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Fixed point theorems for left amenable semigroups of non-Lipschitzian mappings in Banach spaces

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Abstract
In this paper, we provide the existence and convergence theorems of fixed points for left amenable semigroups of asymptotically nonexpansive type mappings in general Banach spaces, which extend and improve many recent results in this area.
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Keywords: asymptotically nonexpansive type mapping; left amenable semigroup; reversible semigroup; fixed point

1 Introduction

Let E be a Banach space and C a nonempty bounded closed convex subset of E. A mapping T on C is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A well-known result of Browder [1] asserts that if E is uniformly convex, then every nonexpansive mapping on C has a fixed point. Kirk [2], Belluce and Kirk [3] extended this result to the case that X has a normal structure or Opial's property. Goebel and Kirk [4] proved that if E is a uniformly convex Banach space, then every asymptotically nonexpansive mapping on C has a fixed point.

As is well known, not every semigroup of nonexpansive mappings on a subset of a Banach space has a fixed point [5]. The existence and convergence of fixed points for semigroups of various mappings have been studied extensively [6–10]. Recently, Suzuki and Takahashi [8], Takahashi and Zembayashi [9], Zhu and Li [10] proved the existence theorems of fixed points for semigroups $\Im = \{T(t) : t \geq 0\}$ of nonexpansive, asymptotically nonexpansive and asymptotically nonexpansive type mappings, respectively. For instance, in [9], Takahashi and Zembayashi proved the following theorem:

Theorem 1.1 [9] *Let C be a nonempty compact convex subset of a Banach space E and $\Im = \{T(t) : t \geq 0\}$ be a semigroup of asymptotically nonexpansive mappings on C, then the set of common fixed points $F(\Im)$ of $\Im$ is nonempty.*

Many results are known in the case that the semigroup G is commutative, amenable or reversible [11–24]. In the case of an amenable semigroup, the first result was established by Takahashi [21] where he proved:

Theorem 1.2 [21] *Let C be a nonempty compact convex subset of a Banach space E. Let $\Im = \{T(t) : t \in G\}$ be an amenable semigroup of nonexpansive mappings on C. Then C contains a common fixed point for $\Im$.*

Theorem 1.2 was proved for a commutative semigroup by DeMarr [11]. Later in [13], Lau showed that the fixed point property is equivalent to the existence of a left invariant mean on $AP(\Im)$, the space of almost periodic functions on the semigroup $\Im$. It should be pointed out that if $\Im$ is left reversible, then $AP(\Im)$ always has a left invariant mean [13], but the converse is false [14]. And in [16], Lau, Miyake and Takahashi gave the following existence theorem:

Theorem 1.3 [16] *Let C be a nonempty weakly compact convex subset of a Banach space E. Let G be a left reversible semigroup (with identity) and $\Im = \{T(t) : t \in G\}$ be a semigroup of nonexpansive mappings on C. Let X be a left invariant $\Im$-stable subspace of $l^\infty(G)$ containing* 1*, and μ be a left invariant mean on X. Then $F(\Im) = F(T_\mu) \cap C_a$, where C_a denotes the set of almost periodic elements in C, i.e., all $x \in C$ such that $\{T(s)x : s \in G\}$ is relatively compact in the norm topology of E. Further, if C is compact, then the set $F(\Im)$ is nonempty.*

In [20], Saeidi extended Theorem 1.3 to the case for left reversible semigroups of asymptotically nonexpansive mappings. Inspired and motivated by [8–10, 16, 20, 21, 23], we investigate the existence and convergence of fixed points for left amenable semigroups of asymptotically nonexpansive type mappings in Banach spaces. We first provide the existence theorem of fixed points for left amenable semigroups of asymptotically nonexpansive type mappings in Banach spaces. Utilizing this result, we obtain a strong convergence theorem of iterative sequences for left amenable semigroups of asymptotically nonexpansive type mappings. The results obtained in this paper extend and improve many recent results in [8–10, 16, 20, 23].

