) \in G(M)$  imply  $f \in M(x)$ .

Let  $B$  be an inverse-strongly monotone mapping of  $C$  into  $H$  and let  $N_C v$  be normal cone to  $C$  at  $v \in C$ , i.e.,  $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}$ , and define

$$Mv = \begin{cases} Bv + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then  $M$  is a maximal monotone and  $\theta \in Mv$  if and only if  $v \in VI(C, B)$  (see [11]).

Let  $M : H \rightarrow 2^H$  be a set-valued maximal monotone mapping, then the single-valued mapping  $J_{M,\lambda} : H \rightarrow H$  defined by

$$J_{M,\lambda}(x) = (I + \lambda M)^{-1}(x), \quad x \in H \tag{1.2}$$

is called the *resolvent operator* associated with  $M$ , where  $\lambda$  is any positive number and  $I$  is the identity mapping. In the worth mentioning that the resolvent operator is nonexpansive, 1-inverse-strongly monotone and that a solution of problem (1.1) is a fixed point of the operator  $J_{M,\lambda}(I - \lambda B)$  for all  $\lambda > 0$  (see [12]).

Let  $F$  be a bifunction of  $C \times C$  into  $\mathcal{R}$ , where  $\mathcal{R}$  is the set of real numbers,  $\Psi : C \rightarrow H$  be a mapping and  $\psi : C \rightarrow \mathcal{R}$  be a real-valued function. The *generalized mixed equilibrium problem* for finding  $x \in C$  such that

$$F(x, y) + \langle \Psi x, y - x \rangle + \psi(y) - \psi(x) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by  $GMEP(F, \psi, \Psi)$ , that is

$$GMEP(F, \psi, \Psi) = \{x \in C : F(x, y) + \langle \Psi x, y - x \rangle + \psi(y) - \psi(x) \geq 0, \forall y \in C\}.$$

If  $\Psi \equiv 0$  and  $\psi \equiv 0$ , the problem (1.3) is reduced into the *equilibrium problem* (see also [13]) for finding  $x \in C$  such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by  $EP(F)$ , that is

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}.$$

This problem contains fixed point problems, includes as special cases numerous problems in physics, optimization and economics. Some methods have been proposed to solve the equilibrium problem, please consult [14–16].

If  $F \equiv 0$  and  $\psi \equiv 0$ , the problem (1.3) is reduced into the *Hartmann-Stampacchia variational inequality* [17] for finding  $x \in C$  such that

$$\langle \Psi x, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by  $VI(C, \Psi)$ . The variational inequality has been extensively studied in the literature [18].

If  $F \equiv 0$  and  $\Psi \equiv 0$ , the problem (1.3) is reduced into the *minimize problem* for finding  $x \in C$  such that

$$\psi(y) - \psi(x) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solutions of (1.6) is denoted by  $Argmin(\psi)$ . Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space  $H$ :

$$\theta(x) = \frac{1}{2} \langle Ax, x \rangle - \langle x, y \rangle, \quad \forall x \in F(S), \tag{1.7}$$

where  $A$  is a linear bounded operator,  $F(S)$  is the fixed point set of a nonexpansive mapping  $S$  and  $y$  is a given point in  $H$  [19].

In 2000, Moudafi [20] introduced the viscosity approximation method for nonexpansive mapping and prove that if  $H$  is a real Hilbert space, the sequence  $\{x_n\}$  defined by the iterative method below, with the initial guess  $x_0 \in C$  is chosen arbitrarily,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) Sx_n, \quad n \geq 0, \tag{1.8}$$

where  $\{\alpha_n\} \subset (0, 1)$  satisfies certain conditions, converge strongly to a fixed point of  $S$  (say  $\bar{x} \in C$ ) which is the unique solution of the following variational inequality:

$$\langle (I - f)\bar{x}, x - \bar{x} \rangle \geq 0, \quad \forall x \in F(S). \tag{1.9}$$

In 2005, Iiduka and Takahashi [21] introduced following iterative process  $x_0 \in C$ ,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 0, \tag{1.10}$$

where  $u \in C$ ,  $\{\alpha_n\} \subset (0, 1)$  and  $\{\lambda_n\} \subset [a, b]$  for some  $a, b$  with  $0 < a < b < 2\beta$ . They proved that under certain appropriate conditions imposed on  $\{\alpha_n\}$  and  $\{\lambda_n\}$ , the sequence  $\{x_n\}$

generated by (1.10) converges strongly to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly monotone mapping (say  $\bar{x} \in C$ ) which solve some variational inequality

$$\langle \bar{x} - u, x - \bar{x} \rangle \geq 0, \quad \forall x \in F(S). \tag{1.11}$$

In 2006, Marino and Xu [19] introduced a general iterative method for nonexpansive mapping. They defined the sequence  $\{x_n\}$  generated by the algorithm  $x_0 \in C$ ,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S x_n, \quad n \geq 0, \tag{1.12}$$

where  $\{\alpha_n\} \subset (0, 1)$  and  $A$  is a strongly positive linear bounded operator. They proved that if  $C = H$  and the sequence  $\{\alpha_n\}$  satisfies appropriate conditions, then the sequence  $\{x_n\}$  generated by (1.12) converge strongly to a fixed point of  $S$  (say  $\bar{x} \in H$ ) which is the unique solution of the following variational inequality:

$$\langle (A - \gamma f)\bar{x}, x - \bar{x} \rangle \geq 0, \quad \forall x \in F(S), \tag{1.13}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{1.14}$$

where  $h$  is a potential function for  $\gamma f$  (i.e.,  $h'(x) = \gamma f(x)$  for  $x \in H$ ).

In 2008, Su *et al.* [22] introduced the following iterative scheme by the viscosity approximation method in a real Hilbert space:  $x_1, u_n \in H$

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) SP_C(u_n - \lambda_n A u_n), \end{cases} \tag{1.15}$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\} \subset [0, 1)$  and  $\{r_n\} \subset (0, \infty)$  satisfy some appropriate conditions. Furthermore, they proved  $\{x_n\}$  and  $\{u_n\}$  converge strongly to the same point  $z \in F(S) \cap VI(C, A) \cap EP(F)$  where  $z = P_{F(S) \cap VI(C, A) \cap EP(F)} f(z)$ .

In 2011, Tan and Chang [10] introduced following iterative process for  $\{T_n : C \rightarrow C\}$  is a sequence of nonexpansive mappings. Let  $\{x_n\}$  be the sequence defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) (SP_C((1 - t_n) J_{M, \lambda} (I - \lambda A) T_n (I - \mu B) x_n)), \quad \forall n \geq 0, \tag{1.16}$$

where  $\{\alpha_n\} \subset (0, 1)$ ,  $\lambda \in (0, 2\alpha]$  and  $\mu \in (0, 2\beta]$ . The sequence  $\{x_n\}$  defined by (1.16) converges strongly to a common element of the set of fixed points of nonexpansive mappings, the set of solutions of the variational inequality and the generalized equilibrium problem.

In this article, we mixed and modified the iterative methods (1.12), (1.15) and (1.16) by purposing the following new general viscosity iterative method:  $x_0, u_n \in C$  and

$$\begin{cases} u_n = T_{r_n}^{(F_1, \psi_1)}(x_n - r_n B_1 x_n), \\ v_n = T_{s_n}^{(F_2, \psi_2)}(x_n - s_n B_2 x_n), \\ x_{n+1} = \xi_n P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A) S J_{M, \lambda} (I - \lambda B) u_n] + (1 - \xi_n) v_n, \quad n \geq 0, \end{cases}$$

where  $\{\alpha_n\}, \{\xi_n\} \subset (0, 1)$ ,  $\lambda \in (0, 2\beta)$  such that  $0 < a \leq \lambda \leq b < 2\beta$ ,  $\{r_n\} \in (0, 2\eta)$  with  $0 < c \leq d \leq 1 - \eta$  and  $\{s_n\} \in (0, 2\rho)$  with  $0 < e \leq f \leq 1 - \rho$  satisfy some appropriate conditions. The purpose of this article, we show that under some control conditions the sequence  $\{x_n\}$  converges strongly to a common element of the set of fixed points of nonexpansive mappings, the common solutions of the generalized mixed equilibrium problem and the set of solutions of the variational inclusion in a real Hilbert space.

## 2 Preliminaries

Let  $H$  be a real Hilbert space with the inner product  $\langle \cdot, \cdot \rangle$  and the norm  $\| \cdot \|$ , respectively. Let  $C$  be a nonempty closed convex subset of  $H$ . Recall that the metric (nearest point) projection  $P_C$  from  $H$  onto  $C$  assigns to each  $x \in H$ , the unique point in  $P_C x \in C$  satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection  $P_C$ . We recall some lemmas which will be needed in the rest of this article.

**Lemma 2.1** *The function  $u \in C$  is a solution of the variational inequality (1.5) if and only if  $u \in C$  satisfies the relation  $u = P_C(u - \lambda \Psi u)$  for all  $\lambda > 0$ .*

**Lemma 2.2** *For a given  $z \in H$ ,  $u \in C$ ,  $u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0, \forall v \in C$ .*

*It is well known that  $P_C$  is a firmly nonexpansive mapping of  $H$  onto  $C$  and satisfies*

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H. \tag{2.1}$$

*Moreover,  $P_C x$  is characterized by the following properties:  $P_C x \in C$  and for all  $x \in H, y \in C$ ,*

$$\langle x - P_C x, y - P_C x \rangle \leq 0. \tag{2.2}$$

**Lemma 2.3** ([23]) *Let  $M : H \rightarrow 2^H$  be a maximal monotone mapping and let  $B : H \rightarrow H$  be a monotone and Lipschitz continuous mapping. Then the mapping  $L = M + B : H \rightarrow 2^H$  is a maximal monotone mapping.*

**Lemma 2.4** ([24]) *Each Hilbert space  $H$  satisfies Opial's condition, that is, for any sequence  $\{x_n\} \subset H$  with  $x_n \rightharpoonup x$ , the inequality  $\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$ , hold for each  $y \in H$  with  $y \neq x$ .*

**Lemma 2.5** ([25]) *Assume  $\{a_n\}$  is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad \forall n \geq 0,$$

*where  $\{\gamma_n\} \subset (0, 1)$  and  $\{\delta_n\}$  is a sequence in  $\mathcal{R}$  such that*

- (i)  $\sum_{n=1}^{\infty} \gamma_n = \infty$ ;
- (ii)  $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\gamma_n} \leq 0$  or  $\sum_{n=1}^{\infty} |\delta_n| < \infty$ .

