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Some new common fixed point results through generalized altering distances on partial metric spaces

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Abstract

We establish common fixed point results for two pairs of weakly compatible mappings on a partial metric space, satisfying a weak contractive condition involving generalized control functions. The presented theorems extend and unify various known fixed point results. Examples are given to show that our results are proper extensions of the known ones.

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1 Introduction

In [1], Matthews introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. He showed that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification.

Subsequently, several authors (see, *e.g.*, Altun and Erduran [2], Oltra *et al.* [3], Romaguera and Schellekens [4], Romaguera and Valero [5], Rus [6], Djukić *et al.* [7], Nashine *et al.* [8], Di Bari and Vetro [9], Paesano and Vetro [10], Shatanawi *et al.* [11], Shatanawi and Nashine [12], Aydi *et al.* [13]) derived fixed point theorems in partial metric spaces.

Altering distance functions (also called control functions) were introduced by Khan *et al.* [14]. Subsequently, they were used by many authors to obtain fixed point results, including those in partial metric spaces (*e.g.*, Abdeljawad [15], Abdeljawad *et al.* [16, 17], Altun *et al.* [18], Ćirić *et al.* [19], Karapinar and Yüksel [20]). Generalized altering distance functions with several variables were used on metric spaces by Berinde [21], Choudhury [22] and Rao *et al.* [23].

In this paper, an attempt has been made to derive some common fixed point theorems for two pairs of weakly compatible mappings on partial metric spaces, satisfying a weak contractive condition involving generalized control functions. The presented theorems extend and unify various known fixed point results. Examples are given to show that our results are proper extensions of the known ones.

2 Preliminaries

The following definitions and details about partial metrics can be seen, *e.g.*, in [1, 24–28].

Definition 1 A *partial metric* on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \iff p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \leq p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X, p) is called a *partial metric space*.

It is clear that, if $p(x,y) = 0$, then from (p_1) and (p_2), it follows that $x = y$. But $p(x,x)$ may not be 0.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$, with respect to τ_p, if $\lim_{n\to\infty} p(x, x_n) = p(x, x)$. This will be denoted as $x_n \to x$, $n \to \infty$ or $\lim_{n\to\infty} x_n = x$. If (X, p) is a partial metric space, and $T : X \to X$ is a mapping, continuous at $x_0 \in X$ (in τ_p) then, for each sequence $\{x_n\}$ in X, we have

$$x_n \to x_0 \quad \Rightarrow \quad Tx_n \to Tx_0.$$

Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(\cdot, \cdot)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ implies $p(x_n, y_n) \to p(x, y)$.

Definition 2 Let (X, p) be a partial metric space. Then:

1 A sequence $\{x_n\}$ in (X, p) is called a *Cauchy sequence* if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite).
2 The space (X, p) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

It is easy to see that every closed subset of a complete partial metric space is complete. If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{2.1}$$

is a metric on X. Furthermore, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if

$$p(x,x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.2}$$

Lemma 1 *Let (X, p) be a partial metric space.*

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).

(b) The space (X, p) is complete if and only if the metric space (X, p^s) is complete.

Definition 3 ([22, 23]) A function $\psi : [0, +\infty)^n \to [0, +\infty)$ is said to be a *generalized altering distance function* if:

1 ψ is continuous;
2 ψ is increasing in each of its variables;
3 $\psi(t_1, \ldots, t_n) = 0$ if and only if $t_1 = \cdots = t_n = 0$.

The set of generalized altering distance functions with n variables will be denoted by $\mathcal{F}_n$. If $\psi \in \mathcal{F}_n$, we will write $\Psi(t) = \psi(t, t, \ldots, t)$ (obviously, this function belongs to $\mathcal{F}_1$).

Simple examples of generalized altering distance functions with, say, four variables are:

$$\psi(t_1, t_2, t_3, t_4) = k \max\{t_1, t_2, t_3, t_4\}, \quad k > 0;$$

$$\psi(t_1, t_2, t_3, t_4) = t_1^p + t_2^q + t_3^r + t_4^s, \quad p, q, r, s \geq 1.$$

Recall also the following notions. Let X be a nonempty set and $T_1, T_2 : X \to X$ be given self-maps on X. If $w = T_1 x = T_2 x$ for some $x \in X$, then x is called a coincidence point of T_1 and T_2, and w is called a point of coincidence of T_1 and T_2. The pair $\{T_1, T_2\}$ is said to be weakly compatible if $T_1 T_2 t = T_2 T_1 t$, whenever $T_1 t = T_2 t$ for some t in X.

3 Results
3.1 Some auxiliary results
Assertions similar to the following lemma (see, *e.g.*, [29]) were used (and proved) in the course of proofs of several fixed point results in various papers.