2 Preliminaries

Let C be a nonempty bounded subset of a Banach space E. Let G be a semitopological semigroup, *i.e.*, G is a semigroup with a Hausdorff topology such that for $s \in G$ the mappings $s \mapsto st$ and $s \mapsto ts$ from G to G are continuous. Let $\Im = \{T(t) : t \in G\}$ be a continuous representation of G on C, *i.e.*, $T(ts)x = T(t)T(s)x$, $t, s \in G$, $x \in C$ and the mapping $(t, x) \mapsto T(t)x$ from $G \times C$ into C is continuous when $G \times C$ has the product topology. Recall that $\Im$ is said to be

(1) nonexpansive if for all $x, y \in C$ and $t \in G$,

$$\|T(t)x - T(t)y\| \leq \|x - y\|;$$

(2) asymptotically nonexpansive [25–27] if there exists a function $k : G \mapsto [0, +\infty)$ with $\inf_{s\in G} \sup_{t\in G} k(ts) \leq 1$ such that for all $x, y \in C$ and $t \in G$,

$$\|T(t)x - T(t)y\| \leq k(t)\|x - y\|;$$

(3) asymptotically nonexpansive type [25–27] if for each $x \in C$, there exists a function $r(\cdot, x) : G \mapsto [0, +\infty)$ with $\inf_{s\in G} \sup_{t\in G} r(ts, x) = 0$ such that for all $x, y \in C$ and $t \in G$,

$$\|T(t)x - T(t)y\| \leq \|x - y\| + r(t, x).$$

It is easily seen that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and that both inclusions are proper [25–27].

Let $l^\infty(G)$ be the Banach space of all bounded real valued functions on G with the supremum norm. Then, for each $s \in G$ and $f \in l^\infty(G)$, we can define $l_s f$ in $l^\infty(G)$ by $(l_s f)(t) = f(st)$ for all $t \in G$. Let X be a subspace of $l^\infty(G)$ containing 1 and X^* be its dual space. An element $\mu \in X^*$ is called a mean on X if $\|\mu\| = \mu(1) = 1$. We always denote the value of μ at $f \in X$ by $\mu_t\langle f(t)\rangle = \mu(f)$. Let X be left invariant, *i.e.*, $l_s(X) \subset X$ for all $s \in G$. A mean μ on X is said to be left invariant if $\mu(l_s f) = \mu(f)$ for all $s \in G$ and $f \in X$. Further, X is called left amenable if X has a left invariant mean. In this case, we also say that G is a left amenable semigroup. Recall that a semigroup G is called left reversible if any two closed right ideals of G have nonvoid intersection. In this case, $(G, \leq)$ is a directed system when the binary relation $\leq$ on G is defined by $s \leq t$ if and only if $\{s\} \cup \overline{sG} \supseteq \{t\} \cup \overline{tG}$, $s, t \in G$. As is well known, the class of left reversible semigroups includes all commutative semigroups and if a semigroup G is left amenable, then G is left reversible. But the converse is false [28].

Let $\Im = \{T(t) : t \in G\}$ be an asymptotically nonexpansive type semigroup on C. Let $F(\Im)$ denote the set of all fixed points of $\Im$, *i.e.*, $F(\Im) = \{x \in C : T(s)x = x, \forall s \in G\}$. A subspace X of $l^\infty(G)$ is called $\Im$-stable if functions $s \mapsto \langle T(s)x, x^*\rangle$ and $s \mapsto \|T(s)x - y\|$ on G are in X for all $x, y \in C$ and $x^* \in E^*$. We know that if μ is a mean on X and if for each $x^* \in E^*$ the function $s \mapsto \langle T(s)x, x^*\rangle$ is contained in X and C is weakly compact, then there exists a unique point x_0 of E such that $\mu_s\langle T(s)x, x^*\rangle = \langle x_0, x^*\rangle$ for all $x^* \in E^*$. Such a point x_0 is always denoted by $T_\mu x$. Obviously, $T_\mu x = x$ for each $x \in F(\Im)$.