*Then  $\lim_{n \rightarrow \infty} a_n = 0$ .*

**Lemma 2.6** ([26]) *Let  $C$  be a closed convex subset of a real Hilbert space  $H$  and let  $T : C \rightarrow C$  be a nonexpansive mapping. Then  $I - T$  is demiclosed at zero, that is,*

$$x_n \rightharpoonup x, \quad x_n - Tx_n \rightarrow 0$$

*implies  $x = Tx$ .*

For solving the generalized mixed equilibrium problem, let us assume that the bifunction  $F : C \times C \rightarrow \mathcal{R}$ , the nonlinear mapping  $\Psi : C \rightarrow H$  is continuous monotone and  $\psi : C \rightarrow \mathcal{R}$  satisfies the following conditions:

- (A1)  $F(x, x) = 0$  for all  $x \in C$ ;
- (A2)  $F$  is monotone, i.e.,  $F(x, y) + F(y, x) \leq 0$  for any  $x, y \in C$ ;
- (A3) for each fixed  $y \in C$ ,  $x \mapsto F(x, y)$  is weakly upper semicontinuous;
- (A4) for each fixed  $x \in C$ ,  $y \mapsto F(x, y)$  is convex and lower semicontinuous;
- (B1) for each  $x \in C$  and  $r > 0$ , there exist a bounded subset  $D_x \subseteq C$  and  $y_x \in C$  such that for any  $z \in C \setminus D_x$ ,

$$F(z, y_x) + \psi(y_x) - \psi(z) + \frac{1}{r} \langle y_x - z, z - x \rangle < 0, \tag{2.3}$$

- (B2)  $C$  is a bounded set.

**Lemma 2.7** ([27]) *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $F : C \times C \rightarrow \mathcal{R}$  be a bifunction mapping satisfies (A1)-(A4) and let  $\psi : C \rightarrow \mathcal{R}$  is convex and lower semicontinuous such that  $C \cap \text{dom } \psi \neq \emptyset$ . Assume that either (B1) or (B2) holds. For  $r > 0$  and  $x \in H$ , then there exists  $u \in C$  such that*

$$F(u, y) + \psi(y) - \psi(u) + \frac{1}{r} \langle y - u, u - x \rangle \geq 0.$$

*Define a mapping  $T_r^{(F, \psi)} : H \rightarrow C$  as follows:*

$$T_r^{(F, \psi)}(x) = \left\{ u \in C : F(u, y) + \psi(y) - \psi(u) + \frac{1}{r} \langle y - u, u - x \rangle \geq 0, \forall y \in C \right\} \tag{2.4}$$

*for all  $x \in H$ . Then, the following hold:*

- (i)  $T_r^{(F, \psi)}$  is single-valued;
- (ii)  $T_r^{(F, \psi)}$  is firmly nonexpansive, i.e., for any  $x, y \in H$ ,

$$\|T_r^{(F, \psi)}x - T_r^{(F, \psi)}y\|^2 \leq \langle T_r^{(F, \psi)}x - T_r^{(F, \psi)}y, x - y \rangle;$$

- (iii)  $F(T_r^{(F, \psi)}) = \text{MEP}(F, \psi)$ ;
- (iv)  $\text{MEP}(F, \psi)$  is closed and convex.

**Lemma 2.8** ([19]) *Assume  $A$  is a strongly positive linear bounded operator on a Hilbert space  $H$  with coefficient  $\bar{\gamma} > 0$  and  $0 < \rho \leq \|A\|^{-1}$ , then  $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$ .*

**Lemma 2.9** ([28]) *Let  $H$  be a real Hilbert space and  $A : H \rightarrow H$  a mapping.*

- (i) *If  $A$  is  $\delta$ -strongly monotone and  $\mu$ -strictly pseudo-contraction with  $\delta + \mu > 1$ , then  $I - A$  is contraction with constant  $\sqrt{(1 - \delta)/\mu}$ .*

(ii) If  $A$  is  $\delta$ -strongly monotone and  $\mu$ -strictly pseudo-contraction with  $\delta + \mu > 1$ , then for any fixed number  $\tau \in (0, 1)$ ,  $I - \tau A$  is contraction with constant  $1 - \tau(1 - \sqrt{(1 - \delta)/\mu})$ .

### 3 Strong convergence theorems

In this section, we show a strong convergence theorem which solves the problem of finding a common element of  $F(S)$ ,  $\text{GMEP}(F_1, \psi_1, B_1)$ ,  $\text{GMEP}(F_2, \psi_2, B_2)$  and  $I(B, M)$  of an inverse-strongly monotone mapping in a real Hilbert space.

**Theorem 3.1** *Let  $H$  be a real Hilbert space,  $C$  be a closed convex subset of  $H$ . Let  $F_1, F_2$  be two bifunctions of  $C \times C$  into  $\mathcal{R}$  satisfying (A1)-(A4) and  $B, B_1, B_2 : C \rightarrow H$  be  $\beta, \eta, \rho$ -inverse-strongly monotone mappings,  $\psi_1, \psi_2 : C \rightarrow \mathcal{R}$  be convex and lower semicontinuous function,  $f : C \rightarrow C$  be a contraction with coefficient  $\alpha$  ( $0 < \alpha < 1$ ),  $M : H \rightarrow 2^H$  be a maximal monotone mapping and  $A$  be a  $\delta$ -strongly monotone and  $\mu$ -strictly pseudo-contraction mapping with  $\delta + \mu > 1$ ,  $\gamma$  is a positive real number such that  $\gamma < \frac{1}{\alpha}(1 - \sqrt{\frac{1-\delta}{\mu}})$ . Assume that either (B1) or (B2) holds. Let  $S$  be a nonexpansive mapping of  $H$  into itself such that*

$$\Theta := F(S) \cap \text{GMEP}(F_1, \psi_1, B_1) \cap \text{GMEP}(F_2, \psi_2, B_2) \cap I(B, M) \neq \emptyset.$$

Suppose  $\{x_n\}$  is a sequence generated by the following algorithm  $x_0 \in C$  arbitrarily:

$$\begin{cases} u_n = T_{r_n}^{(F_1, \psi_1)}(x_n - r_n B_1 x_n), \\ v_n = T_{s_n}^{(F_2, \psi_2)}(x_n - s_n B_2 x_n), \\ x_{n+1} = \xi_n P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A) S J_{M, \lambda}(I - \lambda B) u_n] + (1 - \xi_n) v_n, \quad n \geq 0, \end{cases} \quad (3.1)$$

where  $\{\alpha_n\}, \{\xi_n\} \subset (0, 1)$ ,  $\lambda \in (0, 2\beta)$  such that  $0 < a \leq \lambda \leq b < 2\beta$ ,  $\{r_n\} \in (0, 2\eta)$  with  $0 < c \leq d \leq 1 - \eta$  and  $\{s_n\} \in (0, 2\rho)$  with  $0 < e \leq f \leq 1 - \rho$  satisfy the following conditions:

- (C1):  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ,  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ,
- (C2):  $0 < \liminf_{n \rightarrow \infty} \xi_n < \limsup_{n \rightarrow \infty} \xi_n < 1$ ,  $\sum_{n=1}^{\infty} |\xi_{n+1} - \xi_n| < \infty$ ,
- (C3):  $\liminf_{n \rightarrow \infty} r_n > 0$  and  $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$ ,
- (C4):  $\liminf_{n \rightarrow \infty} s_n > 0$  and  $\lim_{n \rightarrow \infty} |s_{n+1} - s_n| = 0$ .

Then  $\{x_n\}$  converges strongly to  $q \in \Theta$ , where  $q = P_{\Theta}(\gamma f + I - A)(q)$  which solves the following variational inequality:

$$\langle (\gamma f - A)q, p - q \rangle \leq 0, \quad \forall p \in \Theta$$

which is the optimality condition for the minimization problem

$$\min_{q \in \Theta} \frac{1}{2} \langle Aq, q \rangle - h(q), \quad (3.2)$$

where  $h$  is a potential function for  $\gamma f$  (i.e.,  $h'(q) = \gamma f(q)$  for  $q \in H$ ).

*Proof* Since  $B$  is  $\beta$ -inverse-strongly monotone mappings, we have

$$\begin{aligned} \|(I - \lambda B)x - (I - \lambda B)y\|^2 &= \|(x - y) - \lambda(Bx - By)\|^2 \\ &= \|x - y\|^2 - 2\lambda \langle x - y, Bx - By \rangle + \lambda^2 \|Bx - By\|^2 \end{aligned}$$

$$\begin{aligned} &\leq \|x - y\|^2 + \lambda(\lambda - 2\beta)\|Bx - By\|^2 \\ &\leq \|x - y\|^2. \end{aligned} \tag{3.3}$$

And  $B_1, B_2$  are  $\eta, \rho$ -inverse-strongly monotone mappings, we have

$$\begin{aligned} \|(I - r_n B_1)x - (I - r_n B_1)y\|^2 &= \|(x - y) - r_n(B_1x - B_1y)\|^2 \\ &= \|x - y\|^2 - 2r_n\langle x - y, B_1x - B_1y \rangle + r_n^2\|B_1x - B_1y\|^2 \\ &\leq \|x - y\|^2 + r_n(r_n - 2\eta)\|B_1x - B_1y\|^2 \\ &\leq \|x - y\|^2. \end{aligned} \tag{3.4}$$

In similar way, we can obtain

$$\|(I - s_n B_2)x - (I - s_n B_2)y\|^2 \leq \|x - y\|^2. \tag{3.5}$$

It is clear that if  $0 < \lambda < 2\beta$ ,  $0 < r_n < 2\eta$ ,  $0 < s_n \leq 2\rho$  then  $I - \lambda B, I - r_n B_1, I - s_n B_2$  are all nonexpansive. We will divide the proof into six steps.