Lemma 2 *Let (X, d) be a metric space and let $\{y_n\}$ be a sequence in X such that $\{d(y_{n+1}, y_n)\}$ is nonincreasing and*

$$\lim_{n \to \infty} d(y_{n+1}, y_n) = 0.$$

If $\{y_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following four sequences tend to ε when $k \to \infty$:

$$d(y_{2m(k)}, y_{2n(k)}), \qquad d(y_{2m(k)}, y_{2n(k)+1}), \qquad d(y_{2m(k)-1}, y_{2n(k)}), \qquad d(y_{2m(k)-1}, y_{2n(k)+1}).$$

As a corollary (putting $d = p^s$ for a partial metric p), we obtain

Lemma 3 *Let (X, p) be a partial metric space and let $\{y_n\}$ be a sequence in X such that $\{p(y_{n+1}, y_n)\}$ is nonincreasing and*

$$\lim_{n \to \infty} p(y_{n+1}, y_n) = 0. \tag{3.1}$$

If $\{y_{2n}\}$ is not a Cauchy sequence in (X, p), then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following four sequences

tend to ε when $k \to \infty$:

$$p(y_{2m(k)}, y_{2n(k)}), \qquad p(y_{2m(k)}, y_{2n(k)+1}),$$
$$p(y_{2m(k)-1}, y_{2n(k)}), \qquad p(y_{2m(k)-1}, y_{2n(k)+1}). \tag{3.2}$$

3.2 Main results

Theorem 1 *Let (X, p) be a complete partial metric space. Let $T, S, I, J : X \to X$ be given mappings satisfying for every pair $(x, y) \in X \times X$:*

$$\Psi_1\big(p(Sx, Ty)\big) \leq \psi_1\bigg(p(Ix, Jy), p(Ix, Sx), p(Jy, Ty), \frac{1}{2}\big[p(Ix, Ty) + p(Jy, Sx)\big]\bigg)$$
$$- \psi_2\big(p(Ix, Jy), p(Ix, Sx), p(Jy, Ty)\big), \tag{3.3}$$

where $\psi_1 \in \mathcal{F}_4$ and $\psi_2 \in \mathcal{F}_3$ are generalized altering distance functions, and $\Psi_1(t) = \psi_1(t, t, t, t)$. Suppose that

  (i) $TX \subseteq IX$ *and* $SX \subseteq JX$;

  (ii) one of the ranges IX, JX, TX and SX is a closed subset of (X, p).

Then

  (a) I and S have a coincidence point,

  (b) J and T have a coincidence point.

Moreover, if the pairs $\{I, S\}$ and $\{J, T\}$ are weakly compatible, then I, J, T and S have a unique common fixed point.

Proof Let x_0 be an arbitrary point in X. Since $TX \subseteq IX$ and $SX \subseteq JX$, we can define sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_{2n-1} = Sx_{2n-2} = Jx_{2n-1}, \qquad y_{2n} = Tx_{2n-1} = Ix_{2n}, \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Without loss of the generality, we may assume that

$$p(y_{2n}, y_{2n+1}) > 0, \quad \forall n \in \mathbb{N}. \tag{3.5}$$

If not, then $p(y_{2n}, y_{2n+1}) = 0$ and hence $y_{2n} = y_{2n+1}$, for some n. Taking $x = x_{2n}$ and $y = x_{2n+1}$, from (3.4) and the considered contraction condition (3.3), we have

$$\Psi_1\big(p(y_{2n+1}, y_{2n+2})\big) = \Psi_1\big(p(Sx_{2n}, Tx_{2n+1})\big)$$
$$\leq \psi_1\bigg(p(Ix_{2n}, Jx_{2n+1}), p(Ix_{2n}, Sx_{2n}), p(Jx_{2n+1}, Tx_{2n+1}),$$
$$\frac{1}{2}\big[p(Ix_{2n}, Tx_{2n+1}) + p(Jx_{2n+1}, Sx_{2n})\big]\bigg)$$
$$- \psi_2\big(p(Ix_{2n}, Jx_{2n+1}), p(Ix_{2n}, Sx_{2n}), p(Jx_{2n+1}, Tx_{2n+1})\big)$$
$$= \psi_1\bigg(p(y_{2n}, y_{2n+1}), p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2}),$$
$$\frac{1}{2}\big[p(y_{2n}, y_{2n+2}) + p(y_{2n+1}, y_{2n+1})\big]\bigg)$$

$$- \psi_2\big(p(y_{2n},y_{2n+1}),p(y_{2n},y_{2n+1}),p(y_{2n+1},y_{2n+2})\big)$$

$$\leq \psi_1\bigg(p(y_{2n},y_{2n+1}),p(y_{2n},y_{2n+1}),p(y_{2n+1},y_{2n+2}),$$

$$\frac{1}{2}\big[p(y_{2n},y_{2n+1}) + p(y_{2n+1},y_{2n+2})\big]\bigg)$$

$$- \psi_2\big(p(y_{2n},y_{2n+1}),p(y_{2n},y_{2n+1}),p(y_{2n+1},y_{2n+2})\big), \tag{3.6}$$

since

$$p(y_{2n},y_{2n+2}) + p(y_{2n+1},y_{2n+1}) \leq p(y_{2n},y_{2n+1}) + p(y_{2n+1},y_{2n+2}).$$