3 Main results

Lemma 3.1 *Let C be a nonempty weakly compact convex subset of a Banach space E. Let G be a left reversible semigroup and $\Im = \{T(t) : t \in G\}$ be a continuous representation of G as asymptotically nonexpansive type mappings on C, with the condition* $\limsup_{s\in G} r(s, x) = 0$ *for all $x \in C$. Let X be a left invariant $\Im$-stable subspace of $l^\infty(G)$ containing* 1, *and μ be a left invariant mean on X. Then $F(\Im) = F(T_\mu) \cap C_a$.*

Proof If $F(T_\mu) \cap C_a$ is empty, then so is $F(\Im)$ as $F(\Im) \subset F(T_\mu) \cap C_a$. Let $z \in F(T_\mu) \cap C_a$ and define $d = \mu_s\|T(s)z - z\|$, then for all $t \in G$, we have

$$\begin{aligned}
\|T(t)z - z\| &= \|T(t)z - T_\mu z\| = \sup\{|\langle T(t)z - T_\mu z, x^*\rangle| : x^* \in E^*, \|x^*\| = 1\} \\
&= \sup\{|\mu_s\langle T(t)z - T(s)z, x^*\rangle| : x^* \in E^*, \|x^*\| = 1\} \\
&\leq \sup\{\mu_s\|T(t)z - T(s)z\| \cdot \|x^*\| : x^* \in E^*, \|x^*\| = 1\} \\
&= \mu_s\|T(t)z - T(s)z\| = \mu_s\|T(t)z - T(ts)z\| \quad (\text{by } \mu\text{-left invariant}) \\
&\leq \mu_s\|T(s)z - z\| + r(t, z) = d + r(t, z),
\end{aligned}$$

i.e., for all $t \in G$,

$$\|T(t)z - z\| \leq d + r(t, z). \tag{3.1}$$

Next, we shall show $d = 0$. In fact, if $d > 0$, then for each $t \in G$,

$$d = \mu_s\|T(s)z - z\| = \mu_s\|T(ts)z - z\| \leq \sup_{s\in G}\|T(ts)z - z\|,$$

i.e.,

$$\sup_{s\in G}\|T(ts)z-z\|\geq d,\quad \forall t\in G. \tag{3.2}$$

By $\limsup_{s\in G} r(s,z)=0$, then for any $n\in N$, there exists $s_n\in G$ such that

$$\sup_{t\geq s_n} r(t,z)<\frac{1}{4n}. \tag{3.3}$$

It follows from (3.2) that we can choose a cluster point u_1 of the net $\{T(s)z: s\in G\}$ in the set C with $\|u_1-z\|\geq d$ and there exists $t_n^{(1)}\in G$ satisfying $t_n^{(1)}\geq s_n$ and $\|T(t_n^{(1)})z-u_1\|<\frac{1}{4n}$. Combining it with (3.1) and (3.3), we get

$$\begin{aligned}\|u_1-z\| &\leq \|u_1-T(t_n^{(1)})z\|+\|T(t_n^{(1)})z-z\| \\ &\leq \frac{1}{4n}+d+r(t_n^{(1)},z)\leq d+\frac{1}{2n}\quad (\text{by } t_n^{(1)}\geq s_n).\end{aligned}$$

Hence $\|u_1-z\|\leq d$ and so $\|u_1-z\|=d$. It follows from (3.1) and (3.3) that

$$\begin{aligned}\|T(t_n^{(1)}s_ns)z-u_1\| &\leq \|T(t_n^{(1)}s_ns)z-T(t_n^{(1)})z\|+\|T(t_n^{(1)})z-u_1\| \\ &\leq \|T(s_ns)z-z\|+r(t_n^{(1)},z)+\|T(t_n^{(1)})z-u_1\| \\ &\leq d+r(s_ns,z)+\frac{1}{2n}\leq d+\frac{3}{4n}\end{aligned} \tag{3.4}$$