Step 1. We will show  $\{x_n\}$  is bounded. Put  $y_n = J_{M,\lambda}(u_n - \lambda B u_n)$ ,  $n \geq 0$ . It follows that

$$\begin{aligned} \|y_n - q\| &= \|J_{M,\lambda}(u_n - \lambda B u_n) - J_{M,\lambda}(q - \lambda B q)\| \\ &\leq \|u_n - q\|. \end{aligned} \tag{3.6}$$

By Lemma 2.7, we have  $u_n = T_{r_n}^{(F_1, \psi_1)}(x_n - r_n B_1 x_n)$  for all  $n \geq 0$ . Then, we note that

$$\begin{aligned} \|u_n - q\|^2 &= \|T_{r_n}^{(F_1, \psi_1)}(x_n - r_n B_1 x_n) - T_{r_n}^{(F_1, \psi_1)}(q - r_n B_1 q)\|^2 \\ &\leq \|(x_n - r_n B_1 x_n) - (q - r_n B_1 q)\|^2 \\ &\leq \|x_n - q\|^2 + r_n(r_n - 2\eta)\|B_1 x_n - B_1 q\|^2 \\ &\leq \|x_n - q\|^2. \end{aligned} \tag{3.7}$$

In similar way, we can obtain

$$\begin{aligned} \|v_n - q\|^2 &= \|T_{s_n}^{(F_2, \psi_2)}(x_n - s_n B_2 x_n) - T_{s_n}^{(F_2, \psi_2)}(q - s_n B_2 q)\|^2 \\ &\leq \|(x_n - s_n B_2 x_n) - (q - s_n B_2 q)\|^2 \\ &\leq \|x_n - q\|^2 + s_n(s_n - 2\rho)\|B_2 x_n - B_2 q\|^2 \\ &\leq \|x_n - q\|^2. \end{aligned} \tag{3.8}$$

Put  $z_n = P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A)S y_n]$  for all  $n \geq 0$ . From (3.1) and Lemma 2.9(ii), we deduce that

$$\begin{aligned} &\|x_{n+1} - q\| \\ &= \|\xi_n(z_n - q) + (1 - \xi_n)(v_n - q)\| \\ &\leq \xi_n \|P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A)S y_n] - P_C q\| + (1 - \xi_n)\|v_n - q\| \end{aligned}$$



$$\begin{aligned}
 &\leq \xi_n \|\alpha_n \gamma f(x_n) + (I - \alpha_n A) S y_n - q\| + (1 - \xi_n) \|v_n - q\| \\
 &= \xi_n \|\alpha_n (\gamma f(x_n) - Aq) + (I - \alpha_n A) (S y_n - q)\| + (1 - \xi_n) \|v_n - q\| \\
 &\leq \xi_n \alpha_n \|\gamma f(x_n) - Aq\| + \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1 - \delta}{\mu}}\right)\right) \|y_n - q\| + (1 - \xi_n) \|v_n - q\| \\
 &\leq \xi_n \alpha_n \gamma \alpha \|x_n - q\| + \xi_n \alpha_n \|\gamma f(q) - Aq\| + \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1 - \delta}{\mu}}\right)\right) \|x_n - q\| \\
 &\quad + (1 - \xi_n) \|x_n - q\| \\
 &= \left(1 - \left(1 - \sqrt{\frac{1 - \delta}{\mu}} - \gamma \alpha\right) \xi_n \alpha_n\right) \|x_n - q\| + \xi_n \alpha_n \|\gamma f(q) - Aq\| \\
 &\leq \left(1 - \left(1 - \sqrt{\frac{1 - \delta}{\mu}} - \gamma \alpha\right) \xi_n \alpha_n\right) \|x_n - q\| \\
 &\quad + \left(1 - \sqrt{\frac{1 - \delta}{\mu}} - \gamma \alpha\right) \xi_n \alpha_n \frac{\|\gamma f(q) - Aq\|}{\left(1 - \sqrt{\frac{1 - \delta}{\mu}} - \gamma \alpha\right)} \\
 &\leq \max \left\{ \|x_n - q\|, \frac{\|\gamma f(q) - Aq\|}{1 - \sqrt{\frac{1 - \delta}{\mu}} - \gamma \alpha} \right\}. \tag{3.9}
 \end{aligned}$$

It follows from induction that

$$\|x_n - q\| \leq \max \left\{ \|x_0 - q\|, \frac{\|\gamma f(q) - Aq\|}{1 - \sqrt{\frac{1 - \delta}{\mu}} - \gamma \alpha} \right\}, \quad n \geq 0.$$

Therefore  $\{x_n\}$  is bounded, so are  $\{v_n\}$ ,  $\{y_n\}$ ,  $\{z_n\}$ ,  $\{S y_n\}$ ,  $\{f(x_n)\}$  and  $\{A S y_n\}$ .

Step 2. We claim that  $\lim_{n \rightarrow \infty} \|x_{n+2} - x_{n+1}\| = 0$ . From (3.1), we have

$$\begin{aligned}
 \|x_{n+2} - x_{n+1}\| &= \|\xi_{n+1} z_{n+1} + (1 - \xi_{n+1}) v_{n+1} - \xi_n z_n - (1 - \xi_n) v_n\| \\
 &= \|\xi_{n+1} (z_{n+1} - z_n) + (\xi_{n+1} - \xi_n) z_n \\
 &\quad + (1 - \xi_{n+1}) (v_{n+1} - v_n) + (\xi_{n+1} - \xi_n) v_n\| \\
 &\leq \xi_{n+1} \|z_{n+1} - z_n\| + (1 - \xi_{n+1}) \|v_{n+1} - v_n\| \\
 &\quad + |\xi_{n+1} - \xi_n| (\|z_n\| + \|v_n\|). \tag{3.10}
 \end{aligned}$$

We will estimate  $\|v_{n+1} - v_n\|$ . On the other hand, from  $v_{n-1} = T_{s_{n-1}}^{(F_2, \psi_2)}(x_{n-1} - s_{n-1} B_2 x_{n-1})$  and  $v_n = T_{s_n}^{(F_2, \psi_2)}(x_n - s_n B_2 x_n)$ , it follows that

$$\begin{aligned}
 &F_2(v_{n-1}, y) + \langle B_2 x_{n-1}, y - v_{n-1} \rangle + \psi_2(y) - \psi_2(v_{n-1}) \\
 &\quad + \frac{1}{s_{n-1}} \langle y - v_{n-1}, v_{n-1} - x_{n-1} \rangle \geq 0, \quad \forall y \in C \tag{3.11}
 \end{aligned}$$

and

$$F_2(v_n, y) + \langle B_2 x_n, y - v_n \rangle + \psi_2(y) - \psi_2(v_n) + \frac{1}{s_n} \langle y - v_n, v_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.12}$$

Substituting  $y = v_n$  in (3.11) and  $y = v_{n-1}$  in (3.12), we get

$$F_2(v_{n-1}, v_n) + \langle B_2x_{n-1}, v_n - v_{n-1} \rangle + \psi_2(v_n) - \psi_2(v_{n-1}) + \frac{1}{s_{n-1}} \langle v_n - v_{n-1}, v_{n-1} - x_{n-1} \rangle \geq 0$$

and

$$F_2(v_n, v_{n-1}) + \langle B_2x_n, v_{n-1} - v_n \rangle + \psi_2(v_{n-1}) - \psi_2(v_n) + \frac{1}{s_n} \langle v_{n-1} - v_n, v_n - x_n \rangle \geq 0.$$

From (A2), we obtain

$$\left\langle v_n - v_{n-1}, B_2x_{n-1} - B_2x_n + \frac{v_{n-1} - x_{n-1}}{s_{n-1}} - \frac{v_n - x_n}{s_n} \right\rangle \geq 0,$$

and then

$$\left\langle v_n - v_{n-1}, s_{n-1}(B_2x_{n-1} - B_2x_n) + v_{n-1} - x_{n-1} - \frac{s_{n-1}}{s_n}(v_n - x_n) \right\rangle \geq 0,$$

so

$$\left\langle v_n - v_{n-1}, s_{n-1}B_2x_{n-1} - s_{n-1}B_2x_n + v_{n-1} - v_n + v_n - x_{n-1} - \frac{s_{n-1}}{s_n}(v_n - x_n) \right\rangle \geq 0.$$

It follows that

$$\begin{aligned} & \left\langle v_n - v_{n-1}, (I - s_{n-1}B_2)x_n - (I - s_{n-1}B_2)x_{n-1} + v_{n-1} - v_n + v_n - x_n - \frac{s_{n-1}}{s_n}(v_n - x_n) \right\rangle \geq 0, \\ & \langle v_n - v_{n-1}, v_{n-1} - v_n \rangle + \left\langle v_n - v_{n-1}, x_n - x_{n-1} + \left(1 - \frac{s_{n-1}}{s_n}\right)(v_n - x_n) \right\rangle \geq 0. \end{aligned}$$

Without loss of generality, let us assume that there exists a real number  $e$  such that  $s_{n-1} > e > 0$ , for all  $n \in \mathbb{N}$ . Then, we have

$$\begin{aligned} \|v_n - v_{n-1}\|^2 & \leq \left\langle v_n - v_{n-1}, x_n - x_{n-1} + \left(1 - \frac{s_{n-1}}{s_n}\right)(v_n - x_n) \right\rangle \\ & \leq \|v_n - v_{n-1}\| \left\{ \|x_n - x_{n-1}\| + \left|1 - \frac{s_{n-1}}{s_n}\right| \|v_n - x_n\| \right\} \end{aligned}$$

and hence

$$\begin{aligned} \|v_n - v_{n-1}\| & \leq \|x_n - x_{n-1}\| + \frac{1}{s_n} |s_n - s_{n-1}| \|v_n - x_n\| \\ & \leq \|x_n - x_{n-1}\| + \frac{M_1}{e} |s_n - s_{n-1}|, \end{aligned} \tag{3.13}$$

where  $M_1 = \sup\{\|v_n - x_n\| : n \in \mathbb{N}\}$ . Substituting (3.13) into (3.10) that

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| & \leq \xi_{n+1} \|z_{n+1} - z_n\| + (1 - \xi_{n+1}) \left\{ \|x_{n+1} - x_n\| + \frac{M_1}{e} |s_n - s_{n-1}| \right\} \\ & \quad + |\xi_{n+1} - \xi_n| (\|z_n\| + \|v_n\|). \end{aligned} \tag{3.14}$$

We note that

$$\begin{aligned}
 \|z_{n+1} - z_n\| &= \|P_C[\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}A)Sy_{n+1}] - P_C[\alpha_n\gamma f(x_n) - (I - \alpha_nA)Sy_n]\| \\
 &\leq \|\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}A)Sy_{n+1} - (\alpha_n\gamma f(x_n) - (I - \alpha_nA)Sy_n)\| \\
 &\leq \|\alpha_{n+1}\gamma(f(x_{n+1}) - f(x_n)) + (\alpha_{n+1} - \alpha_n)\gamma f(x_n) + (I - \alpha_{n+1}A)(Sy_{n+1} - Sy_n) \\
 &\quad + (\alpha_n - \alpha_{n+1})ASy_n\| \\
 &\leq \alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|\gamma f(x_n)\| \\
 &\quad + \left(1 - \alpha_{n+1}\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|y_{n+1} - y_n\| \\
 &\quad + |\alpha_{n+1} - \alpha_n|\|ASy_n\| \\
 &\leq \alpha_{n+1}\gamma\alpha\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|(\|\gamma f(x_n)\| + \|ASy_n\|) \\
 &\quad + \left(1 - \alpha_{n+1}\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\|y_{n+1} - y_n\|. \tag{3.15}
 \end{aligned}$$