Suppose that $p(y_{2n+1},y_{2n+2}) > 0$. Using (3.6) together with $p(y_{2n},y_{2n+1}) = 0$ and the properties of the generalized altering distance functions ψ_1, ψ_2, we get

$$\Psi_1\big(p(y_{2n+1},y_{2n+2})\big) \leq \psi_1\bigg(0,0,p(y_{2n+1},y_{2n+2}),\frac{1}{2}p(y_{2n+1},y_{2n+2})\bigg)$$

$$- \psi_2\big(0,0,p(y_{2n+1},y_{2n+2})\big),$$

$$< \psi_1\bigg(0,0,p(y_{2n+1},y_{2n+2}),\frac{1}{2}p(y_{2n+1},y_{2n+2})\bigg)$$

$$\leq \Psi_1\big(p(y_{2n+1},y_{2n+2})\big),$$

which is a contradiction. It follows that $p(y_{2n+1},y_{2n+2}) = 0$ and hence $y_{2n+1} = y_{2n+2}$. Following similar arguments, we obtain $y_{2n+2} = y_{2n+3}$. Thus $\{y_n\}$ becomes an eventually constant sequence and y_{2n} is a point of coincidence of I and S, while y_{2n+1} is a point of coincidence of J and T.

Assume further that (3.5) holds. We claim that

$$\lim_{n\to\infty} p(y_{n+1},y_{n+2}) = 0. \tag{3.7}$$

Suppose that, for some $n \in \mathbb{N}$,

$$p(y_{2n+1},y_{2n+2}) > p(y_{2n},y_{2n+1}).$$

Using this together with the properties of generalized altering distance functions ψ_1, ψ_2, we get from (3.6) that

$$\Psi_1\big(p(y_{2n+1},y_{2n+2})\big)$$

$$\leq \psi_1\big(p(y_{2n},y_{2n+1}),p(y_{2n},y_{2n+1}),p(y_{2n+1},y_{2n+2}),p(y_{2n+1},y_{2n+2})\big)$$

$$- \psi_2\big(p(y_{2n},y_{2n+1}),p(y_{2n},y_{2n+1}),p(y_{2n+1},y_{2n+2})\big)$$

$$\leq \Psi_1\big(p(y_{2n+1},y_{2n+2})\big) - \psi_2\big(p(y_{2n},y_{2n+1}),p(y_{2n},y_{2n+1}),p(y_{2n+1},y_{2n+2})\big).$$

This implies that

$$\psi_2\big(p(y_{2n},y_{2n+1}),p(y_{2n},y_{2n+1}),p(y_{2n+1},y_{2n+2})\big) = 0,$$

which yields that $p(y_{2n+1}, y_{2n}) = 0$. Hence, we obtain a contradiction with (3.5). We deduce that

$$p(y_{2n+1}, y_{2n+2}) \leq p(y_{2n}, y_{2n+1}), \quad \forall n \in \mathbb{N}. \tag{3.8}$$

By a similar reasoning, we obtain that

$$p(y_{2n+2}, y_{2n+3}) \leq p(y_{2n+1}, y_{2n+2}), \quad \forall n \in \mathbb{N}. \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$p(y_{n+1}, y_{n+2}) \geq p(y_{n+2}, y_{n+3}), \quad \forall n \in \mathbb{N}.$$

Then, $\{p(y_{n+1}, y_{n+2})\}$ is a nonincreasing sequence of positive real numbers. This implies that there exists $r \geq 0$ such that

$$\lim_{n \to \infty} p(y_{n+1}, y_{n+2}) = r. \tag{3.10}$$

By (3.6), we have

$$
\begin{aligned}
\Psi_1\big(p(y_{2n+1}, y_{2n+2})\big) \\
&\leq \psi_1\big(p(y_{2n}, y_{2n+1}), p(y_{2n}, y_{2n+1}), p(y_{2n}, y_{2n+1}), p(y_{2n}, y_{2n+1})\big) \\
&\quad - \psi_2\big(p(y_{2n}, y_{2n+1}), p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2})\big), \\
&= \Psi_1\big(p(y_{2n}, y_{2n+1})\big) - \psi_2\big(p(y_{2n}, y_{2n+1}), p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2})\big). \tag{3.11}
\end{aligned}
$$

Letting $n \to \infty$ in (3.11) and using continuity of Ψ_1 and ψ_2, we obtain

$$\Psi_1(r) \leq \Psi_1(r) - \psi_2(r, r, r),$$

which implies that $\psi_2(r, r, r) = 0$, and thus $r = 0$. Hence, (3.7) is proved.