for all $s\in G$. Noting

$$\begin{aligned}d &= \|u_1-z\|=\|u_1-T_\mu z\| \\ &= \sup\{|\langle u_1-T_\mu z,x^*\rangle| : x^*\in E^*, \|x^*\|=1\} \\ &= \sup\{|\mu_s\langle u_1-T(s)z,x^*\rangle| : x^*\in E^*, \|x^*\|=1\} \\ &\leq \mu_s\|u_1-T(s)z\|,\end{aligned} \tag{3.5}$$

we obtain

$$\begin{aligned}&\mu_s(\|T(t_n^{(1)}s_ns)z-z\|+\|T(t_n^{(1)}s_ns)z-u_1\|) \\ &\quad =\mu_s\|T(t_n^{(1)}s_ns)z-z\|+\mu_s\|T(t_n^{(1)}s_ns)z-u_1\| \\ &\quad =\mu_s\|T(s)z-z\|+\mu_s\|T(s)z-u_1\|\geq 2d.\end{aligned}$$

This implies that

$$\sup_{s\in G}[\|T(t_n^{(1)}s_ns)z-z\|+\|T(t_n^{(1)}s_ns)z-u_1\|]\geq 2d.$$

Thus there exists $s_n^{(1)}\in G$ such that

$$\|T(t_n^{(1)}s_ns_n^{(1)})z-z\|+\|T(t_n^{(1)}s_ns_n^{(1)})z-u_1\|\geq 2d-\frac{1}{4n}. \tag{3.6}$$

Since $\{T(s)z : s \in G\}$ is a relatively compact set, $\{T(t_n^{(1)} s_n s_n^{(1)})z\}$, as a subset of $\{T(s)z : s \in G\}$, has a strong convergent subsequence. Without loss of generality, we can assume that $T(t_n^{(1)} s_n s_n^{(1)})z \to u_2 \in C$. Setting $t_n^{(2)} = t_n^{(1)} s_n s_n^{(1)}$, then $t_n^{(2)} \geq t_n^{(1)} \geq s_n$, $T(t_n^{(2)})z \to u_2$ and by (3.6),

$$\|u_2 - z\| + \|u_2 - u_1\| \geq 2d. \tag{3.7}$$

On the other hand,

$$\begin{aligned} \|u_2 - z\| &\leq \left\|u_2 - T\big(t_n^{(2)}\big)z\right\| + \left\|T\big(t_n^{(2)}\big)z - z\right\| \\ &\leq \left\|u_2 - T\big(t_n^{(2)}\big)z\right\| + d + r\big(t_n^{(2)}, z\big) \quad \text{(by (3.1))} \\ &\leq \left\|u_2 - T\big(t_n^{(2)}\big)z\right\| + d + \frac{1}{4n} \quad \text{(by (3.3))} \end{aligned}$$

and

$$\begin{aligned} \|u_2 - u_1\| &\leq \left\|u_2 - T\big(t_n^{(2)}\big)z\right\| + \left\|T\big(t_n^{(2)}\big)z - u_1\right\| \\ &= \left\|u_2 - T\big(t_n^{(2)}\big)z\right\| + \left\|T\big(t_n^{(1)} s_n s_n^{(1)}\big)z - u_1\right\| \\ &\leq \left\|u_2 - T\big(t_n^{(2)}\big)z\right\| + d + \frac{3}{4n} \quad \text{(by (3.4))}. \end{aligned}$$

Thus we can conclude $\|u_2 - z\| \leq d$ and $\|u_2 - u_1\| \leq d$. So by (3.7),

$$\|u_2 - z\| = \|u_2 - u_1\| = d.$$

Similar to the proof of (3.5), we can prove $\mu_s \|u_2 - T(s)z\| \geq d$ and

$$\begin{aligned} &\mu_s\Big(\left\|T\big(t_n^{(2)} s_n s\big)z - z\right\| + \left\|T\big(t_n^{(2)} s_n s\big)z - u_1\right\| + \left\|T\big(t_n^{(2)} s_n s\big)z - u_2\right\|\Big) \\ &\quad = \mu_s\left\|T\big(t_n^{(2)} s_n s\big)z - z\right\| + \mu_s\left\|T\big(t_n^{(2)} s_n s\big)z - u_1\right\| + \mu_s\left\|T\big(t_n^{(2)} s_n s\big)z - u_2\right\| \\ &\quad = \mu_s\left\|T(s)z - z\right\| + \mu_s\left\|T(s)z - u_1\right\| + \mu_s\left\|T(s)z - u_2\right\| \geq 3d. \end{aligned}$$