Since  $I - \lambda B$  be nonexpansive, we have

$$\begin{aligned}
 \|y_{n+1} - y_n\| &= \|J_{M,\lambda}(u_{n+1} - \lambda B u_{n+1}) - J_{M,\lambda}(u_n - \lambda B u_n)\| \\
 &\leq \|(u_{n+1} - \lambda B u_{n+1}) - (u_n - \lambda B u_n)\| \\
 &\leq \|u_{n+1} - u_n\|. \tag{3.16}
 \end{aligned}$$

On the other hand, from  $u_{n-1} = T_{r_{n-1}}^{(F_1, \psi_1)}(x_{n-1} - r_{n-1}B_1x_{n-1})$  and  $u_n = T_{r_n}^{(F_1, \psi_1)}(x_n - r_nB_1x_n)$ , it follows that

$$\begin{aligned}
 &F_1(u_{n-1}, y) + \langle B_1x_{n-1}, y - u_{n-1} \rangle + \psi_1(y) - \psi_1(u_{n-1}) \\
 &\quad + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1} \rangle \geq 0, \quad \forall y \in C \tag{3.17}
 \end{aligned}$$

and

$$F_1(u_n, y) + \langle B_1x_n, y - u_n \rangle + \psi_1(y) - \psi_1(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.18}$$

Substituting  $y = u_n$  in (3.17) and  $y = u_{n-1}$  in (3.18), we get

$$F_1(u_{n-1}, u_n) + \langle B_1x_{n-1}, u_n - u_{n-1} \rangle + \psi_1(u_n) - \psi_1(u_{n-1}) + \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1} \rangle \geq 0$$

and

$$F_1(u_n, u_{n-1}) + \langle B_1x_n, u_{n-1} - u_n \rangle + \psi_1(u_{n-1}) - \psi_1(u_n) + \frac{1}{r_n}\langle u_{n-1} - u_n, u_n - x_n \rangle \geq 0.$$

From (A2), we obtain

$$\left\langle u_n - u_{n-1}, B_1x_{n-1} - B_1x_n + \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n} \right\rangle \geq 0,$$

and then

$$\left\langle u_n - u_{n-1}, r_{n-1}(B_1x_{n-1} - B_1x_n) + u_{n-1} - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n) \right\rangle \geq 0,$$

so

$$\left\langle u_n - u_{n-1}, r_{n-1}B_1x_{n-1} - r_{n-1}B_1x_n + u_{n-1} - u_n + u_n - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n) \right\rangle \geq 0.$$

It follows that

$$\begin{aligned} &\left\langle u_n - u_{n-1}, (I - r_{n-1}B_1)x_n - (I - r_{n-1}B_1)x_{n-1} + u_{n-1} - u_n + u_n - x_n - \frac{r_{n-1}}{r_n}(u_n - x_n) \right\rangle \geq 0, \\ &\langle u_n - u_{n-1}, u_{n-1} - u_n \rangle + \left\langle u_n - u_{n-1}, x_n - x_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)(u_n - x_n) \right\rangle \geq 0. \end{aligned}$$

Without loss of generality, let us assume that there exists a real number  $c$  such that  $r_{n-1} > c > 0$ , for all  $n \in \mathbb{N}$ . Then, we have

$$\begin{aligned} \|u_n - u_{n-1}\|^2 &\leq \left\langle u_n - u_{n-1}, x_n - x_{n-1} + \left(1 - \frac{r_{n-1}}{r_n}\right)(u_n - x_n) \right\rangle \\ &\leq \|u_n - u_{n-1}\| \left\{ \|x_n - x_{n-1}\| + \left|1 - \frac{r_{n-1}}{r_n}\right| \|u_n - x_n\| \right\} \end{aligned}$$

and hence

$$\begin{aligned} \|u_n - u_{n-1}\| &\leq \|x_n - x_{n-1}\| + \frac{1}{r_n} |r_n - r_{n-1}| \|u_n - x_n\| \\ &\leq \|x_n - x_{n-1}\| + \frac{M_2}{c} |r_n - r_{n-1}|, \end{aligned} \tag{3.19}$$

where  $M_2 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$ . Substituting (3.19) into (3.16), we have

$$\|y_n - y_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{M_2}{c} |r_n - r_{n-1}|. \tag{3.20}$$

Substituting (3.20) into (3.15), we obtain that

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq \alpha_{n+1}\gamma\alpha \|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n| (\|\gamma f(x_n)\| + \|ASy_n\|) \\ &\quad + \left(1 - \alpha_{n+1} \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \left\{ \|x_n - x_{n-1}\| + \frac{M_2}{c} |r_n - r_{n-1}| \right\}. \end{aligned} \tag{3.21}$$

And substituting (3.13), (3.21) into (3.10), we get

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &\leq \xi_{n+1} \left\{ \alpha_{n+1}\gamma\alpha \|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n| (\|\gamma f(x_n)\| + \|ASy_n\|) \right. \\ &\quad \left. + \left(1 - \alpha_{n+1} \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|x_n - x_{n-1}\| + \frac{M_2}{c} |r_n - r_{n-1}| \right\} \\ &\quad + (1 - \xi_{n+1}) \left\{ \|x_n - x_{n-1}\| + \frac{M_1}{e} |s_n - s_{n-1}| \right\} + |\xi_{n+1} - \xi_n| (\|z_n\| + \|v_n\|) \end{aligned}$$

$$\begin{aligned} &\leq \left(1 - \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right) - \gamma\alpha\right) \xi_{n+1}\alpha_{n+1} \|x_{n+1} - x_n\| + (|\alpha_{n+1} - \alpha_n| \\ &\quad + |\xi_{n+1} - \xi_n|)M_3 + \frac{M_1}{e}|s_n - s_{n-1}| + \frac{M_2}{c}|r_n - r_{n-1}|, \end{aligned} \tag{3.22}$$

where  $M_3 > 0$  is a constant satisfying

$$\sup_n \{ \|\gamma f(x_n)\| + \|ASy_n\|, \|z_n\| + \|v_n\| \} \leq M_3.$$

This together with (C1)-(C4) and Lemma 2.5, imply that

$$\lim_{n \rightarrow \infty} \|x_{n+2} - x_{n+1}\| = 0. \tag{3.23}$$

From (3.20), we also have  $\|y_{n+1} - y_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

Step 3. We show the followings:

- (i)  $\lim_{n \rightarrow \infty} \|Bu_n - Bq\| = 0$ ;
- (ii)  $\lim_{n \rightarrow \infty} \|B_1x_n - B_1q\| = 0$ ;
- (iii)  $\lim_{n \rightarrow \infty} \|B_2x_n - B_2q\| = 0$ .

For  $q \in \Theta$  and  $q = J_{M,\lambda}(q - \lambda Bq)$ , then we get

$$\begin{aligned} \|y_n - q\|^2 &= \|J_{M,\lambda}(u_n - \lambda Bu_n) - J_{M,\lambda}(q - \lambda Bq)\|^2 \\ &\leq \|(u_n - \lambda Bu_n) - (q - \lambda Bq)\|^2 \\ &\leq \|u_n - q\|^2 + \lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2 \\ &\leq \|x_n - q\|^2 + \lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2. \end{aligned} \tag{3.24}$$

It follows that

$$\begin{aligned} \|z_n - q\|^2 &= \|P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)Sy_n) - P_C(q)\|^2 \\ &\leq \|\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(Sy_n - q)\|^2 \\ &\leq \alpha_n \|\gamma f(x_n) - Aq\|^2 + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|y_n - q\|^2 \\ &\quad + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\ &\leq \alpha_n \|\gamma f(x_n) - Aq\|^2 + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\ &\quad + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \{ \|x_n - q\|^2 + \lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2 \} \\ &\leq \alpha_n \|\gamma f(x_n) - Aq\|^2 + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\ &\quad + \|x_n - q\|^2 + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2. \end{aligned} \tag{3.25}$$

By the convexity of the norm  $\| \cdot \|$ , we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\xi_n z_n + (1 - \xi_n)v_n - q\|^2 \\ &\leq \|\xi_n(z_n - q) + (1 - \xi_n)(v_n - q)\|^2 \\ &\leq \xi_n \|z_n - q\|^2 + (1 - \xi_n) \|v_n - q\|^2. \end{aligned} \tag{3.26}$$

Substituting (3.8), (3.25) into (3.26), we obtain

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &\leq \xi_n \left\{ \alpha_n \|\gamma f(x_n) - Aq\|^2 + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \right. \\ &\quad \left. + \|x_n - q\|^2 + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda(\lambda - 2\beta) \|Bu_n - Bq\|^2 \right\} + (1 - \xi_n) \|x_n - q\|^2 \\ &\leq \xi_n \alpha_n \|\gamma f(x_n) - Aq\|^2 + 2\xi_n \alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\ &\quad + \xi_n \|x_n - q\|^2 + \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda(\lambda - 2\beta) \|Bu_n - Bq\|^2 \\ &\quad + (1 - \xi_n) \|x_n - q\|^2. \end{aligned}$$

So, we obtain

$$\begin{aligned} &\xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda(2\beta - \lambda) \|Bu_n - Bq\|^2 \\ &\leq \xi_n \alpha_n \|\gamma f(x_n) - Aq\|^2 + \epsilon_n \\ &\quad + \|x_n - x_{n+1}\| (\|x_n - q\| + \|x_{n+1} - q\|), \end{aligned}$$

where  $\epsilon_n = 2\xi_n \alpha_n (1 - \alpha_n (1 - \sqrt{\frac{1-\delta}{\mu}})) \|\gamma f(x_n) - Aq\| \|y_n - q\|$ . Since conditions (C1)-(C3) and  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ , then we obtain that  $\|Bu_n - Bq\| \rightarrow 0$  as  $n \rightarrow \infty$ . We consider this inequality in (3.25) that

$$\begin{aligned} \|z_n - q\|^2 &\leq \alpha_n \|\gamma f(x_n) - Aq\|^2 + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|y_n - q\|^2 \\ &\quad + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\|. \end{aligned} \tag{3.27}$$

Substituting (3.6) and (3.8) into (3.27), we have

$$\begin{aligned} \|z_n - q\|^2 &\leq \alpha_n \|\gamma f(x_n) - Aq\|^2 + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \\ &\quad \times \{ \|x_n - q\|^2 + r_n(r_n - 2\eta) \|B_1 x_n - B_1 q\|^2 \} \end{aligned}$$

$$\begin{aligned}
 & + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\
 = & \alpha_n \|\gamma f(x_n) - Aq\|^2 + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|x_n - q\|^2 \\
 & + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n(r_n - 2\eta) \|B_1x_n - B_1q\|^2 \\
 & + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\
 \leq & \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 \\
 & + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n(r_n - 2\eta) \|B_1x_n - B_1q\|^2 \\
 & + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\|. \tag{3.28}
 \end{aligned}$$