Next, we claim that $\{y_n\}$ is a Cauchy sequence in the space (X, p) (and also in the metric space (X, p^s) by Lemma 1). For this it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose that this is not the case. Then, using Lemma 3 we get that there exist $\varepsilon > 0$ and two sequences $\{m(i)\}$ and $\{n(i)\}$ of positive integers such that $n(i) > m(i) > i$ and sequences (3.2) tend to ε when $i \to \infty$. Applying condition (3.3) to elements $x = x_{2m(i)}$ and $y = x_{2n(i)-1}$, we get that

$$
\begin{aligned}
\Psi_1\big(p(y_{2m(i)+1}, y_{2n(i)})\big) &= \Psi_1\big(p(Sx_{2m(i)}, Tx_{2n(i)-1})\big) \\
&\leq \psi_1\bigg(p(y_{2m(i)}, y_{2n(i)-1}), p(y_{2m(i)}, y_{2m(i)+1}), p(y_{2n(i)-1}, y_{2n(i)}), \\
&\qquad \frac{1}{2}\big[p(y_{2m(i)}, y_{2n(i)}) + p(y_{2n(i)-1}, y_{2m(i)+1})\big]\bigg) \\
&\quad - \psi_2\big(p(y_{2m(i)}, y_{2n(i)-1}), p(y_{2m(i)}, y_{2m(i)+1}), p(y_{2n(i)-1}, y_{2n(i)})\big).
\end{aligned}
$$

Passing to the limit as $i \to \infty$ in the last inequality (and using the continuity of the functions ψ_1, ψ_2), we obtain

$$\Psi_1(\varepsilon) \le \psi_1(\varepsilon, 0, 0, \epsilon) - \psi_2(\varepsilon, 0, 0) \le \Psi_1(\varepsilon) - \psi_2(\varepsilon, 0, 0),$$

which implies that $\psi_2(\varepsilon, 0, 0) = 0$, that is a contradiction since $\varepsilon > 0$. We deduce that $\{y_n\}$ is a Cauchy sequence.

Finally, we prove the existence of a common fixed point of the four mappings I, J, S and T.

Since (X, p) is complete, then from Lemma 1, (X, p^s) is a complete metric space. Therefore, the sequence $\{y_n\}$ p^s-converges to some $z \in X$ that is, $\lim_{n \to \infty} p^s(y_n, z) = 0$. From (2.2), we have

$$p(z, z) = \lim_{n \to \infty} p(y_n, z) = \lim_{m \ge n \to \infty} p(y_n, y_m). \tag{3.12}$$

Moreover, since $\{y_n\}$ is a Cauchy sequence in the metric space (X, p^s), then $\lim_{n,m \to \infty} p^s(y_n, y_m) = 0$. On the other hand, by (p$_2$) and (3.7), we have $p(y_n, y_n) \le p(y_n, y_{n+1}) \to 0$, $n \to \infty$ and hence

$$\lim_{n \to \infty} p(y_n, y_n) = 0. \tag{3.13}$$

Thus from the definition of p^s and (3.13), we have $\lim_{m \ge n \to \infty} p(y_n, y_m) = 0$. Therefore, from (3.12), we have

$$p(z, z) = \lim_{n \to \infty} p(y_n, z) = \lim_{m \ge n \to \infty} p(y_n, y_m) = 0. \tag{3.14}$$

This implies that

$$\lim_{n \to \infty} p(y_{2n}, z) = \lim_{n \to \infty} p(y_{2n+1}, z) = 0. \tag{3.15}$$

Thus we have

$$\lim_{n \to \infty} p(Tx_{2n-1}, z) = \lim_{n \to \infty} p(Ix_{2n}, z) = 0$$

and

$$\lim_{n \to \infty} p(Sx_{2n}, z) = \lim_{n \to \infty} p(Jx_{2n+1}, z) = 0.$$

Now we can suppose, without loss of generality, that IX is a closed subset of the partial metric space (X, p). From (3.15), there exists $u \in X$ such that $z = Iu$. We claim that $Su = z$. Suppose, to the contrary, that $p(Su, z) > 0$. By (p$_4$) we get

$$p(z, Su) \le p(z, Tx_{2n-1}) + p(Su, Tx_{2n-1}) - p(Tx_{2n-1}, Tx_{2n-1})$$

$$\le p(z, y_{2n}) + p(Su, y_{2n}).$$

It follows by (3.15) that

$$p(z, Su) \le \limsup_{n \to \infty} p(Su, y_{2n}).$$

Then, since Ψ_1 is increasing and continuous, we get that

$$\Psi_1\big(p(z, Su)\big) \leq \limsup_{n\to\infty} \Psi_1\big(p(Su, y_{2n})\big). \tag{3.16}$$