This means

$$\sup_{s \in G}\Big(\left\|T\big(t_n^{(2)} s_n s\big)z - z\right\| + \left\|T\big(t_n^{(2)} s_n s\big)z - u_1\right\| + \left\|T\big(t_n^{(2)} s_n s\big)z - u_2\right\|\Big) \geq 3d.$$

Thus there exists $s_n^{(2)} \in G$ such that

$$\left\|T\big(t_n^{(2)} s_n s_n^{(2)}\big)z - z\right\| + \left\|T\big(t_n^{(2)} s_n s_n^{(2)}\big)z - u_1\right\| + \left\|T\big(t_n^{(2)} s_n s_n^{(2)}\big)z - u_2\right\| \geq 3d - \frac{1}{n}.$$

Therefore,

$$\begin{aligned}
\|T(t_n^{(2)}s_ns_n^{(2)})z - z\| &\le d + r(t_n^{(2)}s_ns_n^{(2)}, z) \le d + \frac{1}{4n},\\
\|T(t_n^{(2)}s_ns_n^{(2)})z - u_2\| &\le \|T(t_n^{(2)}s_ns_n^{(2)})z - T(t_n^{(2)})z\| + \|T(t_n^{(2)})z - u_2\|\\
&\le \|T(s_ns_n^{(2)})z - z\| + r(t_n^{(2)}, z) + \|T(t_n^{(2)})z - u_2\|\\
&\le d + r(s_ns_n^{(2)}, z) + r(t_n^{(2)}, z) + \|T(t_n^{(2)})z - u_2\|\\
&\le d + \frac{1}{2n} + \|T(t_n^{(2)})z - u_2\|
\end{aligned}$$

and

$$\begin{aligned}
\|T(t_n^{(2)}s_ns_n^{(2)})z - u_1\| &\le \|T(t_n^{(2)}s_ns_n^{(2)})z - T(t_n^{(1)})z\| + \|T(t_n^{(1)})z - u_1\|\\
&\le \|T(t_n^{(1)}s_ns_n^{(1)}s_ns_n^{(2)})z - T(t_n^{(1)})z\| + \|T(t_n^{(1)})z - u_1\|\\
&\le \|T(s_ns_n^{(1)}s_ns_n^{(2)})z - z\| + r(t_n^{(1)}, z) + \|T(t_n^{(1)})z - u_1\|\\
&\le d + r(s_ns_n^{(1)}s_ns_n^{(2)}, z) + r(t_n^{(1)}, z) + \|T(t_n^{(1)})z - u_1\|\\
&\le d + \frac{1}{2n} + \|T(t_n^{(1)})z - u_1\|.
\end{aligned}$$

Since $\{T(t_n^{(2)}s_ns_n^{(2)})z\}$ has a strong convergent subsequence, without loss of generality, we can assume that $T(t_n^{(2)}s_ns_n^{(2)})z \to u_3 \in C$. Setting $t_n^{(3)} = t_n^{(2)}s_ns_n^{(2)}$, then $t_n^{(3)} \ge t_n^{(2)}$, $T(t_n^{(3)})z \to u_3$,

$$\|u_3 - z\| \le d, \qquad \|u_3 - u_2\| \le d, \qquad \|u_3 - u_1\| \le d$$

and

$$\|u_3 - z\| + \|u_3 - u_1\| + \|u_3 - u_2\| \ge 3d.$$

Thus we have found $u_3 \in C$ such that

$$\|u_3 - u_1\| = \|u_3 - u_2\| = \|u_3 - z\| = d.$$

Now, by mathematical induction, we can find a sequence $\{u_i\} \subset C$ satisfying

$$\|u_i - z\| = d, \qquad \|u_i - u_j\| = d \quad (\forall i, j \in N, i \ne j).$$