Substituting (3.7) and (3.28) into (3.26), we obtain

$$\begin{aligned}
 \|x_{n+1} - q\|^2 \leq & \xi_n \left\{ \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 \right. \\
 & + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n(r_n - 2\eta) \|B_1x_n - B_1q\|^2 \\
 & \left. + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \right\} \\
 & + (1 - \xi_n) \|x_n - q\|^2 \\
 = & \xi_n \alpha_n \|\gamma f(x_n) - Aq\|^2 \\
 & + \xi_n \|x_n - q\|^2 + \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n(r_n - 2\eta) \|B_1x_n - B_1q\|^2 \\
 & + 2\xi_n \alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\
 & + (1 - \xi_n) \|x_n - q\|^2. \tag{3.29}
 \end{aligned}$$

So, we also have

$$\begin{aligned}
 & \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n(2\eta - r_n) \|B_1x_n - B_1q\|^2 \\
 & \leq \xi_n \alpha_n \|\gamma f(x_n) - Aq\|^2 + \epsilon_n \\
 & \quad + \|x_n - x_{n+1}\| (\|x_n - q\| + \|x_{n+1} - q\|),
 \end{aligned}$$

where  $\epsilon_n = 2\xi_n \alpha_n (1 - \alpha_n (1 - \sqrt{\frac{1-\delta}{\mu}})) \|\gamma f(x_n) - Aq\| \|y_n - q\|$ . Since conditions (C1)-(C3),  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ , then we obtain that  $\|B_1x_n - B_1q\| \rightarrow 0$  as  $n \rightarrow \infty$ . Substituting

(3.24) into (3.27), we have

$$\begin{aligned}
 & \|z_n - q\|^2 \\
 & \leq \alpha_n \|\gamma f(x_n) - Aq\|^2 \\
 & \quad + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \{ \|x_n - q\|^2 + \lambda(\lambda - 2\beta) \|Bu_n - Bq\|^2 \} \\
 & \quad + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\
 & \leq \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 \\
 & \quad + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda(\lambda - 2\beta) \|Bu_n - Bq\|^2 \\
 & \quad + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\|. \tag{3.30}
 \end{aligned}$$

Substituting (3.8) and (3.30) into (3.26), we obtain

$$\begin{aligned}
 & \|x_{n+1} - q\|^2 \\
 & \leq \xi_n \left\{ \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 \right. \\
 & \quad + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda(\lambda - 2\beta) \|Bu_n - Bq\|^2 \\
 & \quad \left. + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \right\} \\
 & \quad + (1 - \xi_n) \{ \|x_n - q\|^2 + s_n(s_n - 2\rho) \|B_2x_n - B_2q\|^2 \} \\
 & = \xi_n \alpha_n \|\gamma f(x_n) - Aq\|^2 \\
 & \quad + \xi_n \|x_n - q\|^2 + \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda(\lambda - 2\beta) \|Bu_n - Bq\|^2 \\
 & \quad + 2\xi_n \alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\
 & \quad + (1 - \xi_n) \|x_n - q\|^2 + (1 - \xi_n) s_n(s_n - 2\rho) \|B_2x_n - B_2q\|^2 \\
 & = \xi_n \alpha_n \|\gamma f(x_n) - Aq\|^2 \\
 & \quad + \|x_n - q\|^2 + \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda(\lambda - 2\beta) \|Bu_n - Bq\|^2 \\
 & \quad + 2\xi_n \alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\
 & \quad + (1 - \xi_n) s_n(s_n - 2\rho) \|B_2x_n - B_2q\|^2. \tag{3.31}
 \end{aligned}$$



So, we also have

$$\begin{aligned} & (1 - \xi_n)s_n(2\rho - s_n)\|B_2x_n - B_2q\|^2 \\ & \leq \xi_n\alpha_n\|\gamma f(x_n) - Aq\|^2 + \epsilon_n + \|x_n - x_{n+1}\|(\|x_n - q\| + \|x_{n+1} - q\|) \\ & \quad + \xi_n\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right)\lambda(\lambda - 2\beta)\|Bu_n - Bq\|^2, \end{aligned}$$

where  $\epsilon_n = 2\xi_n\alpha_n(1 - \alpha_n(1 - \sqrt{\frac{1-\delta}{\mu}}))\|\gamma f(x_n) - Aq\|\|y_n - q\|$ . Since conditions (C1), (C2), (C4),  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$  and  $\lim_{n \rightarrow \infty} \|Bu_n - Bq\| = 0$ , then we obtain that  $\|B_2x_n - B_2q\| \rightarrow 0$  as  $n \rightarrow \infty$ .

Step 4. We show the followings:

- (i)  $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$ ;
- (ii)  $\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0$ ;
- (iii)  $\lim_{n \rightarrow \infty} \|y_n - Sy_n\| = 0$ .

Since  $T_{r_n}^{(F_1, \psi_1)}$  is firmly nonexpansive, we observe that

$$\begin{aligned} \|u_n - q\|^2 &= \|T_{r_n}^{(F_1, \psi_1)}(x_n - r_n B_1 x_n) - T_{r_n}^{(F_1, \psi_1)}(q - r_n B_1 q)\|^2 \\ &\leq \langle (x_n - r_n B_1 x_n) - (q - r_n B_1 q), u_n - q \rangle \\ &= \frac{1}{2}(\|(x_n - r_n B_1 x_n) - (q - r_n B_1 q)\|^2 + \|u_n - q\|^2 \\ &\quad - \|(x_n - r_n B_1 x_n) - (q - r_n B_1 q) - (u_n - q)\|^2) \\ &\leq \frac{1}{2}(\|x_n - q\|^2 + \|u_n - q\|^2 - \|(x_n - u_n) - r_n(B_1 x_n - B_1 q)\|^2) \\ &= \frac{1}{2}(\|x_n - q\|^2 + \|u_n - q\|^2 - \|x_n - u_n\|^2 \\ &\quad + 2r_n \langle B_1 x_n - B_1 q, x_n - u_n \rangle - r_n^2 \|B_1 x_n - B_1 q\|^2). \end{aligned}$$

Hence, we have

$$\|u_n - q\|^2 \leq \|x_n - q\|^2 - \|x_n - u_n\|^2 + 2r_n \|B_1 x_n - B_1 q\| \|x_n - u_n\|. \tag{3.32}$$

Since  $J_{M, \lambda}$  is 1-inverse-strongly monotone, we compute

$$\begin{aligned} \|y_n - q\|^2 &= \|J_{M, \lambda}(u_n - \lambda B u_n) - J_{M, \lambda}(q - \lambda B q)\|^2 \\ &\leq \langle (u_n - \lambda B u_n) - (q - \lambda B q), y_n - q \rangle \\ &= \frac{1}{2}(\|(u_n - \lambda B u_n) - (q - \lambda B q)\|^2 + \|y_n - q\|^2 \\ &\quad - \|(u_n - \lambda B u_n) - (q - \lambda B q) - (y_n - q)\|^2) \\ &= \frac{1}{2}(\|u_n - q\|^2 + \|y_n - q\|^2 - \|(u_n - y_n) - \lambda(B u_n - B q)\|^2) \\ &\leq \frac{1}{2}(\|u_n - q\|^2 + \|y_n - q\|^2 - \|u_n - y_n\|^2 \\ &\quad + 2\lambda \langle u_n - y_n, B u_n - B q \rangle - \lambda^2 \|B u_n - B q\|^2), \end{aligned} \tag{3.33}$$

which implies that

$$\|y_n - q\|^2 \leq \|u_n - q\|^2 - \|u_n - y_n\|^2 + 2\lambda \|u_n - y_n\| \|Bu_n - Bq\|. \tag{3.34}$$

Substitute (3.32) into (3.34), we have

$$\begin{aligned} \|y_n - q\|^2 \leq & \{ \|x_n - q\|^2 - \|x_n - u_n\|^2 + 2r_n \|B_1x_n - B_1q\| \|x_n - u_n\| \} \\ & - \|u_n - y_n\|^2 + 2\lambda \|u_n - y_n\| \|Bu_n - Bq\|. \end{aligned} \tag{3.35}$$

Substitute (3.35) into (3.27), we have

$$\begin{aligned} \|z_n - q\|^2 \leq & \alpha_n \|\gamma f(x_n) - Aq\|^2 + \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \{ \|x_n - q\|^2 - \|x_n - u_n\|^2 \\ & + 2r_n \|B_1x_n - B_1q\| \|x_n - u_n\| - \|u_n - y_n\|^2 + 2\lambda \|u_n - y_n\| \|Bu_n - Bq\| \} \\ & + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\ \leq & \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_n - u_n\|^2 \\ & + 2 \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) r_n \|B_1x_n - B_1q\| \|x_n - u_n\| - \|u_n - y_n\|^2 \\ & + 2 \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda \|u_n - y_n\| \|Bu_n - Bq\| \\ & + 2\alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\|. \end{aligned} \tag{3.36}$$

Since  $T_{s_n}^{(F_2, \psi_2)}$  is firmly nonexpansive, we observe that

$$\begin{aligned} \|v_n - q\|^2 &= \|T_{s_n}^{(F_2, \psi_2)}(x_n - s_n B_2 x_n) - T_{s_n}^{(F_2, \psi_2)}(q - s_n B_2 q)\|^2 \\ &\leq \langle (x_n - s_n B_2 x_n) - (q - s_n B_2 q), v_n - q \rangle \\ &= \frac{1}{2} (\| (x_n - s_n B_2 x_n) - (q - s_n B_2 q) \|^2 + \|v_n - q\|^2 \\ &\quad - \| (x_n - s_n B_2 x_n) - (q - s_n B_2 q) - (v_n - q) \|^2) \\ &\leq \frac{1}{2} (\|x_n - q\|^2 + \|v_n - q\|^2 - \| (x_n - v_n) - s_n (B_2 x_n - B_2 q) \|^2) \\ &= \frac{1}{2} (\|x_n - q\|^2 + \|v_n - q\|^2 - \|x_n - v_n\|^2 \\ &\quad + 2s_n \langle B_2 x_n - B_2 q, x_n - v_n \rangle - s_n^2 \|B_2 x_n - B_2 q\|^2). \end{aligned}$$

Hence, we have

$$\|v_n - q\|^2 \leq \|x_n - q\|^2 - \|x_n - v_n\|^2 + 2s_n \|B_2 x_n - B_2 q\| \|x_n - v_n\|. \tag{3.37}$$