Now, from (3.3)

$$
\begin{aligned}
\Psi_1\big(p(Su, y_{2n})\big) &= \Psi_1\big(p(Su, Tx_{2n-1})\big) \\
&\leq \psi_1\bigg(p(Iu, Jx_{2n-1}), p(Iu, Su), p(Jx_{2n-1}, Tx_{2n-1}), \\
&\qquad \frac{1}{2}\big[p(Iu, Tx_{2n-1}) + p(Su, Jx_{2n-1})\big]\bigg) \\
&\qquad - \psi_2\big(p(Iu, Jx_{2n-1}), p(Iu, Su), p(Jx_{2n-1}, Tx_{2n-1})\big) \\
&\leq \psi_1\bigg(p(z, y_{2n-1}), p(z, Su), p(y_{2n-1}, y_{2n}), \\
&\qquad \frac{1}{2}\big[p(z, y_{2n}) + p(Su, z) + p(z, y_{2n-1}) - p(z, z)\big]\bigg) \\
&\qquad - \psi_2\big(p(z, y_{2n-1}), p(z, Su), p(y_{2n-1}, y_{2n})\big). \tag{3.17}
\end{aligned}
$$

Passing to the upper limit as $n \to \infty$ in (3.17), we obtain using (3.14) and the continuity of ψ_1, ψ_2 that

$$
\begin{aligned}
\limsup_{n\to\infty} \Psi_1\big(p(Su, y_{2n})\big) &\leq \psi_1\bigg(0, p(z, Su), 0, \frac{1}{2}p(Su, z)\bigg) - \psi_2\big(0, p(z, Su), 0\big) \\
&< \Psi_1\big(p(z, Su)\big).
\end{aligned}
$$

Therefore, from (3.16) we have

$$\Psi_1\big(p(z, Su)\big) < \Psi_1\big(p(z, Su)\big),$$

which is a contradiction. Thus we deduce that

$$p(z, Su) = 0 \quad \text{and} \quad z = Su. \tag{3.18}$$

We get that $Su = Iu = z$, so u is a coincidence point of I and S.

From $SX \subset JX$ and (3.18), we have $z \in JX$. Hence we deduce that there exists $v \in X$ such that $z = Jv$. We claim that $Tv = z$. Suppose, to the contrary, that $p(Tv, z) > 0$. From (3.3), we have

$$
\begin{aligned}
\Psi_1\big(p(z, Tv)\big) &= \Psi_1\big(p(Su, Tv)\big) \\
&\leq \psi_1\bigg(p(Iu, Jv), p(Iu, Su), p(Jv, Tv), \frac{1}{2}\big[p(Iu, Tv) + p(Su, Jv)\big]\bigg) \\
&\qquad - \psi_2\big(p(Iu, Jv), p(Iu, Su), p(Jv, Tv)\big) \\
&= \psi_1\bigg(p(z, z), p(z, z), p(z, Tv), \frac{1}{2}\big[p(z, Tv) + p(z, z)\big]\bigg)
\end{aligned}
$$

$$- \psi_2\big(p(z,z), p(z,z), p(z, Tv)\big)$$

$$< \psi_1\left(0, 0, p(z, Tv), \frac{1}{2}p(z, Tv)\right) \leq \Psi_1\big(p(z, Tv)\big),$$

which is a contradiction. Then, we deduce that

$$p(z, Tv) = 0 \quad \text{and} \quad z = Tv. \tag{3.19}$$

We get that $Jv = Tv = z$, so v is a coincidence point of J and S.

Since the pair $\{S, I\}$ is weakly compatible, from (3.18), we have $Sz = SIu = ISu = Iz$. We claim that $Sz = z$. Suppose, to the contrary, that $p(Sz, z) > 0$. Then we have

$$p(Sz, z) \leq p(Sz, y_{2n}) + p(y_{2n}, z) = p(Sz, Tx_{2n-1}) + p(y_{2n}, z).$$

Again from (3.15) we get that

$$p(Sz, z) \leq \limsup_{n \to \infty} p(Sz, Tx_{2n-1}).$$

Then, since Ψ_1 is increasing and continuous, we get

$$\Psi_1\big(p(Sz, z)\big) \leq \limsup_{n \to \infty} \Psi_1\big(p(Sz, Tx_{2n-1})\big). \tag{3.20}$$

Now, from (3.3)

$$\Psi_1\big(p(Sz, Tx_{2n-1})\big) \leq \psi_1\bigg(p(Iz, Jx_{2n-1}), p(Iz, Sz), p(Jx_{2n-1}, Tx_{2n-1}),$$

$$\frac{1}{2}\big[p(Iz, Tx_{2n-1}) + p(Sz, Jx_{2n-1})\big]\bigg)$$

$$- \psi_2\big(p(Iz, Jx_{2n-1}), p(Iz, Sz), p(Jx_{2n-1}, Tx_{2n-1})\big)$$

$$\leq \psi_1\bigg(\big[p(Sz, z) + p(z, y_{2n-1}) - p(z, z)\big], p(Sz, Sz), p(y_{2n-1}, y_{2n}),$$