Since $T(t_n^{(i)})z \to u_i$, we can seek out $t_{n_i}^{(i)} \in G$ with $\|T(t_{n_i}^{(i)})z - u_i\| \le \frac{d}{4}$. Thus

$$\|T(t_{n_i}^{(i)})z - T(t_{n_j}^{(j)})z\| \ge \frac{d}{2} \quad (\forall i, j \in N, i \ne j),$$

which is a contradiction with the relative compactness of $\{T(t_{n_i}^{(i)})z : i \in N\}$. Therefore, we can conclude $d = 0$.

In the following, we shall show $z \in F(\Im)$. Indeed, for any $h \in G$, $T(h) : C \to C$ is continuous at z, then for all $\varepsilon > 0$, there exists a $\delta > 0$ ($\delta < \varepsilon$) such that for all $x \in C$ with $\|x - z\| < \delta$,

$$\|T(h)x - T(h)z\| < \varepsilon.$$

By (3.1) and the definition of $r(\cdot, z)$, we can get

$$\inf_{s\in G}\sup_{t\in G}\|T(ts)z - z\| \le d + \inf_{s\in G}\sup_{t\in G} r(ts, z) = 0$$

and so we can find a $s_\delta \in G$ such that $\sup_{t\in G}\|T(ts_\delta)z - z\| < \delta$, *i.e.*, for all $t \in G$,

$$\|T(ts_\delta)z - z\| < \delta.$$

Hence

$$\begin{aligned}\|T(h)z - z\| &\le \|T(h)z - T(h)T(ts_\delta)z\| + \|T(h)T(ts_\delta)z - z\| \\ &= \|T(h)z - T(h)T(ts_\delta)z\| + \|T(hts_\delta)z - z\| \\ &< \varepsilon + \delta < 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $z \in F(\Im)$. This completes the proof. □

Now we can give the existence theorem of fixed points for left amenable semigroups of non-Lipschitzian mappings in Banach spaces.

Theorem 3.1 *Let C be a nonempty compact convex subset of a Banach space E. Let G be a left reversible semigroup and $\Im = \{T(t) : t \in G\}$ be a continuous representation of G as asymptotically nonexpansive type mappings on C, with the condition $\limsup_{s\in G} r(s,x) = 0$ for all $x \in C$. Let X be a left invariant $\Im$-stable subspace of $l^\infty(G)$ containing 1, and μ be a left invariant mean on X. Then the set $F(\Im)$ is nonempty.*

Proof For all $x, y \in C$ and $t \in G$, we have

$$\begin{aligned}\|T_\mu x - T_\mu y\| &= \sup\{|\langle T_\mu x - T_\mu y, x^*\rangle| : x^* \in E^*, \|x^*\| = 1\} \\ &= \sup\{|\mu_s\langle T(s)x - T(s)y, x^*\rangle| : x^* \in E^*, \|x^*\| = 1\} \\ &\le \sup\{\mu_s\|T(s)x - T(s)y\| \cdot \|x^*\| : x^* \in E^*, \|x^*\| = 1\} \\ &= \mu_s\|T(s)x - T(s)y\| = \mu_s\|T(ts)x - T(ts)y\| \\ &\le \sup_{s\in G}\|T(ts)x - T(ts)y\| \le \|x - y\| + \sup_{s\in G} r(ts, z),\end{aligned}$$

and so by $\limsup_{s\in G} r(s,z) = 0$, we get $\|T_\mu x - T_\mu y\| \le \|x - y\|$, *i.e.*, T_μ is a nonexpansive mapping from C into itself. Since a nonexpansive mapping of a compact convex subset of a Banach space into itself has a fixed point [29], T_μ has a fixed point z. By Lemma 3.1, $z \in F(\Im)$. This completes the proof. □

Remark 3.1 Theorem 3.1 is an extension of the main results in [8–10, 16, 20, 23] to the case for left amenable semigroups of asymptotically nonexpansive type mappings in Banach spaces.