Substitute (3.36) and (3.37) into (3.26), we obtain

$$\begin{aligned}
 & \|x_{n+1} - q\|^2 \\
 & \leq \xi_n \|z_n - q\|^2 + (1 - \xi_n) \|v_n - q\|^2 \\
 & \leq \xi_n \left\{ \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_n - u_n\|^2 - \|u_n - y_n\|^2 \right. \\
 & \quad + 2 \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\mu}} \right) \right) r_n \|B_1 x_n - B_1 q\| \|x_n - u_n\| \\
 & \quad + 2 \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\mu}} \right) \right) \lambda \|u_n - y_n\| \|Bu_n - Bq\| \\
 & \quad \left. + 2\alpha_n \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\mu}} \right) \right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \right\} \\
 & \quad + (1 - \xi_n) \{ \|x_n - q\|^2 - \|x_n - v_n\|^2 + 2s_n \|B_2 x_n - B_2 q\| \|x_n - v_n\| \} \\
 & \leq \xi_n \alpha_n \|\gamma f(x_n) - Aq\|^2 + \xi_n \|x_n - q\|^2 - \|x_n - u_n\|^2 - \|u_n - y_n\|^2 \\
 & \quad + 2\xi_n \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\mu}} \right) \right) r_n \|B_1 x_n - B_1 q\| \|x_n - u_n\| \\
 & \quad + 2\xi_n \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\mu}} \right) \right) \lambda \|u_n - y_n\| \|Bu_n - Bq\| \\
 & \quad + 2\xi_n \alpha_n \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\mu}} \right) \right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\
 & \quad + (1 - \xi_n) \|x_n - q\|^2 - \|x_n - v_n\|^2 + 2(1 - \xi_n) s_n \|B_2 x_n - B_2 q\| \|x_n - v_n\|. \tag{3.38}
 \end{aligned}$$

Then, we derive

$$\begin{aligned}
 & \|x_n - u_n\|^2 + \|u_n - y_n\|^2 + \|x_n - v_n\|^2 \\
 & \leq \xi_n \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
 & \quad + 2\xi_n \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\mu}} \right) \right) r_n \|B_1 x_n - B_1 q\| \|x_n - u_n\| \\
 & \quad + 2\xi_n \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\mu}} \right) \right) \lambda \|u_n - y_n\| \|Bu_n - Bq\| \\
 & \quad + 2\xi_n \alpha_n \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\mu}} \right) \right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\
 & \quad + 2(1 - \xi_n) s_n \|B_2 x_n - B_2 q\| \|x_n - v_n\| \\
 & \leq \xi_n \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_{n+1} - x_n\| (\|x_n - q\| + \|x_{n+1} - q\|) \\
 & \quad + 2\xi_n \left( 1 - \alpha_n \left( 1 - \sqrt{\frac{1 - \delta}{\mu}} \right) \right) r_n \|B_1 x_n - B_1 q\| \|x_n - u_n\|
 \end{aligned}$$

$$\begin{aligned}
 &+ 2\xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \lambda \|u_n - y_n\| \|Bu_n - Bq\| \\
 &+ 2\xi_n \alpha_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)\right) \|\gamma f(x_n) - Aq\| \|y_n - q\| \\
 &+ 2(1 - \xi_n) s_n \|B_2 x_n - B_2 q\| \|x_n - v_n\|.
 \end{aligned} \tag{3.39}$$

By conditions (C1)-(C4),  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ ,  $\lim_{n \rightarrow \infty} \|Bu_n - Bq\| = 0$ ,  $\lim_{n \rightarrow \infty} \|B_1 x_n - B_1 q\| = 0$  and  $\lim_{n \rightarrow \infty} \|B_2 x_n - B_2 q\| = 0$ . So, we have  $\|x_n - u_n\| \rightarrow 0$ ,  $\|u_n - y_n\| \rightarrow 0$ ,  $\|x_n - v_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . We note that  $x_{n+1} - x_n = \xi_n(z_n - x_n) + (1 - \xi_n)(v_n - x_n)$ . From  $\lim_{n \rightarrow \infty} \|x_n - v_n\| = 0$ ,  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ , and hence

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0. \tag{3.40}$$

It follows that

$$\|x_n - y_n\| \leq \|x_n - u_n\| + \|u_n - y_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{3.41}$$

Since

$$\|z_n - y_n\| \leq \|z_n - x_n\| + \|x_n - y_n\|.$$

So, by (3.40) and  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ , we obtain

$$\lim_{n \rightarrow \infty} \|z_n - y_n\| = 0. \tag{3.42}$$

Therefore, we observe that

$$\begin{aligned}
 \|S y_n - z_n\| &= \|P_C S y_n - P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A) S y_n)\| \\
 &\leq \|S y_n - \alpha_n \gamma f(x_n) - (I - \alpha_n A) S y_n\| \\
 &= \alpha_n \|\gamma f(x_n) - A S y_n\|.
 \end{aligned} \tag{3.43}$$

By condition (C1), we have  $\|S y_n - z_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Next, we observe that

$$\|S y_n - y_n\| \leq \|S y_n - z_n\| + \|z_n - y_n\|.$$

By (3.42) and (3.43), we have  $\|S y_n - y_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .

Step 5. We show that  $q \in \Theta := F(S) \cap \text{GMEP}(F_1, \psi_1, B_1) \cap \text{GMEP}(F_2, \psi_2, B_2) \cap I(B, M)$  and  $\limsup_{n \rightarrow \infty} \langle (\gamma f - A)q, S y_n - q \rangle \leq 0$ . It is easy to see that  $P_\Theta(\gamma f + (I - A))$  is a contraction of  $H$  into itself. In fact, from Lemma 2.9, we have

$$\begin{aligned}
 &\|P_\Theta(\gamma f + (I - A))x - P_\Theta(\gamma f + (I - A))y\| \\
 &\leq \|(\gamma f + (I - A))x - (\gamma f + (I - A))y\| \\
 &\leq \gamma \|f(x) - f(y)\| + (I - A)\|x - y\|
 \end{aligned}$$

$$\begin{aligned} &\leq \gamma\alpha\|x - y\| + \left(1 - \left(1 - \sqrt{\frac{1 - \delta}{\mu}}\right)\right)\|x - y\| \\ &= \left(\sqrt{\frac{1 - \delta}{\mu}} + \gamma\alpha\right)\|x - y\|. \end{aligned}$$

Hence  $H$  is complete, there exists a unique fixed point  $q \in H$  such that  $q = P_{\Theta}(\gamma f + (I - A)(q))$ . By Lemma 2.2 we obtain that  $\langle(\gamma f - A)q, w - q\rangle \leq 0$  for all  $w \in \Theta$ .

Next, we show that  $\limsup_{n \rightarrow \infty} \langle(\gamma f - A)q, Sy_n - q\rangle \leq 0$ , where  $q = P_{\Theta}(\gamma f + I - A)(q)$  is the unique solution of the variational inequality  $\langle(\gamma f - A)q, p - qr\rangle \geq 0, \forall p \in \Theta$ . We can choose a subsequence  $\{y_{n_i}\}$  of  $\{y_n\}$  such that

$$\limsup_{n \rightarrow \infty} \langle(\gamma f - A)q, Sy_n - q\rangle = \lim_{i \rightarrow \infty} \langle(\gamma f - A)q, Sy_{n_i} - q\rangle.$$

As  $\{y_{n_i}\}$  is bounded, there exists a subsequence  $\{y_{n_{i_j}}\}$  of  $\{y_{n_i}\}$  which converges weakly to  $w$ . We may assume without loss of generality that  $y_{n_i} \rightharpoonup w$ . We claim that  $w \in \Theta$ . Since  $\|y_n - Sy_n\| \rightarrow 0$  and by Lemma 2.6, we have  $w \in F(S)$ .

Next, we show that  $w \in \text{GMEP}(F_1, \psi_1, B_1)$ . Since  $u_n = T_{r_n}^{(F_1, \psi_1)}(x_n - r_n B_1 x_n)$ , we know that

$$F_1(u_n, y) + \psi_1(y) - \psi_1(u_n) + \langle B_1 x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

It follows by (A2) that

$$\psi_1(y) - \psi_1(u_n) + \langle B_1 x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq F_1(y, u_n), \quad \forall y \in C.$$

Hence,

$$\psi_1(y) - \psi_1(u_{n_i}) + \langle B_1 x_{n_i}, y - u_{n_i} \rangle + \frac{1}{r_{n_i}} \langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle \geq F_1(y, u_{n_i}), \quad \forall y \in C. \quad (3.44)$$

For  $t \in (0, 1]$  and  $y \in H$ , let  $y_t = ty + (1 - t)w$ . From (3.44), we have

$$\begin{aligned} &\langle y_t - u_{n_i}, B_1 y_t \rangle \\ &\geq \langle y_t - u_{n_i}, B_1 y_t \rangle - \psi_1(y_t) + \psi_1(u_{n_i}) - \langle B_1 x_{n_i}, y_t - u_{n_i} \rangle \\ &\quad - \frac{1}{r_{n_i}} \langle y_t - u_{n_i}, u_{n_i} - x_{n_i} \rangle + F_1(y_t, u_{n_i}) \\ &= \langle y_t - u_{n_i}, B_1 y_t - B_1 u_{n_i} \rangle + \langle y_t - u_{n_i}, B_1 u_{n_i} - B_1 x_{n_i} \rangle - \psi_1(y_t) + \psi_1(u_{n_i}) \\ &\quad - \frac{1}{r_{n_i}} \langle y_t - u_{n_i}, u_{n_i} - x_{n_i} \rangle + F_1(y_t, u_{n_i}). \end{aligned}$$

From  $\|u_{n_i} - x_{n_i}\| \rightarrow 0$ , we have  $\|B_1 u_{n_i} - B_1 x_{n_i}\| \rightarrow 0$ . Further, from (A4) and the weakly lower semicontinuity of  $\psi_1$ ,  $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rightarrow 0$  and  $u_{n_i} \rightharpoonup w$ , we have

$$\langle y_t - w, B_1 y_t \rangle \geq -\psi_1(y_t) + \psi_1(w) + F_1(y_t, w). \quad (3.45)$$

From (A1), (A4) and (3.45), we have

$$\begin{aligned} 0 &= F_1(y_t, y_t) - \psi_1(y_t) + \psi_1(y_t) \\ &\leq tF_1(y_t, y) + (1-t)F_1(y_t, w) + t\psi_1(y) + (1-t)\psi_1(w) - \psi_1(y_t) \\ &= t[F_1(y_t, y) + \psi_1(y) - \psi_1(y_t)] + (1-t)[F_1(y_t, w) + \psi_1(w) - \psi_1(y_t)] \\ &\leq t[F_1(y_t, y) + \psi_1(y) - \psi_1(y_t)] + (1-t)\langle y_t - w, B_1y_t \rangle \\ &= t[F_1(y_t, y) + \psi_1(y) - \psi_1(y_t)] + (1-t)t\langle y - w, B_1y_t \rangle, \end{aligned}$$

and hence

$$0 \leq F_1(y_t, y) + \psi_1(y) - \psi_1(y_t) + (1-t)\langle y - w, B_1y_t \rangle.$$

Letting  $t \rightarrow 0$ , we have, for each  $y \in C$ ,

$$F_1(w, y) + \psi_1(y) - \psi_1(w) + \langle y - w, B_1w \rangle \geq 0.$$

This implies that  $w \in \text{GMEP}(F_1, \psi_1, B_1)$ . By following the same arguments, we can show that  $w \in \text{GMEP}(F_2, \psi_2, B_2)$ .