$$\frac{1}{2}\big[p(Sz, y_{2n}) + p(Sz, z) + p(z, y_{2n-1}) - p(z, z)\big]\bigg)$$

$$- \psi_2\big(p(Sz, y_{2n-1}), p(Sz, Sz), p(y_{2n-1}, y_{2n})\big).$$

Passing to the upper limit as $n \to \infty$, we obtain (since $p(Sz, Sz) \leq p(Sz, z)$)

$$\limsup_{n \to \infty} \Psi_1\big(p(Sz, Tx_{2n-1})\big) \leq \psi_1\big(p(Sz, z), p(Sz, Sz), 0, p(Sz, z)\big)$$

$$- \psi_2\big(p(Sz, z), p(Sz, Sz), 0\big)$$

$$< \Psi_1\big(p(Sz, z)\big).$$

Therefore, from (3.20) we have

$$\Psi_1\big(p(Sz, z)\big) < \Psi_1\big(p(Sz, z)\big),$$

a contradiction. This implies that

$$p(Sz, z) = 0 \quad \text{and} \quad z = Sz.$$

Hence, we have

$$Sz = z = Iz. \tag{3.21}$$

Since the pair $\{T, J\}$ is weakly compatible, from (3.19), we have $Tz = TJv = JTv = Jz$. We claim that $Tz = z$. Suppose, to the contrary, that $p(Tz, z) > 0$, then by (3.3), we have

$$
\begin{aligned}
\Psi_1\big(p(z, Tz)\big) &= \Psi_1\big(p(Sz, Tz)\big) \\
&\leq \psi_1\bigg(p(Iz, Jz), p(Iz, Sz), p(Jz, Tz), \frac{1}{2}\big[p(Iz, Tz) + p(Sz, Jz)\big]\bigg) \\
&\quad - \psi_2\big(p(Iz, Jz), p(Iz, Sz), p(Jz, Tz)\big) \\
&= \psi_1\bigg(p(z, Tz), 0, p(Tz, Tz), \frac{1}{2}\big[p(z, Tz) + p(z, Tz)\big]\bigg) \\
&\quad - \psi_2\big(p(z, Tz), 0, p(Tz, Tz)\big) \\
&\leq \Psi_1\big(p(z, Tz)\big) - \psi_2\big(p(z, Tz), 0, p(Tz, Tz)\big).
\end{aligned}
$$

Therefore, $\psi_2(p(Tz, z), 0, p(Tz, Tz)) = 0$. Hence, we have $p(z, Tz) = 0$ and

$$Tz = z = Jz. \tag{3.22}$$

Now, combining (3.21) and (3.22), we deduce

$$z = Iz = Sz = Tz = Jz,$$

so z is a common fixed point of the four mappings I, J, S and T.

We claim that there is a unique common fixed point of S, T, I and J. Assume on contrary that, $Su = Tu = Iu = Ju = u$ and $Sv = Tv = Iv = Jv = v$ with $p(u, v) > 0$. By supposition, we can replace x by u and y by v in (3.3) to obtain

$$
\begin{aligned}
\Psi_1\big(p(u, v)\big) &= \Psi_1\big(p(Su, Tv)\big) \\
&\leq \psi_1\bigg(p(Iu, Jv), p(Iu, Su), p(Jv, Tv), \frac{1}{2}\big[p(Iu, Tv) + p(Su, Jv)\big]\bigg) \\
&\quad - \psi_2\big(p(Iu, Jv), p(Iu, Su), p(Jv, Tv)\big) \\
&= \psi_1\big(p(u, v), 0, 0, p(u, v)\big) - \psi_2\big(p(u, v), 0, 0\big) \\
&< \Psi_1\big(p(u, v)\big),
\end{aligned}
$$

a contradiction. Hence $p(u, v) = 0$, that is, $u = v$. We conclude that S, T, I and J have only one common fixed point in X. The proof is complete. □

It is easy to state the corollary of Theorem 1 involving a contraction of integral type.

Corollary 1 *Let T, S, I and J as well as ψ_1, ψ_2 satisfy the conditions of Theorem 1, except that condition (3.3) is replaced by the following: there exists a positive Lebesgue integrable function u on $\mathbb{R}^+$ such that $\int_0^\varepsilon u(t)\,dt > 0$ for each $\varepsilon > 0$ and that*

$$\int_0^{\Psi_1(p(Sx,Ty))} u(t)\,dt \leq \int_0^{\psi_1(p(Ix,Jy),p(Ix,Sx),p(Jy,Ty),\frac{1}{2}[p(Ix,Ty)+p(Jy,Sx)])} u(t)\,dt$$
$$- \int_0^{\psi_2((p(Ix,Jy),p(Ix,Sx),p(Jy,Ty))} u(t)\,dt,$$

for all $x,y \in X$. Then, S, T, I and J have a unique common fixed point.