Recall that for each $s \in G$, we define a point evaluation δ_s on X by $\delta_s(f) = f(s)$ for every $f \in X$. A convex combination of point evaluation is called a finite mean on G. If λ is a finite mean on G, say $\lambda = \Sigma_{i=1}^n a_i \delta_{s_i}$, where $s_i \in G$, $a_i \geq 0$, $i = 1, 2, \ldots, n$, and $\Sigma_{i=1}^n a_i = 1$, then $\lambda(t)\langle T(t)x, x^*\rangle = \Sigma_{i=1}^n a_i \langle T(s_i)x, x^*\rangle = \langle \Sigma_{i=1}^n a_i T(s_i)x, x^*\rangle$ for all $x^* \in E^*$. For convenience, we denote it by $\lambda(t)\langle T(t)x\rangle = \Sigma_{i=1}^n a_i T(s_i)x$. A net $\{\lambda_\alpha : \alpha \in I\}$ of finite means on G is said to be strongly left regular if

$$\lim_{\alpha \in I} \| \lambda_\alpha - l_s^* \lambda_\alpha \| = 0$$

for all $s \in G$, where A is a directed system and l_s^* is the conjugate operator of l_s.

Corollary 3.1 *Let C be a nonempty compact convex subset of a Banach space E. Let G be a left reversible semigroup and* $\Im = \{T(t) : t \in G\}$ *be a continuous representation of G as asymptotically nonexpansive type mappings on C, with the condition* $\limsup_{s \in G} r(s, x) = 0$ *for all* $x \in C$*. Let X be a left invariant* $\Im$*-stable subspace of* $l^\infty(G)$ *containing* 1 *and* $\{\lambda_\alpha : \alpha \in I\}$ *be a net of strongly left regular finite means on G. If* $z \in C$ *satisfies*

$$\liminf_{\alpha \in I} \| \lambda_\alpha(t) \langle T(t)z \rangle - z \| = 0,$$

then $z \in F(\Im)$.

Proof Since $\liminf_{\alpha \in I} \|\lambda_\alpha(t)\langle T(t)z\rangle - z\| = 0$ and $\{\lambda_\alpha : \alpha \in I\} \subset D^*$, we can find a subnet $\{\lambda_{\alpha_\beta} : \beta \in A\}$ of $\{\lambda_\alpha : \alpha \in I\}$ such that $\lim_{\beta \in A} \lambda_{\alpha_\beta}(t)\langle T(t)z\rangle = z$ and $\omega^* - \lim_{\beta \in A} \lambda_{\alpha_\beta} = \mu$, where A is a directed system. Hence μ is a left invariant mean on X (see [30]) and $T_\mu z = z$, which implies $z \in F(\Im)$. This completes the proof. □

Remark 3.2 Corollary 3.1 is an extension of the main results in [8–10, 23].

Next we shall prove the strong convergence theorem for the iterative sequences of left reversible semigroups of asymptotically nonexpansive type mappings. We need a lemma which plays a crucial role in the proof of Theorem 3.2.

Lemma 3.2 [30] *Let* z_n *and* w_n *be bounded sequences in a Banach space X and let* α_n *be a sequence in* $(0, 1)$ *with* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$*. Suppose that* $z_{n+1} = \alpha_n w_n + (1 - \alpha_n) z_n$ *for all* $n \in N$ *and*

$$\limsup_{n\to\infty} \left(\|w_n - w_{n+k}\| - \|z_n - z_{n+k}\| \right) \leq 0$$

for all $k \in N$*. Then* $\liminf_{n\to\infty} \|w_n - z_n\| = 0$.