Lastly, we show that  $w \in I(B, M)$ . In fact, since  $B$  is a  $\beta$ -inverse-strongly monotone,  $B$  is monotone and Lipschitz continuous mapping. It follows from Lemma 2.3 that  $M + B$  is a maximal monotone. Let  $(v, g) \in G(M + B)$ , since  $g - Bv \in M(v)$ . Again since  $y_{n_i} = J_{M, \lambda}(u_{n_i} - \lambda Bu_{n_i})$ , we have  $u_{n_i} - \lambda Bu_{n_i} \in (I + \lambda M)(y_{n_i})$ , that is,  $\frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda Bu_{n_i}) \in M(y_{n_i})$ . By virtue of the maximal monotonicity of  $M + B$ , we have

$$\left\langle v - y_{n_i}, g - Bv - \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda Bu_{n_i}) \right\rangle \geq 0,$$

and hence

$$\begin{aligned} \langle v - y_{n_i}, g \rangle &\geq \left\langle v - y_{n_i}, Bv + \frac{1}{\lambda}(u_{n_i} - y_{n_i} - \lambda Bu_{n_i}) \right\rangle \\ &= \langle v - y_{n_i}, Bv - By_{n_i} \rangle + \langle v - y_{n_i}, By_{n_i} - Bu_{n_i} \rangle \\ &\quad + \left\langle v - y_{n_i}, \frac{1}{\lambda}(u_{n_i} - y_{n_i}) \right\rangle. \end{aligned}$$

It follows from  $\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0$ , we have  $\lim_{n \rightarrow \infty} \|Bu_n - By_n\| = 0$  and  $y_{n_i} \rightharpoonup w$  that

$$\limsup_{n \rightarrow \infty} \langle v - y_n, g \rangle = \langle v - w, g \rangle \geq 0.$$

It follows from the maximal monotonicity of  $B + M$  that  $\theta \in (M + B)(w)$ , that is,  $w \in I(B, M)$ . Therefore,  $w \in \Theta$ . It follows that

$$\limsup_{n \rightarrow \infty} \langle (\gamma f - A)q, Sy_n - q \rangle = \lim_{i \rightarrow \infty} \langle (\gamma f - A)q, Sy_{n_i} - q \rangle = \langle (\gamma f - A)q, w - q \rangle \leq 0.$$

Step 6. We prove  $x_n \rightarrow q$ . By using (3.1) and together with Schwarz inequality, we have

$$\begin{aligned}
 \|x_{n+1} - q\|^2 &= \|\xi_n P_C[(\alpha_n \gamma f(x_n) + (I - \alpha_n A)S y_n) - q] + (1 - \xi_n)(v_n - q)\|^2 \\
 &\leq \xi_n \|P_C[(\alpha_n \gamma f(x_n) + (I - \alpha_n A)S y_n) - P_C(q)]\|^2 + (1 - \xi_n) \|v_n - q\|^2 \\
 &\leq \xi_n \|\alpha_n (\gamma f(x_n) - Aq) + (I - \alpha_n A)(S y_n - q)\|^2 + (1 - \xi_n) \|x_n - q\|^2 \\
 &\leq \xi_n (I - \alpha_n A)^2 \|S y_n - q\|^2 + \xi_n \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\
 &\quad + 2\xi_n \alpha_n \langle (I - \alpha_n A)(S y_n - q), \gamma f(x_n) - Aq \rangle + (1 - \xi_n) \|x_n - q\|^2 \\
 &\leq \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1 - \delta}{\mu}}\right)\right)^2 \|y_n - q\|^2 + \xi_n \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\
 &\quad + 2\xi_n \alpha_n \langle S y_n - q, \gamma f(x_n) - Aq \rangle - 2\xi_n \alpha_n^2 \langle A(S y_n - q), \gamma f(x_n) - Aq \rangle \\
 &\quad + (1 - \xi_n) \|x_n - q\|^2 \\
 &\leq \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1 - \delta}{\mu}}\right)\right)^2 \|x_n - q\|^2 + \xi_n \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\
 &\quad + 2\xi_n \alpha_n \langle S y_n - q, \gamma f(x_n) - \gamma f(q) \rangle + 2\xi_n \alpha_n \langle S y_n - q, \gamma f(q) - Aq \rangle \\
 &\quad - 2\xi_n \alpha_n^2 \langle A(S y_n - q), \gamma f(x_n) - Aq \rangle + (1 - \xi_n) \|x_n - q\|^2 \\
 &\leq \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1 - \delta}{\mu}}\right)\right)^2 \|x_n - q\|^2 + \xi_n \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\
 &\quad + 2\xi_n \alpha_n \|S y_n - q\| \|\gamma f(x_n) - \gamma f(q)\| + 2\xi_n \alpha_n \langle S y_n - q, \gamma f(q) - Aq \rangle \\
 &\quad - 2\xi_n \alpha_n^2 \langle A(S y_n - q), \gamma f(x_n) - Aq \rangle + (1 - \xi_n) \|x_n - q\|^2 \\
 &\leq \xi_n \left(1 - \alpha_n \left(1 - \sqrt{\frac{1 - \delta}{\mu}}\right)\right)^2 \|x_n - q\|^2 + \xi_n \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 \\
 &\quad + 2\xi_n \gamma \alpha \alpha_n \|y_n - q\| \|x_n - q\| + 2\xi_n \alpha_n \langle S y_n - q, \gamma f(q) - Aq \rangle \\
 &\quad - 2\xi_n \alpha_n^2 \langle A(S y_n - q), \gamma f(x_n) - Aq \rangle + (1 - \xi_n) \|x_n - q\|^2 \\
 &\leq \left(\xi_n - 2\xi_n \alpha_n \left(1 - \sqrt{\frac{1 - \delta}{\mu}}\right) + \xi_n \alpha_n^2 \left(1 - \sqrt{\frac{1 - \delta}{\mu}}\right)^2\right) \|x_n - q\|^2 \\
 &\quad + \xi_n \alpha_n^2 \|\gamma f(x_n) - Aq\|^2 + 2\xi_n \gamma \alpha \alpha_n \|x_n - q\|^2 \\
 &\quad + 2\xi_n \alpha_n \langle S y_n - q, \gamma f(q) - Aq \rangle - 2\xi_n \alpha_n^2 \langle A(S y_n - q), \gamma f(x_n) - Aq \rangle \\
 &\quad + (1 - \xi_n) \|x_n - q\|^2 \\
 &\leq \left(1 - 2\xi_n \alpha_n \left(1 - \sqrt{\frac{1 - \delta}{\mu}}\right) + 2\xi_n \gamma \alpha \alpha_n\right) \|x_n - q\|^2 \\
 &\quad + \alpha_n \left\{ \xi_n \alpha_n \|\gamma f(x_n) - Aq\|^2 \right. \\
 &\quad + 2\xi_n \langle S y_n - q, \gamma f(q) - Aq \rangle - 2\xi_n \alpha_n \|A(S y_n - q)\| \|\gamma f(x_n) - Aq\| \\
 &\quad \left. + \xi_n \alpha_n \left(1 - \sqrt{\frac{1 - \delta}{\mu}}\right)^2 \|x_n - q\|^2 \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \left(1 - 2 \left( \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right) - \gamma\alpha \right) \xi_n \alpha_n \right) \|x_n - q\|^2 + \alpha_n \left\{ \xi_n \alpha_n \|\gamma f(x_n) - Aq\|^2 \right. \\
 &\quad + 2\xi_n \langle Sy_n - q, \gamma f(q) - Aq \rangle - 2\xi_n \alpha_n \|A(Sy_n - q)\| \|\gamma f(x_n) - Aq\| \\
 &\quad \left. + \xi_n \alpha_n \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right)^2 \|x_n - q\|^2 \right\}.
 \end{aligned}$$

Since  $\{x_n\}$  is bounded, where  $\eta \geq \xi_n \|\gamma f(x_n) - Aq\|^2 - 2\xi_n \|A(Sy_n - q)\| \|\gamma f(x_n) - Aq\| + \xi_n (1 - \sqrt{\frac{1-\delta}{\mu}})^2 \|x_n - q\|^2$  for all  $n \geq 0$ . It follows that

$$\|x_{n+1} - q\|^2 \leq \left(1 - 2 \left( \left(1 - \sqrt{\frac{1-\delta}{\mu}}\right) - \gamma\alpha \right) \xi_n \alpha_n \right) \|x_n - q\|^2 + \alpha_n \varsigma_n, \tag{3.46}$$

where  $\varsigma_n = 2\xi_n \langle Sy_n - q, \gamma f(q) - Aq \rangle + \eta \alpha_n$ . By  $\limsup_{n \rightarrow \infty} \langle (\gamma f - A)q, Sy_n - q \rangle \leq 0$ , we get  $\limsup_{n \rightarrow \infty} \varsigma_n \leq 0$ . Applying Lemma 2.5, we can conclude that  $x_n \rightarrow q$ . This completes the proof.  $\square$

**Corollary 3.2** *Let  $H$  be a real Hilbert space,  $C$  be a closed convex subset of  $H$ . Let  $F_1, F_2$  be two bifunctions of  $C \times C$  into  $\mathcal{R}$  satisfying (A1)-(A4) and  $B, B_1, B_2 : C \rightarrow H$  be  $\beta, \eta, \rho$ -inverse-strongly monotone mappings,  $\psi_1, \psi_2 : C \rightarrow \mathcal{R}$  be convex and lower semicontinuous function,  $f : C \rightarrow C$  be a contraction with coefficient  $\alpha$  ( $0 < \alpha < 1$ ),  $M : H \rightarrow 2^H$  be a maximal monotone mapping. Assume that either (B1) or (B2) holds. Let  $S$  be a nonexpansive mapping of  $H$  into itself such that*

$$\Theta := F(S) \cap \text{GMEP}(F_1, \psi_1, B_1) \cap \text{GMEP}(F_2, \psi_2, B_2) \cap I(B, M) \neq \emptyset.$$

Suppose  $\{x_n\}$  is a sequence generated by the following algorithm  $x_0 \in C$  arbitrarily:

$$\begin{cases}
 u_n = T_{r_n}^{(F_1, \psi_1)}(x_n - r_n B_1 x_n), \\
 v_n = T_{s_n}^{(F_2, \psi_2)}(x_n - s_n B_2 x_n), \\
 x_{n+1} = \xi_n P_C[\alpha_n f(x_n) + (I - \alpha_n) S J_{M, \lambda}(I - \lambda B)u_n] + (1 - \xi_n)v_n,
 \end{cases}$$

where  $\{\alpha_n\}, \{\xi_n\} \subset (0, 1)$ ,  $\lambda \in (0, 2\beta)$  such that  $0 < a \leq \lambda \leq b < 2\beta$ ,  $\{r_n\} \in (0, 2\eta)$  with  $0 < c \leq d \leq 1 - \eta$  and  $\{s_n\} \in (0, 2\rho)$  with  $0 < e \leq f \leq 1 - \rho$  satisfy the conditions (C1)-(C4).