If in Theorem 1 $I = J$ is the identity mapping on X, then we have the following consequence:

Theorem 2 *Let (X,p) be a complete partial metric space. Let $T,S : X \to X$ be given mappings satisfying for every pair $(x,y) \in X \times X$*

$$\Psi_1(p(Sx,Ty)) \leq \psi_1\left(p(x,y),p(x,Sx),p(y,Ty),\frac{1}{2}[p(x,Ty)+p(y,Sx)]\right)$$
$$- \psi_2(p(x,y),p(x,Sx),p(y,Ty)), \tag{3.23}$$

where $\psi_1 \in \mathcal{F}_4$ and $\psi_2 \in \mathcal{F}_3$ are altering distance functions, and $\Psi_1(t) = \psi_1(t,t,t,t)$. Then T and S have a unique common fixed point.

Remark 1 Several corollaries of Theorems 1 and 2 could be derived for particular choices of ψ_1 and ψ_2. We state some of them.

Putting $\psi_1(t_1,t_2,t_3,t_4) = \psi(\max\{t_1,t_2,t_3,t_4\})$ and $\psi_2(t_1,t_2,t_3) = \phi(\max\{t_1,t_2,t_3\})$ for $\psi,\phi \in \mathcal{F}_1$, [15, Theorem 9] is obtained.

It is clear from the proof of Theorem 1 that condition (3.3), resp. (3.23), can be replaced by

$$\Psi_1(p(Sx,Ty))$$
$$\leq \psi_1\left(p(Ix,Jy),p(Ix,Sx),p(Jy,Ty),\frac{1}{2}[p(Ix,Ty)+p(Jy,Sx)]\right)$$
$$- \psi_2\left(p(Ix,Jy),p(Ix,Sx),p(Jy,Ty),\frac{1}{2}[p(Ix,Ty)+p(Jy,Sx)]\right), \tag{3.24}$$

resp.

$$\Psi_1(p(Sx,Ty))$$
$$\leq \psi_1\left(p(x,y),p(x,Sx),p(y,Ty),\frac{1}{2}[p(x,Ty)+p(y,Sx)]\right)$$
$$- \psi_2\left(p(x,y),p(x,Sx),p(y,Ty),\frac{1}{2}[p(x,Ty)+p(y,Sx)]\right), \tag{3.25}$$

where $\psi_2 \in \mathcal{F}_4$. Hence, Theorem 1 can be considered an extension of [23, Theorem 2.1] to the frame of partial metric spaces (since semi-compatible mappings are weakly compatible).

Putting $\psi_1(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$ and $\psi_2 = (1 - r)\psi_1$, with $0 \leq r < 1$, in Theorem 2 (with condition (3.25)), we obtain [20, Theorem 8]. The same substitution in Theorem 1 (with (3.24)) gives an improvement of [20, Theorem 12] (since only weak compatibility and not commutativity of the respective mappings is assumed).

Putting $\psi_1(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$ and $\psi_2 = \varphi \circ \psi_1$ for $\varphi \in \mathcal{F}_1$ in Theorem 2 (with condition (3.25)), [16, Theorem 5] is obtained.

Of course, several known results from the frame of standard metric spaces (see, *e.g.,* [30] and [31]) are also special cases of these theorems. For example, the following corollary can be obtained as a consequence of Theorem 2, which is a generalization and extension of [31, Corollary 3.2].

Corollary 2 *Let (X, p) be a complete partial metric space. Let $T, S : X \to X$ be given mappings satisfying for every pair $(x, y) \in X \times X$*

$$
\begin{aligned}
\Psi_1\big(p\big(S^m x, T^n y\big)\big) \\
\leq \psi_1\bigg(p(x, y), p\big(x, S^m x\big), p\big(y, T^n y\big), \frac{1}{2}\big[p\big(x, T^n y\big) + p\big(y, S^m x\big)\big] \bigg) \\
- \psi_2\big(p(x, y), p\big(x, S^m x\big), p\big(y, T^n y\big)\big),
\end{aligned}
$$

where m and n are positive integers, $\psi_1 \in \mathcal{F}_4$ and $\psi_2 \in \mathcal{F}_3$ are altering distance functions, and $\Psi_1(t) = \psi_1(t, t, t, t)$. Then T and S have a unique common fixed point.

Remark 2 However, it is not possible to use $\psi_1, \psi_2 \in \mathcal{F}_5$ in Theorems 1 and 2, as the following example, adapted from [23, Example 2.3], shows.