Theorem 3.2 *Let C be a nonempty compact convex subset of a Banach space X and G be a left reversible semigroup. Let* $\Im = \{T(t) : t \in G\}$ *be a continuous representation of G as asymptotically nonexpansive type mappings on C, with the condition* $\limsup_{s \in G} r(s, x) = 0$

for all $x \in C$. Let X be a left invariant $\Im$-stable subspace of $l^\infty(G)$ containing 1*, and μ be a left invariant mean on X. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \alpha_n T_\mu x_n + (1 - \alpha_n)x_n,$$

for all $n \in N$, where $\alpha_n \subset [0,1]$ satisfies $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Then x_n converges strongly to a fixed point $z \in F(\Im)$.

Proof It follows from

$$\begin{aligned} \|T_\mu x_{n+1} - x_{n+1}\| &\leq \|T_\mu x_{n+1} - T_\mu x_n\| + \|T_\mu x_n - x_{n+1}\| \\ &= \|T_\mu x_{n+1} - T_\mu x_n\| + (1 - \alpha_n)\|T_\mu x_n - x_n\| \\ &= \|T_\mu x_n - x_n\| + \|T_\mu x_{n+1} - T_\mu x_n\| - \|x_{n+1} - x_n\| \\ &\leq \|T_\mu x_n - x_n\| \end{aligned}$$

that $\lim_{n\to\infty} \|T_\mu x_n - x_n\|$ exists. By Lemma 3.1, we get $\liminf_{n\to\infty} \|T_\mu x_n - x_n\| = 0$ and so $\lim_{n\to\infty} \|T_\mu x_n - x_n\| = 0$. Since C is compact, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to z \in C$. Hence, z is a fixed point of T_μ. By Lemma 3.1, we have $z \in F(\Im)$ and

$$\begin{aligned} \|x_{n+1} - z\| &= \left\|\alpha_n T_\mu x_n + (1 - \alpha_n)x_n - z\right\| \\ &\leq \alpha_n \|T_\mu x_n - z\| + (1 - \alpha_n)\|x_n - z\| \\ &\leq \|x_n - z\|. \end{aligned}$$

Hence $\lim_{n\to\infty} \|x_n - z\|$ exists. Thus $\lim_{n\to\infty} \|x_n - z\| = \lim_{k\to\infty} \|x_{n_k} - z\| = 0$, which implies that x_n converges strongly to $z \in F(\Im)$. This completes the proof. □

In the following, we shall give an example of a semigroup which is asymptotically nonexpansive type but not asymptotically nonexpansive on a compact set.

Example 3.1 [27] Let Δ be the Cantor ternary set. Define the Cantor ternary function

$$\tau(x) = \begin{cases} \sum_{n=1}^{+\infty} \frac{b_n}{2^n}, & x = \sum_{n=1}^{+\infty} \frac{2b_n}{3^n} \in \Delta \ (b_n = 0,1), \\ \sup\left\{\tau(y), y \leq x, y \in \Delta\right\}, & x \in [0,1]\backslash\Delta \end{cases}$$

then $\tau : [0,1] \to [0,1]$ is a continuous and increasing but not absolutely continuous function with $\tau(0) = 0$, $\tau(\frac{1}{2}) = \frac{1}{2}$ (see [31]). Since a Lipschitzian function is absolutely continuous, τ is non-Lipschitzian. For all $t > 0$, we define $T(t) : [0,1] \to [0,1]$ by

$$T(t)x = \begin{cases} \frac{x}{2^t}, & 0 \leq x \leq \frac{1}{2}, \\ \frac{\tau(1-x)}{2^t}, & \frac{1}{2} < x \leq 1. \end{cases}$$

Then $T(t)$ is continuous but not Lipschitzian continuous (since τ is non-Lipschitzian) and for all $x, y \in [0,1]$, $|T(t)x| \leq \frac{1}{2^{t+1}}$,

$$\left|T(t)x - T(t)y\right| \leq \frac{1}{2^t} \leq |x - y| + \frac{1}{2^t}.$$

Therefore, we can conclude that the semigroup $\Im = \{T(t) : t > 0\}$ is asymptotically nonexpansive type but not an asymptotically nonexpansive on $[0,1]$. Also, 0 is a fixed point of $\Im$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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