Then  $\{x_n\}$  converges strongly to  $q \in \Theta$ , where  $q = P_\Theta(f + I)(q)$  which solves the following variational inequality:

$$\langle (f - I)q, p - q \rangle \leq 0, \quad \forall p \in \Theta.$$

*Proof* Putting  $A \equiv I$  and  $\gamma \equiv 1$  in Theorem 3.1, we can obtain desired conclusion immediately.  $\square$

**Corollary 3.3** *Let  $H$  be a real Hilbert space,  $C$  be a closed convex subset of  $H$ . Let  $F_1, F_2$  be two bifunctions of  $C \times C$  into  $\mathcal{R}$  satisfying (A1)-(A4) and  $B, B_1, B_2 : C \rightarrow H$  be  $\beta, \eta, \rho$ -inverse-strongly monotone mappings,  $\psi_1, \psi_2 : C \rightarrow \mathcal{R}$  be convex and lower semicontinuous*



function. Assume that either (B1) or (B2) holds. Let  $S$  be a nonexpansive mapping of  $H$  into itself such that

$$\Theta := F(S) \cap \text{GMEP}(F_1, \psi_1, B_1) \cap \text{GMEP}(F_2, \psi_2, B_2) \cap I(B, M) \neq \emptyset.$$

Suppose  $\{x_n\}$  is a sequence generated by the following algorithm  $x_0 \in C$  arbitrarily:

$$\begin{cases} u_n = T_{r_n}^{(F_1, \psi_1)}(x_n - r_n B_1 x_n), \\ v_n = T_{s_n}^{(F_2, \psi_2)}(x_n - s_n B_2 x_n), \\ x_{n+1} = \xi_n P_C[\alpha_n u + (I - \alpha_n) S J_{M, \lambda}(I - \lambda B) u_n] + (1 - \xi_n) v_n, \end{cases}$$

where  $\{\alpha_n\}, \{\xi_n\} \subset (0, 1), \lambda \in (0, 2\beta)$  such that  $0 < a \leq \lambda \leq b < 2\beta, \{r_n\} \in (0, 2\eta)$  with  $0 < c \leq d \leq 1 - \eta$  and  $\{s_n\} \in (0, 2\rho)$  with  $0 < e \leq f \leq 1 - \rho$  satisfy the conditions (C1)-(C4).

Then  $\{x_n\}$  converges strongly to  $q \in \Theta$ , where  $q = P_\Theta(q)$  which solves the following variational inequality:

$$\langle u - q, p - q \rangle \leq 0, \quad \forall p \in \Theta.$$

*Proof* Putting  $f \equiv u \in C$  is a constant in Corollary 3.2, we can obtain desired conclusion immediately.  $\square$

**Corollary 3.4** Let  $H$  be a real Hilbert space,  $C$  be a closed convex subset of  $H$ . Let  $F_1, F_2$  be two bifunctions of  $C \times C$  into  $\mathcal{R}$  satisfying (A1)-(A4) and  $B, B_1, B_2 : C \rightarrow H$  be  $\beta, \eta, \rho$ -inverse-strongly monotone mappings,  $\psi_1, \psi_2 : C \rightarrow \mathcal{R}$  be convex and lower semicontinuous function,  $f : C \rightarrow C$  be a contraction with coefficient  $\alpha$  ( $0 < \alpha < 1$ ) and  $A$  is  $\delta$ -strongly monotone and  $\mu$ -strictly pseudo-contraction with  $\delta + \mu > 1, \gamma$  is a positive real number such that  $\gamma < \frac{1}{\alpha}(1 - \sqrt{\frac{1-\delta}{\mu}})$ . Assume that either (B1) or (B2) holds. Let  $S$  be a nonexpansive mapping of  $C$  into itself such that

$$\Theta := F(S) \cap \text{GMEP}(F_1, \psi_1, B_1) \cap \text{GMEP}(F_2, \psi_2, B_2) \cap \text{VI}(C, B) \neq \emptyset.$$

Suppose  $\{x_n\}$  is a sequence generated by the following algorithm  $x_0 \in C$  arbitrarily:

$$\begin{cases} u_n = T_{r_n}^{(F_1, \psi_1)}(x_n - r_n B_1 x_n), \\ v_n = T_{s_n}^{(F_2, \psi_2)}(x_n - s_n B_2 x_n), \\ x_{n+1} = \xi_n P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A) S P_C(I - \lambda B) u_n] + (1 - \xi_n) v_n, \end{cases}$$

where  $\{\alpha_n\}, \{\xi_n\} \subset (0, 1), \lambda \in (0, 2\beta)$  such that  $0 < a \leq \lambda \leq b < 2\beta, \{r_n\} \in (0, 2\eta)$  with  $0 < c \leq d \leq 1 - \eta$  and  $\{s_n\} \in (0, 2\rho)$  with  $0 < e \leq f \leq 1 - \rho$  satisfy the conditions (C1)-(C4).

Then  $\{x_n\}$  converges strongly to  $q \in \Theta$ , where  $q = P_\Theta(\gamma f + I - A)(q)$  which solves the following variational inequality:

$$\langle (\gamma f - A)q, p - q \rangle \leq 0, \quad \forall p \in \Theta.$$

*Proof* Taking  $J_{M, \lambda} = P_C$  in Theorem 3.1, we can obtain desired conclusion immediately.  $\square$

**Corollary 3.5** *Let  $H$  be a real Hilbert space,  $C$  be a closed convex subset of  $H$ . Let  $f : C \rightarrow C$  be a contraction with coefficient  $\alpha$  ( $0 < \alpha < 1$ ),  $A$  is  $\delta$ -strongly monotone and  $\mu$ -strictly pseudo-contraction with  $\delta + \mu > 1$ ,  $\gamma$  is a positive real number such that  $\gamma < \frac{1}{\alpha}(1 - \sqrt{\frac{1-\delta}{\mu}})$ . Let  $S$  be a nonexpansive mapping of  $C$  into itself such that*

$$\Theta := F(S) \neq \emptyset.$$

*Suppose  $\{x_n\}$  is a sequence generated by the following algorithm  $x_0 \in C$  arbitrarily:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S x_n,$$

*where  $\{\alpha_n\} \subset (0, 1)$  and satisfy the condition  $\lim_{n \rightarrow \infty} \alpha_n = 0$ . Then  $\{x_n\}$  converges strongly to  $q \in \Theta$ , where  $q = P_{\Theta}(\gamma f + I - A)(q)$  which solves the following variational inequality:*

$$\langle (\gamma f - A)q, p - q \rangle \leq 0, \quad \forall p \in \Theta.$$

*Proof* Taking  $\xi_n \equiv 1$ ,  $P_C \equiv I$  and  $B, B_1, B_2 \equiv 0$  in Corollary 3.4, we can obtain desired conclusion immediately.  $\square$

**Remark 3.6** Corollary 3.5 generalizes and improves the result of Marino and Xu [19].

**Corollary 3.7** *Let  $H$  be a real Hilbert space,  $C$  be a closed convex subset of  $H$ . Let  $F_1, F_2$  be two bifunctions of  $C \times C$  into  $\mathcal{R}$  satisfying (A1)-(A4) and  $B_1, B_2 : C \rightarrow H$  be  $\eta, \rho$ -inverse-strongly monotone mappings,  $\psi_1, \psi_2 : C \rightarrow \mathcal{R}$  be convex and lower semicontinuous function,  $f : C \rightarrow C$  be a contraction with coefficient  $\alpha$  ( $0 < \alpha < 1$ ). Assume that either (B1) or (B2) holds. Let  $S$  be a nonexpansive mapping of  $C$  into itself such that*

$$\Theta := F(S) \cap \text{GMEP}(F_1, \psi_1, B_1) \cap \text{GMEP}(F_2, \psi_2, B_2) \neq \emptyset.$$

*Suppose  $\{x_n\}$  is a sequence generated by the following algorithm  $x_0 \in C$  arbitrarily:*

$$\begin{cases} u_n = T_{r_n}^{(F_1, \psi_1)}(x_n - r_n B_1 x_n), \\ v_n = T_{s_n}^{(F_2, \psi_2)}(x_n - s_n B_2 x_n), \\ x_{n+1} = \xi_n P_C[\alpha_n f(x_n) + (I - \alpha_n) S u_n] + (1 - \xi_n) v_n, \end{cases}$$

*where  $\{\alpha_n\}, \{\xi_n\} \subset (0, 1)$ ,  $\{r_n\} \in (0, 2\eta)$  with  $0 < c \leq d \leq 1 - \eta$  and  $\{s_n\} \in (0, 2\rho)$  with  $0 < e \leq f \leq 1 - \rho$  satisfy the conditions (C1)-(C4).*

*Then  $\{x_n\}$  converges strongly to  $q \in \Theta$ , where  $q = P_{\Theta}(f + I)(q)$  which solves the following variational inequality:*

$$\langle (f - I)q, p - q \rangle \leq 0, \quad \forall p \in \Theta.$$

*Proof* Taking  $\gamma \equiv 1$ ,  $A \equiv I$ ,  $J_{M, \lambda} \equiv I$  and  $B \equiv 0$  in Theorem 3.1, we can obtain desired conclusion immediately.  $\square$

**Remark 3.8** Corollary 3.7 generalizes and improves the result of Yao and Liou [29].

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contribute equally and significantly in this research work. All authors read and approved the final manuscript.

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