Example 1 Let $X = \{1, 2, 3, 4\}$ and $p : X \times X \to X$ be given by $p(x, x) = \frac{1}{2}$ for $x \in X$, $p(1, 2) = p(3, 4) = 2$, $p(1, 3) = p(2, 4) = 1$, $p(1, 4) = p(2, 3) = \frac{3}{2}$ and $p(y, x) = p(x, y)$ for $x, y \in X$. Then (X, p) is a (complete) partial metric space. Consider the mappings $S, T : X \to X$ defined by

$$
S = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 2 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 4 & 3 \end{pmatrix},
$$

and the functions $\psi_1, \psi_2 \in \mathcal{F}_5$ given as $\psi_1(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, t_2, t_3, t_4, t_5\}$ and $\psi_2 = \frac{1}{4}\psi_1$. It is easy to check that

$$
\begin{aligned}
p(Sx, Ty) &\leq \frac{3}{4} \max\big\{p(x, y), p(x, Sx), p(y, Ty), p(x, Ty), p(y, Sx)\big\} \\
&= \psi_1(\cdots) - \psi_2(\cdots)
\end{aligned}
$$

holds for all $x, y \in X$. However, these mappings have no common fixed points; hence, condition (3.23) (or (3.25)) of Theorem 2 cannot be replaced by the respective condition with 5 variables. At the same time, condition (3.25) is not satisfied since, for $x = 3$, $y = 1$, $p(Sx, Ty) = p(1, 4) = \frac{3}{2}$ and

$$
\max\bigg\{p(x, y), p(x, Sx), p(y, Ty), \frac{1}{2}\big[p(x, Ty) + p(y, Sx)\big]\bigg\} = \max\bigg\{1, 1, \frac{3}{2}, \frac{1}{2}\bigg(2 + \frac{1}{2}\bigg)\bigg\} = \frac{3}{2},
$$

hence,

$$\Psi_1\big(p(Sx, Ty)\big) = \frac{3}{2} > \psi_1(\cdots) - \psi_2(\cdots)$$

whatever $\psi_2 \in \mathcal{F}_4$ is chosen.

This example also shows (as in [23, Remark 2.4]) the importance of the second generalized altering distance function ψ_2 in Theorems 1 and 2.

The next example shows that Theorems 1 and 2 are proper extensions of the respective results in standard metric spaces.

Example 2 Let $X = [0, 1]$ be endowed with the partial metric $p(x, y) = \max\{x, y\}$. Consider the mappings $S, T : X \to X$ defined by

$$Sx = Tx = \frac{x^2}{1 + x},$$

and the functions $\psi_1, \psi_2 \in \mathcal{F}_4$, given by

$$\psi_1(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}, \qquad \psi_2(t_1, t_2, t_3, t_4) = \frac{\max\{t_1, t_2, t_3, t_4\}}{1 + \max\{t_1, t_2, t_3, t_4\}}.$$

Take arbitrary elements, say $y \le x$, from X. Then

$$p(Sx, Ty) = \max\left\{\frac{x^2}{1 + x}, \frac{y^2}{1 + y}\right\} = \frac{x^2}{1 + x}, \quad \text{and} \quad \Psi_1\big(p(Sx, Ty)\big) = \frac{x^2}{1 + x}.$$

On the other hand,

$$\max\left(p(x, y), p(x, Sx), p(y, Ty), \frac{1}{2}\big[p(x, Ty) + p(y, Sx)\big]\right)$$

$$= \max\left\{p(x, y), p\left(x, \frac{x^2}{1 + x}\right), p\left(y, \frac{y^2}{1 + y}\right), \frac{1}{2}\left(p\left(x, \frac{y^2}{1 + y}\right) + p\left(y, \frac{x^2}{1 + x}\right)\right)\right\}$$

$$= \max\left\{x, x, y, \frac{1}{2}\left(x + \max\left\{y, \frac{x^2}{1 + x}\right\}\right)\right\} = x,$$

and

$$\psi_1\left(p(x, y), p(x, Sx), p(y, Ty), \frac{1}{2}\big[p(x, Ty) + p(y, Sx)\big]\right)$$

$$- \psi_2\left(p(x, y), p(x, Sx), p(y, Ty), \frac{1}{2}\big[p(x, Ty) + p(y, Sx)\big]\right)$$

$$= x - \frac{x}{1 + x} = \frac{x^2}{1 + x}.$$

Hence, condition (3.25) is satisfied, as well as other conditions of Theorem 2. Mappings S, T have a common fixed point $z = 0$.

On the other hand, consider the same problem in the standard metric $d(x, y) = p^s(x, y) = |x - y|$ and take $x = 1$ and $y = \frac{1}{2}$. Then

$$d(Sx, Ty) = \left| \frac{1}{2} - \frac{1}{6} \right| = \frac{1}{3}$$

and

$$\max\left(d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}\big[d(x, Ty) + d(y, Sx)\big] \right)$$

$$= \max\left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}\left(\frac{5}{6} + 0 \right) \right\} = \frac{1}{2}$$

and hence

$$\psi_1\left(d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}\big[d(x, Ty) + d(y, Sx)\big] \right)$$

$$- \psi_2\left(d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}\big[d(x, Ty) + d(y, Sx)\big] \right)$$

$$= \frac{1}{2} - \frac{\frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{6}.$$

Thus, condition (3.25) for $p = d$ does not hold and the existence of a common fixed point of these mappings cannot be derived from [23, Theorem 2.1].

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